Hyperbolic approximation of the Navier-Stokes-Fourier system

Timothée Crin-Barat

Chair of Dynamics, Control and Numerics Department of Data Science, FAU, Erlangen, Germany

Tongji University, November 24 2023

Joint work with S. Kawashima, J. Xu and E. Zuazua.

Paradox of heat conduction

• One of the most successful models in continuum physics is Fourier's law of heat conduction

$$q = -\kappa \nabla T$$

where q is the thermal flux vector, T is the temperature, and $\kappa > 0$ stands for the thermal conductivity.

Paradox of heat conduction

• One of the most successful models in continuum physics is Fourier's law of heat conduction

$$q = -\kappa \nabla T$$

where q is the thermal flux vector, T is the temperature, and $\kappa > 0$ stands for the thermal conductivity.

• With this law, the widely used full compressible Navier-Stokes system in $\mathbb{R}^d$ reads:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla p = \mathrm{div}\,\tau, \\ \partial_t(\rho T) + \mathrm{div}\,(\rho u T + up) - \kappa \Delta T - \mathrm{div}\,(\tau \cdot u) = 0. \end{cases} \tag{1}$$

• A shortcoming of Fourier's law is that it leads to a parabolic equation for the temperature field: any initial disturbance is felt instantly throughout the entire medium.

Paradox of heat conduction

• One of the most successful models in continuum physics is Fourier's law of heat conduction

$$q = -\kappa \nabla T$$

where q is the thermal flux vector, T is the temperature, and $\kappa > 0$ stands for the thermal conductivity.

• With this law, the widely used full compressible Navier-Stokes system in $\mathbb{R}^d$ reads:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla p = \mathrm{div}\,\tau, \\ \partial_t(\rho T) + \mathrm{div}\,(\rho u T + u p) - \kappa \Delta T - \mathrm{div}\,(\tau \cdot u) = 0. \end{cases} \tag{1}$$

• A shortcoming of Fourier's law is that it leads to a parabolic equation for the temperature field: any initial disturbance is felt instantly throughout the entire medium.

→ Such behavior contradicts the principle of causality.

An alternative: Cattaneo's law

- To correct this unrealistic feature one can use the Maxwell-Cattaneo law:

$$\varepsilon^2 \partial_t q + q = -\kappa \nabla T,$$

  where ε is the thermal relaxation characteristic time

- However, this leads to a non-Galilean invariant model. In '09, Christov formulated the following law

$$\varepsilon^2 \left(\partial_t q + u \cdot \nabla q - q \cdot \nabla u + (\nabla \cdot u) q\right) + q = -\kappa \nabla T. \tag{2}$$

An alternative: Cattaneo's law

- To correct this unrealistic feature one can use the Maxwell-Cattaneo law:

$$\varepsilon^2 \partial_t q + q = -\kappa \nabla T,$$

where ε is the thermal relaxation characteristic time

- However, this leads to a non-Galilean invariant model. In '09, Christov formulated the following law

$$\varepsilon^2 \left(\partial_t q + u \cdot \nabla q - q \cdot \nabla u + (\nabla \cdot u) q\right) + q = -\kappa \nabla T. \tag{2}$$

- Essentially, $-\Delta T$ is now replaced by the first-order coupling (in blue) below:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla p = \mathrm{div}\,\tau, \\ \partial_t(\rho T) + \mathrm{div}\,(\rho u T + up) + \mathrm{div}\, q - \mathrm{div}\,(\tau \cdot u) = 0, \\ \varepsilon^2 \left(\partial_t q + u \cdot \nabla q - q \cdot \nabla u + (\nabla \cdot u) q\right) + q + \kappa \nabla T = 0, \end{cases} \tag{3}$$

An alternative: Cattaneo's law

- To correct this unrealistic feature one can use the Maxwell-Cattaneo law:

$$\varepsilon^2 \partial_t q + q = -\kappa \nabla T,$$

where ε is the thermal relaxation characteristic time

- However, this leads to a non-Galilean invariant model. In '09, Christov formulated the following law

$$\varepsilon^2 \left(\partial_t q + u \cdot \nabla q - q \cdot \nabla u + (\nabla \cdot u) q \right) + q = -\kappa \nabla T. \tag{2}$$

- Essentially, $-\Delta T$ is now replaced by the first-order coupling (in blue) below:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div} \tau, \\ \partial_t(\rho T) + \operatorname{div}(\rho u T + u p) + \operatorname{div} q - \operatorname{div}(\tau \cdot u) = 0, \\ \varepsilon^2 \left(\partial_t q + u \cdot \nabla q - q \cdot \nabla u + (\nabla \cdot u) q \right) + q + \kappa \nabla T = 0, \end{cases} \tag{3}$$

- $\rightarrow$ Finite speed of propagation for the temperature.

