ON THE DECAY OF ONE-DIMENSIONAL LOCALLY AND PARTIALLY DISSIPATED HYPERBOLIC SYSTEMS

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Abstract. We study the time-asymptotic behavior of linear hyperbolic systems subject to partial dissipation that is localized in suitable subsets of the domain. Specifically, we recover the classical decay rates of partially dissipative systems that satisfy the stability condition (SK), with a time-delay that depends only on the velocity of each component and the size of the undamped region. To quantify this delay, we assume that the undamped region is a bounded space interval and that the system, without space-restriction on the dissipation, satisfies the stability condition (SK). The former assumption ensures that the time spent by the characteristics of the system in the undamped region is finite, and the latter ensures that the solutions decay whenever the damping is active. Our approach consists of reformulating the system into n coupled transport equations and showing that the time-decay estimates are delayed by the sum of the times that each characteristic spends in the undamped region.

1. Introduction

We consider linear hyperbolic systems of the form

$$\begin{cases} \partial_t U + A\partial_x U = -BU\mathbb{1}_\omega, & (x,t) \in \mathbb{R}\times(0,\infty), \\ U(0,x) = U_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\omega := [-R,R]^c$, $R>0$, $U : \mathbb{R}\times(0,\infty)\to\mathbb{R}^n$ is the unknown function such that $U=(u_1,u_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ (with $n\in\mathbb{N}^*$ and $n_1+n_2=n$), and A, B are $n\times n$ real symmetric matrices. Specifically, we assume

$$B := \begin{pmatrix} 0_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & D \end{pmatrix}, \tag{1.2}$$

where D is a symmetric definite positive $n_2\times n_2$ matrix.

In system (1.1), the damping term only acts on the n_2 components of the system and is effective exclusively in the region ω. Our main aim is to investigate the impact of this damping term on the long-term behavior of the solution and its decay.

As discussed in [25], the choice of ω as an exterior domain is motivated by a *geometric control condition* (see [2]): if the inclusion $\{|x|\geq R\}\subset\omega$ is not satisfied for some $R>0$, the ray of geometric optics may escape the damping effect, and the solution may not exhibit any decay properties.

In the following analysis, we assume that A is a *strictly hyperbolic matrix* with n real distinct eigenvalues such that

$$\lambda_1 > \cdots > \lambda_p > 0 > \lambda_{p+1} > \cdots > \lambda_n. \tag{1.3}$$

In particular, this implies that A has no zero eigenvalues

$$\lambda_i \neq 0, \qquad i\in\{1,\dots,n\}. \tag{1.4}$$

Similar assumptions are commonly made in studies on the boundary controllability of (systems of) conservation laws (see, e.g., [16]): when there are zero eigenvalues, i.e.

$$\begin{aligned} &\lambda_\ell > \lambda_m \equiv 0 > \lambda_r, \\ &\ell\in\{1,\dots,p\}, \quad m\in\{p+1,\dots,q\}, \quad r\in\{q+1,\dots,n\}, \end{aligned}$$

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there may be standing-wave solutions that cannot reach the boundary. Hence, to achieve exact controllability, suitable boundary control corresponds to non-zero eigenvalues, while suitable internal controls correspond to zero eigenvalues.

When $\omega = \mathbb{R}$, the existence and behavior of solutions to (1.1) are well-established. According to classical theory (see, e.g., [19]), (1.1) generates a semigroup $S_d(t)$ of bounded operators on $L^2(\mathbb{R};\mathbb{R})$. Therefore, given an initial data $U_0 \in L^2(\mathbb{R})$, the system (1.1) has a unique solution $U \in C((0,\infty); L^2(\mathbb{R}))$ such that

$$U(x,t) = S_d(t)U_0(x), \qquad (x,t) \in \mathbb{R}\times(0,\infty).$$

Indeed, applying the Fourier transform (in the space variable) to (1.1) yields, for all $(\xi,t)\in\mathbb{R}\times(0,\infty)$,

$$\partial_t \widehat{U}(t,\xi) + iA\xi\widehat{U}(t,\xi) = -B\widehat{U}(t,\xi),$$

or, in a condensed form,

$$\partial_t \widehat{U}(t,\xi) = E(\xi)\widehat{U}(t,\xi),$$

where $E(\xi) := -B - iA\xi$. Solving this first order ODE, we obtain

$$\widehat{U}(\xi,t) = \exp(E(\xi)t)\widehat{U}_0(\xi).$$

Then the C_0-semigroup S_d acting on $L^2(\mathbb{R})^n$ can be defined as

$$S_d(t)U_0 = e^{-tE}U_0 = \mathcal{F}^{-1}(e^{-tE(\xi)}\widehat{U}_0), \tag{1.5}$$