- To correct this unrealistic feature one can use the Maxwell-Cattaneo law:

$$\varepsilon^2 \partial_t q + q = -\kappa \nabla T,$$

where ε is the thermal relaxation characteristic time

- However, this leads to a non-Galilean invariant model. In '09, Christov formulated the following law

$$\varepsilon^2 \left(\partial_t q + u \cdot \nabla q - q \cdot \nabla u + (\nabla \cdot u) q\right) + q = -\kappa \nabla T. \tag{2}$$

- Essentially, $-\Delta T$ is now replaced by the first-order coupling (in blue) below:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div} \tau, \\ \partial_t(\rho T) + \operatorname{div}(\rho u T + u p) + \operatorname{div} q - \operatorname{div}(\tau \cdot u) = 0, \\ \varepsilon^2 \left(\partial_t q + u \cdot \nabla q - q \cdot \nabla u + (\nabla \cdot u) q\right) + q + \kappa \nabla T = 0, \end{cases} \tag{3}$$

- $\to$ Finite speed of propagation for the temperature.
- Question: How to justify rigorously the limit $\varepsilon \to 0$?
- Element of response to the *paradox of heat conduction*.
- Useful for numerics.

First-order partially dissipative coupling

Porous media approximation

- The compressible Euler equations with damping reads:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \varepsilon^2(\partial_t u + u \cdot \nabla u) + \dfrac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \tag{E}$$

This system can be understood as a hyperbolic approximation, as $\varepsilon \to 0$, of the solution of the porous media equation:

$$\partial_t n - \Delta P(n) = 0.$$

Porous media approximation

- The compressible Euler equations with damping reads:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \varepsilon^2 (\partial_t u + u \cdot \nabla u) + \dfrac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \tag{E}$$

This system can be understood as a hyperbolic approximation, as $\varepsilon \to 0$, of the solution of the porous media equation:

$$\partial_t n - \Delta P(n) = 0.$$

- Numerous results in the 1D case: Jin-Xin '95, Junca-Rascle '02.

- The compressible Euler equations with damping reads:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \varepsilon^2(\partial_t u + u \cdot \nabla u) + \dfrac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \tag{E}$$

This system can be understood as a hyperbolic approximation, as $\varepsilon \to 0$, of the solution of the porous media equation:

$$\partial_t n - \Delta P(n) = 0.$$

- Numerous results in the 1D case: Jin-Xin '95, Junca-Rascle '02.
- Weak convergence result in the multi-dimensional case: Coulombel-Goudon-Lin '07 '13, Fang-Xu '09, Kawashima-Xu '14
- Strong convergence in $\mathbb{R}^d$ with $d \geq 1$ for global-in-time strong solutions being small perturbations of $(\bar{\rho}, \bar{u}) = (\bar{\rho}, 0)$ with $\bar{\rho} > 0$: Danchin-CB '22.

- The compressible Euler equations with damping reads:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \varepsilon^2(\partial_t u + u \cdot \nabla u) + \dfrac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \tag{E}$$

This system can be understood as a hyperbolic approximation, as $\varepsilon \to 0$, of the solution of the porous media equation:

$$\partial_t n - \Delta P(n) = 0.$$

- Numerous results in the 1D case: Jin-Xin '95, Junca-Rascle '02.
- Weak convergence result in the multi-dimensional case: Coulombel-Goudon-Lin '07 '13, Fang-Xu '09, Kawashima-Xu '14
- Strong convergence in $\mathbb{R}^d$ with $d \geq 1$ for global-in-time strong solutions being small perturbations of $(\bar{\rho}, \bar{u}) = (\bar{\rho}, 0)$ with $\bar{\rho} > 0$: Danchin-CB '22.
- **Tools**: Littlewood-Paley, Shizuta-Kawashima's theory and hypocoercivity theory.

Let us have a look at the one-dimensional damped p-system

$$\begin{cases} \partial_t \rho + \partial_x u = 0, \\ \partial_t u + \partial_x \rho + \frac{u}{\varepsilon} = 0. \end{cases}$$

- **Goal:** obtain uniform-in-ε a priori estimates.

Let us have a look at the one-dimensional damped p-system

$$\begin{cases} \partial_t \rho + \partial_x u = 0, \\ \partial_t u + \partial_x \rho + \dfrac{u}{\varepsilon} = 0. \end{cases}$$

- **Goal:** obtain uniform-in-ε a priori estimates.
- **First difficulty:** how to handle the *partially dissipative* structure? Indeed, standard energy estimates leads to:

$$\frac{d}{dt}\|(\rho, u)\|_{L^2}^2 + \frac{1}{\varepsilon}\|u\|_{L^2}^2 \leq 0$$

$\rightarrow$ **lack of coercivity: no time-decay information for ρ.**

Toy-model analysis

Let us have a look at the one-dimensional damped p-system

$$\begin{cases} \partial_t \rho + \partial_x u = 0, \\ \partial_t u + \partial_x \rho + \dfrac{u}{\varepsilon} = 0. \end{cases}$$

- **Goal:** obtain uniform-in-ε a priori estimates.
- **First difficulty:** how to handle the *partially dissipative* structure? Indeed, standard energy estimates leads to:

$$\frac{d}{dt}\|(\rho, u)\|_{L^2}^2 + \frac{1}{\varepsilon}\|u\|_{L^2}^2 \leq 0$$

$\rightarrow$ **lack of coercivity: no time-decay information for ρ.**

- Idea: Inspired by the hypocoercivity theory, consider the following perturbed functional

$$\mathcal{L}^2 = \|(\rho, u, \partial_x \rho, \partial_x u)\|_{L^2}^2 + \varepsilon \int_{\mathbb{R}} u \partial_x \rho.$$

Differentiating in time this functional, one obtains

$$\frac{d}{dt}\mathcal{L}^2 + \frac{1}{\varepsilon}\|(u, \partial_x u)\|_{L^2}^2 + \varepsilon\|\partial_x \rho\|_{L^2}^2 \leq 0.$$

- **Second difficulty**: the decay rates depend on the frequencies and the relaxation parameter ε.

  From the previous estimate, one obtains formally

$$<< \quad \frac{d}{dt}\|(\rho,u)\|_{L^2} + \min(\frac{1}{\varepsilon}, \varepsilon|\xi|^2)\|(\rho,u)\|_{L^2} \leq 0. \quad >>$$

Toy-model analysis (continued)

- **Second difficulty**: the decay rates depend on the frequencies and the relaxation parameter ε.

  From the previous estimate, one obtains formally

  $$<< \quad \frac{d}{dt}\|(\rho,u)\|_{L^2} + \min\left(\frac{1}{\varepsilon}, \varepsilon|\xi|^2\right)\|(\rho,u)\|_{L^2} \leq 0. \quad >>$$

- Therefore, in low frequencies $|\xi| < \dfrac{1}{\varepsilon}$, the solution behaves as the solution of the heat equation.
- And, in high frequencies $|\xi| > \dfrac{1}{\varepsilon}$, the solution is exponentially damped.

- **Second difficulty**: the decay rates depend on the frequencies and the relaxation parameter ε.

  From the previous estimate, one obtains formally

$$<< \quad \frac{d}{dt}\|(\rho,u)\|_{L^2} + \min\left(\frac{1}{\varepsilon}, \varepsilon|\xi|^2\right)\|(\rho,u)\|_{L^2} \leq 0. \quad >>$$

- Therefore, in low frequencies $|\xi| < \dfrac{1}{\varepsilon}$, the solution behaves as the solution of the heat equation.
- And, in high frequencies $|\xi| > \dfrac{1}{\varepsilon}$, the solution is exponentially damped.
- One has

$$\|(\rho,u)^h(t)\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)} \leq Ce^{-\lambda\frac{t}{\varepsilon}}\|(\rho_0,u_0)\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)},$$
$$\|(\rho,u)^\ell(t)\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^n)} \leq C(\varepsilon t)^{-\frac{d}{2}}\|(\rho_0,u_0)\|_{L^1(\mathbb{R}^d,\mathbb{R}^n)}$$

  where $(\rho,u)^h$ and $(\rho,u)^\ell$ correspond, respectively, to the high and low frequencies of the solution.