where $-E = -A\partial_x U + BU$ is the associated generator. Its domain contains the Sobolev space $H^1(\mathbb{R})^n$ and, thanks to the Fourier-Plancherel theorem, the estimate of the semigroup e^{-tE} in L^2 is reduced to the analysis of $e^{-tE(\xi)}$ for $\xi\in\mathbb{R}^*$. For future reference, we introduce $S_d(t,s)$ for $(t,s)\in(0,\infty)^2$ with $s\le t$, which represents the dissipative semigroup associated with the generator $-E$ and acting on the time interval $[s,t]$. If $s=0$, this notation simplifies to $S_d(t)$.

In general, the semigroup S_d is not dissipative and the operator norm satisfies $|||S|||=c$ for some constant $c>0$. Indeed, owing to the symmetry of the matrix A, the classical energy method leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U(\cdot,t)\|^2_{L^2(\mathbb{R})} + (BU(\cdot,t)|U(\cdot,t))_{L^2(\mathbb{R})} = 0, \qquad t>0. \tag{1.6}$$

The structure of B in (1.2) implies that there exists $\kappa_0>0$ such that

$$(BU(\cdot,t)|U(\cdot,t))_{L^2(\mathbb{R})} \ge \kappa_0\|u_2(\cdot,t)\|^2_{L^2(\mathbb{R})}. \tag{1.7}$$

Hence, (1.6) yields L^2-in-time integrability on the component u_2 but not for u_1. To overcome this lack of coercivity issue, Shizuta and Kawashima developed a condition, the well-known stability condition (SK), which ensures the decay of the solution to zero when $t\to\infty$ (see [20]):

$$\{\text{eigenvectors of } A\xi\}\cap\mathrm{Ker}(B) = \{0\}, \qquad \forall\,\xi\in\mathbb{R}^*. \tag{SK}$$

In one spatial dimension, this condition is equivalent to the absence of plane wave solutions to the hyperbolic system propagating to the characteristic directions, thereby guaranteeing the decay of our solutions. In higher dimensions, it is established that the (SK) condition is sufficient to ensure these properties but is not necessary. For further details, we refer to [4].

When $\omega=\mathbb{R}$ and under the (SK) assumption, it is proven in [20, Theorem 1.1] that $\lim_{t\to\infty} e^{-tE(\xi)} = 0$ and more precisely, in terms of the operator norm, we have

$$|||e^{-tE(\xi)}||| \le Ce^{-c\frac{|\xi|^2}{1+|\xi|^2}t}, \qquad t>0. \tag{1.8}$$

As $|\xi|^2/(1+|\xi|^2)\le\min(1,|\xi|^2)$, the behavior of the solution dependents on the frequency regime under consideration (see, e.g., [4, Theorem 1]).

Theorem 1.1 (SK decay estimate)**.** *Let us assume that $\omega=\mathbb{R}$, the matrix A is symmetric and satisfies (1.3), B satisfies (1.2), and the couple (A,B) satisfies the (SK) condition. Let U be the solution of (1.1) associated with the initial data $U_0\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Then the following decay estimates hold:*

$$\|U^h(\cdot,t)\|_{L^2(\mathbb{R})} \le C_K e^{-\gamma t}\|U_0\|_{L^2(\mathbb{R})}, \qquad t>0, \tag{1.9}$$

$$\|U^\ell(\cdot,t)\|_{L^\infty(\mathbb{R})} \le C_K t^{-1/2}\|U_0\|_{L^1(\mathbb{R})}, \qquad t>0, \tag{1.10}$$

where C_K and γ are positive constants depending only on A and B, $\widehat{U}^h(\xi,t) = \widehat{U}(\xi,t)\mathbb{1}_{\{|\xi|>1\}}$, and $\widehat{U}^\ell(\xi,t) = \widehat{U}(\xi,t)\mathbb{1}_{\{|\xi|<1\}}$.

The high frequencies of the solution are exponentially damped, whereas the low frequencies behave similarly to solutions of the heat equation. Additionally, computations presented in [20, Theorem 5] enable the deduction of the existence of global smooth solutions for nonlinear systems associated with initial data close to a constant equilibrium in any dimension (see [4, 11, 5, 22, 9, 8]).