For general partially dissipative hyperbolic systems of the form

$$\partial_t U + A\partial_x U + BU = 0 \quad \text{where} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \text{with } D > 0,$$

the previous idea can also be applied under the following condition:

Definition (Shizuta-Kawashima '80s)

$$\ker B \cap \{\text{eigenvectors of } A\} = \{0\}. \tag{SK}$$

For general partially dissipative hyperbolic systems of the form

$$\partial_t U + A\partial_x U + BU = 0 \quad \text{where} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \text{with } D > 0,$$

the previous idea can also be applied under the following condition:

Definition (Shizuta-Kawashima '80s)

$$\ker B \cap \{\text{eigenvectors of } A\} = \{0\}. \tag{SK}$$

Such condition is actually equivalent to the Kalman rank condition for the couple (A, B).

For general partially dissipative hyperbolic systems of the form

$$\partial_t U + A\partial_x U + BU = 0 \quad \text{where} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \text{with } D > 0,$$

the previous idea can also be applied under the following condition:

Definition (Shizuta-Kawashima '80s)

$$\ker B \cap \{\text{eigenvectors of } A\} = \{0\}. \tag{SK}$$

Such condition is actually equivalent to the Kalman rank condition for the couple (A, B).

Inspired by this fact and the theories of hypocoercivity and hypoellipticity, Beauchard and Zuazua constructed the following Lyapunov functional

$$\mathcal{L}^2 \triangleq \|U\|_{H^1}^2 + \int_{\mathbb{R}^d} \mathcal{I} \quad \text{where} \quad \mathcal{I} \triangleq \Im \sum_{k=1}^{n-1} \varepsilon_k \big(BA^{k-1}\widehat{U} \cdot BA^k \widehat{U}\big).$$

For general partially dissipative hyperbolic systems of the form

$$\partial_t U + A\partial_x U + BU = 0 \quad \text{where} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \text{with } D > 0,$$

the previous idea can also be applied under the following condition:

Definition (Shizuta-Kawashima '80s)

$$\ker B \cap \{\text{eigenvectors of } A\} = \{0\}. \tag{SK}$$

Such condition is actually equivalent to the Kalman rank condition for the couple (A, B).

Inspired by this fact and the theories of hypocoercivity and hypoellipticity, Beauchard and Zuazua constructed the following Lyapunov functional

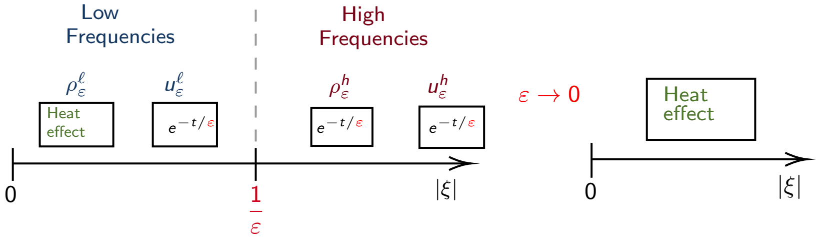
$$\mathcal{L}^2 \triangleq \|U\|_{H^1}^2 + \int_{\mathbb{R}^d} \mathcal{I} \quad \text{where} \quad \mathcal{I} \triangleq \Im \sum_{k=1}^{n-1} \varepsilon_k \big(BA^{k-1}\widehat{U} \cdot BA^k \widehat{U}\big).$$

If the (SK) condition is satisfied, differentiating in time this functional leads to

$$\frac{d}{dt}\mathcal{L} + \kappa \min(1, |\xi|^2)\mathcal{L} \leq 0 \quad \text{and } \mathcal{L} \sim \|U\|_{H^1}$$

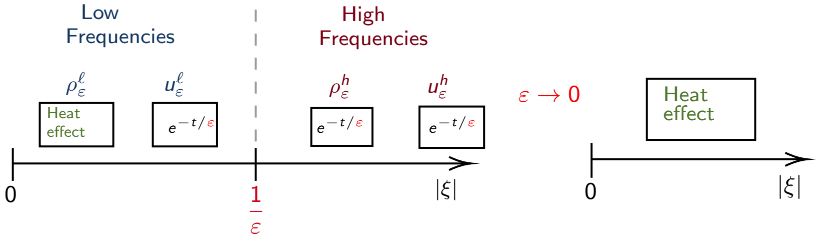
Cattaneo approximation:

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x u_\varepsilon = 0 \\ \varepsilon^2 \partial_t u_\varepsilon + \partial_x \rho_\varepsilon + u_\varepsilon = 0 \end{cases} \quad \underset{\varepsilon \to 0}{\longrightarrow} \quad \partial_t \rho - \Delta \rho = 0$$



Cattaneo approximation:

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x u_\varepsilon = 0 \\ \varepsilon^2 \partial_t u_\varepsilon + \partial_x \rho_\varepsilon + u_\varepsilon = 0 \end{cases} \quad \xrightarrow[\varepsilon \to 0]{} \quad \partial_t \rho - \Delta \rho = 0$$



- We proved the strong relaxation limit in $\mathbb{R}^d$ in various contexts
  - Compressible Euler equations with damping (Danchin-CB, Math. Ann.).
  - Jin-Xin System (Shou-CB, JDE).
  - 2D-Boussinesq system (Bianchini-Paicu-CB, ARMA).
- How to show it for the Navier-Stokes-Cattaneo system?

A (partially) hyperbolic Navier-Stokes system

We have just seen that the equation

$$\partial_t u - \Delta u = 0$$

can be approximated, for a small ε, by the following hyperbolic system

$$\begin{cases} \partial_t u + \operatorname{div} v = 0 \\ \varepsilon^2 \partial_t v + \nabla u + v = 0. \end{cases}$$

- Aim: understand to what extent this approximation can be used to approximate systems modelling physical phenomena.

We have just seen that the equation

$$\partial_t u - \Delta u = 0$$

can be approximated, for a small ε, by the following hyperbolic system

$$\begin{cases} \partial_t u + \operatorname{div} v = 0 \\ \varepsilon^2 \partial_t v + \nabla u + v = 0. \end{cases}$$

- Aim: understand to what extent this approximation can be used to approximate systems modelling physical phenomena.

Performing such approximation for the compressible Navier-Stokes system, one has

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div} \tau, \\ \partial_t(\rho T) + \operatorname{div}(\rho u T + up) + \operatorname{div} q - \operatorname{div}(\tau \cdot u) = 0, \\ \varepsilon^2 \left(\partial_t q + u \cdot \nabla q - q \cdot \nabla u + (\nabla \cdot u) q\right) + q + \kappa \nabla T = 0, \end{cases} \tag{4}$$

Let us now see how to justify that the solution of this system converges to the solution of the classical Navier-Stokes equations.

- **Knowledge on the limit system:** Danchin showed the existence of global-in-time solutions by highlighting different properties for $|\xi| \leq K$ and $|\xi| \geq K$ where K is a large constant.

Frequency splitting

- **Knowledge on the limit system:** Danchin showed the existence of global-in-time solutions by highlighting different properties for $|\xi| \leq K$ and $|\xi| \geq K$ where K is a large constant.
- **Knowledge on the hyperbolic approximation:** It suggests to distinguish two distinct frequency regimes with a threshold located at $\dfrac{1}{\varepsilon}$.

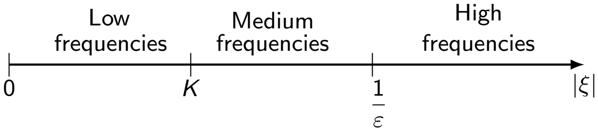
- **Knowledge on the limit system:** Danchin showed the existence of global-in-time solutions by highlighting different properties for $|\xi| \leq K$ and $|\xi| \geq K$ where K is a large constant.
- **Knowledge on the hyperbolic approximation:** It suggests to distinguish two distinct frequency regimes with a threshold located at $\frac{1}{\varepsilon}$.