In their paper [4, Proposition 1], Beauchard and Zuazua observed that the (SK) condition is equivalent to the classical *Kalman rank condition* (see [15, Chapter 2, Theorem 5, pp. 81–82]) in control theory for all pairs $(A(\xi), B)$ with $\xi \neq 0$. They then obtained a simpler proof of the decay estimates (1.9)-(1.10) by constructing an energy functional with additional low-order terms. This construction was motivated by the hypo-coercivity theory of Villani [21] and works on the damped wave equation.

More recently, in [7], Crin-Barat and Danchin developed a method that allows the study of general quasi-linear partially dissipative hyperbolic systems in a critical regularity framework. In addition, they justified the relaxation process associated with such systems by highlighting a purely damped mode in the low-frequency regime, which allows the diagonalization of the system in this regime

In the case of a damping term acting in a region ω instead of the whole space, the tools developed in the above references are not of much help: indeed, they mainly rely on the Fourier transform, which would yield a convolution between $\widehat{\mathbb{1}_\omega}$ and $\widehat{U}$ that seems to mix the frequencies too much to obtain useful information about the dissipative mechanism.

In the model case of the wave equation, the decay of solutions when the damping term acts only in a region of the domain satisfying a suitable geometric condition has been an active area of investigation in the past few decades. In [25], Zuazua proved energy decay for the Klein-Gordon equation with locally distributed dissipation, while [24] obtained a decay estimate for the damped wave equation in a bounded domain. Local energy decay results were subsequently obtained for the linear wave equation in an unbounded exterior domain $\Omega \subset \mathbb{R}^d$ with localized dissipation effective only near a part of the boundary (see [17]). In [18], the case of systems with total dissipation (and in a compact domain) was also dealt with. Finally, in [14, 13], Léautaud and Lerner studied the decay rate for the energy of solutions of a damped wave equation in a situation where the geometric control condition was violated.

In a similar vein, in [6], Coron and Nguyen studied the controllability of general linear hyperbolic systems in one spatial dimension using boundary controls on one side, while, in [1], the authors dealt with the controllability from the interior of a hyperbolic system with a reduced number of controls (which parallels the stabilization from ω using partial damping). For coupled waves, similar results were obtained in [23], while [3, 10] presented results concerning parabolic or parabolic-hyperbolic systems.

In this contribution, we study the problem by relying on a direct method that involves only the consideration of characteristic curves and a semigroup-wise decomposition. Our main theorem provides an analog of Theorem 1.1.

Theorem 1.2 (Decay estimates for locally-undamped partially dissipative systems)**.** *Let us assume that the matrix A is symmetric and satisfies (1.3), B satisfies (1.2), and the couple (A, B) satisfies the (SK) condition. Let U be the solution of (1.1) associated with the initial data $U_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then, for*

$$t \geq \tau, \qquad \text{with } \tau := \max\left(\sum_{i=1}^{p} \frac{2R}{|\lambda_i|}, \sum_{i=p+1}^{n} \frac{2R}{|\lambda_i|}\right),$$

the following decay estimates hold:

$$\|U(\cdot,t)\|_{L^2(\mathbb{R})} \leq \widetilde{C}_K e^{-\gamma(t-\tau)}\|U_0\|_{L^2(\mathbb{R})} + \widetilde{C}_K (t-\tau)^{-1/4}\|U_0\|_{L^2(\mathbb{R})}, \tag{1.11}$$

$$\|U(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \widetilde{C}_K (t-\tau)^{-1/2}\|U_0\|_{L^1(\mathbb{R})}, \tag{1.12}$$

where $\widetilde{C}_K = 6C_K$ and the positive constants C_K and γ are defined in Theorem 1.1 and depend only on A and B.

The result from Theorem 1.2 on the time interval $[\tau, +\infty]$ is natural and optimal (up to the modification of the constant C_K), as it recovers the same decay rates as in the case $\omega = \mathbb{R}$, delayed by the time each characteristic spends in the undamped region ω^c. The modification of the constant C_k is due to a technical difficulty related to the fact that our analysis is based on a precise decomposition of the physical space into six distinct intervals. Also, in (1.11), we could more briefly write $\widetilde{C}'_K := \widetilde{C}_K e^{\gamma\tau}$, but we prefer the form above because it allows us to provide an explicit expression for the delay perceived by the time-decay rates of the energy from Theorem 1.1.

Theorem 1.2 holds without any restrictions on the support of the initial data. However, it could be refined when considering compactly supported initial data. For example, if we consider initial data supported far from ω^c, then the solution would decay during the entire time it takes to reach ω^c. Therefore, the estimates (1.11) and (1.12) would be satisfied, but the decay could be improved for times shorter than τ.