Complete picture: We divide the frequency space as

Frequency splitting

- **Knowledge on the limit system:** Danchin showed the existence of global-in-time solutions by highlighting different properties for $|\xi| \leq K$ and $|\xi| \geq K$ where K is a large constant.
- **Knowledge on the hyperbolic approximation:** It suggests to distinguish two distinct frequency regimes with a threshold located at $\frac{1}{\varepsilon}$.

Complete picture: We divide the frequency space as



Formally, when $\varepsilon \to 0$, it means that:

- The low frequency regime is not modified.
- The mid-frequency regime becomes larger and larger and recovers the high-frequency regime.
- The high frequency regime disappears.

$\to$ We retrieve the behavior of the compressible Navier-Stokes-Fourier system in the limit.

Tools

- We define homogeneous Besov spaces restricted in frequency as follows:

$$\|f\|^{\ell}_{\dot{B}^s_{2,1}} := \sum_{j\leq J_0} 2^{js}\|f_j\|_{L^2}, \qquad \|f\|^{m,\varepsilon}_{\dot{B}^s_{p,1}} := \sum_{J_0\leq j\leq J_\varepsilon} 2^{js}\|f_j\|_{L^p},$$

$$\|f\|^{h,\varepsilon}_{\dot{B}^s_{2,1}} := \sum_{j\geq J_\varepsilon-1} 2^{js}\|f_j\|_{L^2}$$

  where $J_0 = \log_2(K)$, for $K > 0$ a constant, and $J_\varepsilon = -\kappa\log_2(\varepsilon)$.

- In each regime, the partially diffusive and partially dissipative coupling are involved. $\to$ New methods to derive a priori estimates: hypocoercivity + efficient unknowns.

Tools

- We define homogeneous Besov spaces restricted in frequency as follows:

$$\|f\|^{\ell}_{\dot{B}^s_{2,1}} := \sum_{j\leq J_0} 2^{js}\|f_j\|_{L^2}, \qquad \|f\|^{m,\varepsilon}_{\dot{B}^s_{p,1}} := \sum_{J_0\leq j\leq J_\varepsilon} 2^{js}\|f_j\|_{L^p},$$

$$\|f\|^{h,\varepsilon}_{\dot{B}^s_{2,1}} := \sum_{j\geq J_\varepsilon-1} 2^{js}\|f_j\|_{L^2}$$

  where $J_0 = \log_2(K)$, for $K>0$ a constant, and $J_\varepsilon = -\kappa\log_2(\varepsilon)$.

- In each regime, the partially diffusive and partially dissipative coupling are involved. $\rightarrow$ New methods to derive a priori estimates: hypocoercivity + efficient unknowns.

Morale

- The hyperbolic approximation *creates* a temporary high-frequency regime that disappears in the limit.
- The remaining frequency regimes correspond to the behaviour of the limit system.
- Difficulty: justify that the linear and nonlinear analysis can be done in the *new* high-frequency setting.

• First: use our knowledge of the limit system. We know that in high frequencies the Navier-Stokes system can be “partially diagonalized”.

• First: use our knowledge of the limit system. We know that in high frequencies the Navier-Stokes system can be "partially diagonalized".

• Defining the effective velocity, as introduced by Hoff and Haspot, $w = u + (-\Delta)^{-1}\nabla\rho$, in high frequencies, the linear system we are interested in reads

$$\begin{cases} \partial_t\rho + \rho = \operatorname{div} w, \\ \partial_t w - \Delta w = w - (-\Delta)^{-1}\nabla\rho + \nabla\theta, \\ \partial_t\theta + \operatorname{div} q + \operatorname{div} w = 0, \\ \varepsilon^2\partial_t q + q + \nabla\theta = 0, \end{cases} \tag{5}$$

• The equations of ρ and w can be studied separately, we simply need to be careful about the linear source terms.

• First: use our knowledge of the limit system. We know that in high frequencies the Navier-Stokes system can be "partially diagonalized".

• Defining the effective velocity, as introduced by Hoff and Haspot, $w = u + (-\Delta)^{-1}\nabla\rho$, in high frequencies, the linear system we are interested in reads

$$\begin{cases} \partial_t\rho + \rho = \mathrm{div}\, w, \\ \partial_t w - \Delta w = w - (-\Delta)^{-1}\nabla\rho + \nabla\theta, \\ \partial_t\theta + \mathrm{div}\, q + \mathrm{div}\, w = 0, \\ \varepsilon^2\partial_t q + q + \nabla\theta = 0, \end{cases} \tag{5}$$

• The equations of ρ and w can be studied separately, we simply need to be careful about the linear source terms.

• For the Cattaneo part, we introduce the Lyapunov (in the spirit of that of Beauchard and Zuazua and the hypocoercivity theory)

$$\mathcal{L}_j^h = \|(\theta_j, q_j)\|_{L^2}^2 + 2^{-2j}\int_{\mathbb{R}^d} q_j \cdot \nabla\theta_j \quad \text{for } j \geq J_\varepsilon. \tag{6}$$

$\rightarrow$ The blue term allows to recover dissipation for θ. Using that $\mathcal{L}_j^h \sim \|(\theta_j, q_j)\|_{L^2}^2$, direct computations gives

$$\frac{d}{dt}\mathcal{L}_j^h + \mathcal{L}_j^h \leq \|\mathrm{div}\, w_j\|_{L^2}\|\theta_j\|_{L^2}.$$

Some nonlinear analysis

Some nonlinear analysis

- We are working in the $L^2 - L^p$ framework:

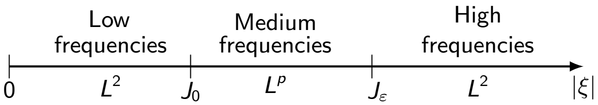


Figure: Frequency domain splitting for Navier-Stokes Cattaneo

- Due to the lack of embedding of the type $B^s_{p,1} \hookrightarrow B^s_{2,1}$ if $p > 2 \rightarrow$ it is difficult to absorb nonlinearities in the high and low-frequency regimes.

- We are working in the $L^2 - L^p$ framework:

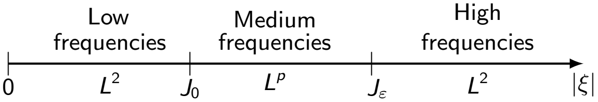


Figure: Frequency domain splitting for Navier-Stokes Cattaneo

- Due to the lack of embedding of the type $B^s_{p,1} \hookrightarrow B^s_{2,1}$ if $p > 2 \rightarrow$ it is difficult to absorb nonlinearities in the high and low-frequency regimes.
- Indeed, the medium frequencies are only bounded in L^p-based spaces.
- $\rightarrow$ Need to develop advanced product laws.

For instance: let $2 \leq p \leq 4$ and $p^* \triangleq 2p/(p-2)$. For all $s > 0$, we have

$$\|ab\|^{h,\varepsilon}_{\dot{B}^s_{2,1}} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|b\|^{h,\varepsilon}_{\dot{B}^s_{2,1}} + \|b\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a\|^{h,\varepsilon}_{\dot{B}^s_{2,1}}$$
$$+ \|a\|^{\ell,\varepsilon}_{\dot{B}^{\frac{d}{p}}_{p,1}} \|b\|^{\ell,\varepsilon}_{\dot{B}^{s+\frac{d}{p}-\frac{d}{2}}_{p,1}} + \|b\|^{\ell,\varepsilon}_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a\|^{\ell,\varepsilon}_{\dot{B}^{s+\frac{d}{p}-\frac{d}{2}}_{p,1}}.$$

Tools: Bony paraproduct decomposition and precise frequency analysis.

Ill-prepared relaxation result in a critical framework

Ill-prepared relaxation result in a critical framework

Theorem (Kawashima-Xu-Zuazua-CB '23)

Let $d \geq 3$, $p \in [2,4]$ *and* $P(\rho,\theta) = \pi(\rho)\theta$, $\bar\rho, \bar\theta > 0$

- *Let* $(\rho^\varepsilon - \bar\rho, v^\varepsilon, \theta^\varepsilon - \bar\theta, q^\varepsilon)$ *be the global solution of Navier-Stokes-Cattaneo (constructed with the previous arguments) with initial data* $(\rho_0^\varepsilon, v_0^\varepsilon, \theta_0^\varepsilon, q_0^\varepsilon)$.
- *Let* $(\rho - \bar\rho, v, \theta - \bar\theta)$ *be the global solution of Navier-Stokes-Fourier with initial data* (ρ_0, v_0, θ_0).