Outline. This paper is organized as follows. Section 2 establishes some preliminary results on hyperbolic systems and outlines the proof strategy for our main theorem. In Section 3, we analyze some case studies, including scalar equations and 2×2 systems. We then generalize the key insights gathered in these cases and prove Theorem 1.2 in Section 4. Finally, in Section 5, we discuss potential extensions of the main results and several open problems.

2. Preliminaries and strategy of proofs

Before discussing the proof strategy of the Theorem 1.2, let us introduce some preliminary notations. As the matrix A is symmetric with n real distinct eigenvalues, there exists a matrix $P \in O(n,\mathbb{R})$ such that

$$P^{-1}AP = \Lambda \quad \text{and} \quad \Lambda = \operatorname{diag}(\lambda_1, ..., \lambda_n).$$

Setting $V = P^{-1}U$, the system (1.1) can be reformulated into

$$\begin{cases} \partial_t V + \Lambda \partial_x V = -P^{-1}BPV, & (x,t) \in \mathbb{R}\times(0,\infty), \\ V(0,x) = V_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

i.e., writing $V = (v_1, \dots, v_n)$ and $V_0 = (v_{1,0}, \dots, v_{n,0})$,

$$\begin{cases} \partial_t v_1 + \lambda_1 \partial_x v_1 = -\sum_{j=1}^n \widetilde{b}_{1,j} v_j\, \mathbb{1}_\omega, & (x,t) \in \mathbb{R}\times(0,\infty), \\ \quad\vdots & \\ \partial_t v_n + \lambda_n \partial_x v_n = -\sum_{j=1}^n \widetilde{b}_{n,j} v_j\, \mathbb{1}_\omega, & (x,t) \in \mathbb{R}\times(0,\infty), \\ v_1(x,0) = v_{1,0}(x), & x \in \mathbb{R} \\ \quad\vdots & \\ v_n(x,0) = v_{n,0}(x), & x \in \mathbb{R}, \end{cases} \tag{2.2}$$

where the $\widetilde{b}_{i,j}$ corresponds to the coefficients of the matrix $P^{-1}BP$. It is sufficient to prove Theorem 1.2 for the unknown V (since this would imply the same conclusion for U up to a multiplicative constant).

Example 2.1 (Damped wave equation). The damped wave equation $\partial^2_{tt}u - \partial^2_{xx}u + \partial_t u = 0$ can be equivalently rewritten as follows:

$$\begin{cases} \partial_t p - \partial_x p = -\frac{1}{2}(p+r), & (x,t) \in \mathbb{R}\times(0,\infty), \\ \partial_t r + \partial_x r = -\frac{1}{2}(p+r), & (x,t) \in \mathbb{R}\times(0,\infty). \end{cases}$$

Remark 2.2 (Commuting matrices). If the matrices A and B commute, we can diagonalize the matrices simultaneously and end up with decoupled equations. Furthermore, under the commutativity assumption, the (SK) condition amounts to $\operatorname{rank}(B) = n$. This reduces the situation to the 'totally dissipative' case.

2.1. Characteristics and propagation times.

For all $i \in \{1, \dots, n\}$, the characteristic lines X_i of each equation of system (2.2) passing through the point $(x_0, t_0) \in \mathbb{R}\times[0,T]$ are given by

$$X_i\,(t, x_0, t_0) := \lambda_i(t - t_0) + x_0, \qquad t \in [0,T].$$

We would like to highlight two key facts that will be integral to our study (cf. Figure 1):

(1) once a characteristic has crossed and exited the undamped region ω^c, it will never cross it again; consequently, the time spent by all characteristics in ω^c is uniformly bounded in x and t;
(2) depending on the sign of the eigenvalues (i.e., on the directions of the characteristics), some characteristics will cross ω^c, while others will not.