We define the error unknowns $(\widetilde\rho, \widetilde v, \widetilde\theta)$ *as*

$$(\widetilde\rho, \widetilde v, \widetilde\theta) := (\rho^\varepsilon - \rho, v^\varepsilon - v, \theta^\varepsilon - \theta).$$

If we assume that

$$\|(\widetilde\rho_0, \widetilde v_0, \widetilde\theta_0)\|^{\ell}_{B^{\frac{d}{2}-1}_{2,1}} + \|\widetilde\rho_0\|^{h}_{B^{\frac{d}{p}-1}_{p,1}} + \|(\widetilde v_0, \widetilde\theta_0)\|^{h}_{B^{\frac{d}{p}-1}_{p,1}} \lesssim \varepsilon. \quad (7)$$

Then, we have the strong convergence result:

$$\|(\widetilde\rho, \widetilde v, \widetilde\theta)\|^{\ell}_{L^\infty_T(B^{\frac{d}{2}-2}_{2,1})} + \|(\widetilde\rho, \widetilde v, \widetilde\theta)\|^{\ell}_{L^1_T(B^{\frac{d}{2}}_{2,1})} + \|q^\varepsilon + \kappa\nabla\theta^\varepsilon\|_{L^1_T(B^{\frac{d}{p}-1}_{p,1})}$$
$$+ \|\widetilde\rho\|^{h}_{L^\infty_T\cap L^1_T(B^{\frac{d}{p}-1}_{p,1})} + \|(\widetilde v, \widetilde\theta)\|^{h}_{L^\infty_T(B^{\frac{d}{p}-2}_{p,1})} + \|(\widetilde v, \widetilde\theta)\|^{h}_{L^1_T(B^{\frac{d}{p}}_{p,1})} \lesssim \varepsilon$$

Extensions

Extensions

- To what extent can this hyperbolic approximation be used? Numerical schemes, PINNs.

Extensions

- To what extent can this hyperbolic approximation be used? Numerical schemes, PINNs.
- What about other operators that the laplacian?

- To what extent can this hyperbolic approximation be used? Numerical schemes, PINNs.
- What about other operators that the laplacian?

  With Roberta Bianchini and Marius Paicu (ARMA '23), we showed that the stably stratified solutions of the incompressible porous media equation:

$$\partial_t \rho - \mathcal{R}_1^2 \rho = 0 \quad \text{with } \mathcal{R}_1 = \frac{\partial_1}{\sqrt{-\Delta}}$$

  can be approximated by the 0-th order stratified Boussinesq system:

$$\begin{cases} \partial_t \rho + \mathcal{R}_1 b = 0, \\ \varepsilon \partial_t b + \mathcal{R}_1 \rho + b = 0. \end{cases} \tag{2DB}$$

  Such justification involves anisotropic Besov spaces so as to recover crucial $L^1_T(W^{1,\infty})$ bounds on the solution.

Extensions

- To what extent can this hyperbolic approximation be used? Numerical schemes, PINNs.
- What about other operators that the laplacian?

  With Roberta Bianchini and Marius Paicu (ARMA '23), we showed that the stably stratified solutions of the incompressible porous media equation:

  $$\partial_t \rho - \mathcal{R}_1^2 \rho = 0 \quad \text{with } \mathcal{R}_1 = \frac{\partial_1}{\sqrt{-\Delta}}$$

  can be approximated by the 0-th order stratified Boussinesq system:

  $$\begin{cases} \partial_t \rho + \mathcal{R}_1 b = 0, \\ \varepsilon \partial_t b + \mathcal{R}_1 \rho + b = 0. \end{cases} \tag{2DB}$$

  Such justification involves anisotropic Besov spaces so as to recover crucial $L^1_T(W^{1,\infty})$ bounds on the solution.
- Question: under what conditions can an operator be approximated in this fashion?
- Interplay of partial dissipation, anisotropy and special structure of the nonlinearities.

Thank you for your attention!