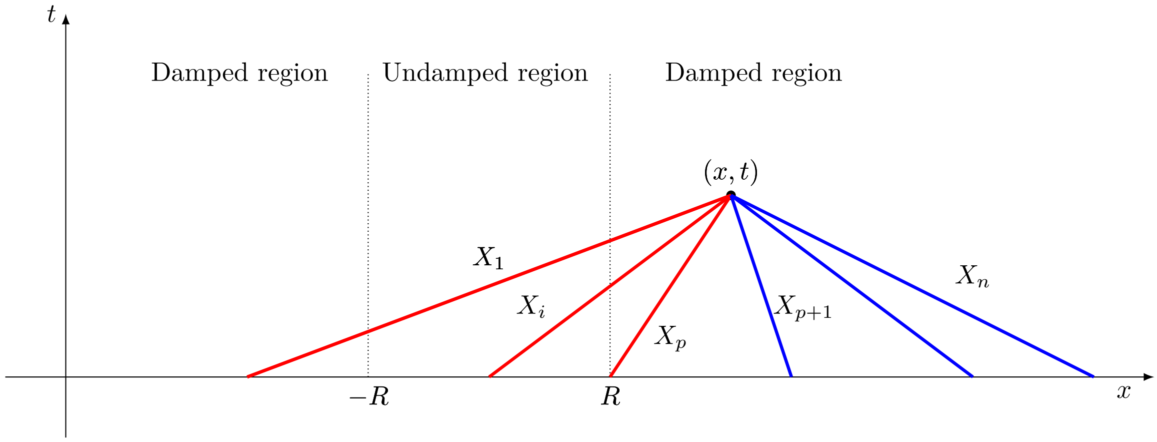

FIGURE 1. Characteristics passing through a point $(x,t) \in \mathbb{R} \times (0,\infty)$.

As the proof of Theorem 1.2 revolves around time-quantities that are related to the characteristics and the undamped region ω^c, we need to introduce the following notation (see Figure 2):

- $t_{i,en}(x_0,t_0)$ ("en" for "enter"): the time it takes the characteristic line $X_i(\cdot,x_0,t_0)$ to intersect $x=-R$ (resp. $x=R$) if $\lambda_i<0$ (resp. $\lambda_i>0$) from the time $t=0$ – that is, to enter the undamped region; if it does not enter ω^c, we set $t_{i,en}(x_0,t_0)=0$. We have
$$\begin{cases} t_{i,en}(x_0,t_0) = \max\left(0, t_0 - \dfrac{x_0+R}{|\lambda_i|}\right), & i \in \{1,\dots,p\},\\ t_{i,en}(x_0,t_0) = \max\left(0, t_0 - \dfrac{x_0-R}{|\lambda_i|}\right), & i \in \{p+1,\dots,n\}; \end{cases}$$
- $t_{i,ex}(x_0,t_0)$ ("ex" for "exit"): the time it takes the characteristic line $X_i(\cdot,x_0,t_0)$ to intersect $x=R$ (resp. $x=-R$) if $\lambda_i<0$ (resp. $\lambda_i>0$) from $t=0$ – that is, to exit the undamped region; if it does not exit ω^c, we set $t_{i,ex}(x_0,t_0)=0$. We have
$$\begin{cases} t_{i,ex}(x_0,t_0) = \max\left(0, t_0 - \dfrac{x_0-R}{|\lambda_i|}\right), & i \in \{1,\dots,p\},\\ t_{i,ex}(x_0,t_0) = \max\left(0, t_0 - \dfrac{x_0+R}{|\lambda_i|}\right), & i \in \{p+1,\dots,n\}; \end{cases}$$
- $\tau_i(x_0,t_0)$: the length of time during which the characteristic line $X_i(\cdot,t_0,x_0)$ is in ω^c, i.e.
$$\tau_i(x_0,t_0) = t_{i,ex}(x_0,t_0) - t_{i,en}(x_0,t_0), \tag{2.3}$$
which is uniformly bounded as
$$\sup_{x_0\in\mathbb{R},\, t_0\in[0,T]} \tau_i(x_0,t_0) \le \frac{2R}{|\lambda_i|}.$$

2.2. **Construction of a solution to system (2.2).** We now turn to the explicit construction of a solution to (2.2) by followings their associated characteristics and expressing it with semigroups.

2.2.1. *Conservative semigroup.* Inside ω^c, the solution of (2.2) shares the properties of the solution of
$$\partial_t V + \Lambda \partial_x V = 0, \qquad (x,t) \in \mathbb{R} \times (0,\infty), \tag{2.4}$$
which does not experience any dissipation.

We define $S_c(t,s)$ ("c" for conservative) as the semigroup associated with (2.4) on the time interval $[s,t]$ for $s \le t$ with $s,t \in (0,\infty)$. For a given initial data $V_0 \in L^2(\mathbb{R})$, the solution of (2.4) can be expressed as
$$V(x,t) = S_c(t,0)V_0(x) = \begin{pmatrix} v_{1,0}(x-\lambda_1 t)\\ \vdots \\ v_{n,0}(x-\lambda_n t)\end{pmatrix}, \qquad (x,t) \in \mathbb{R} \times (0,\infty).$$

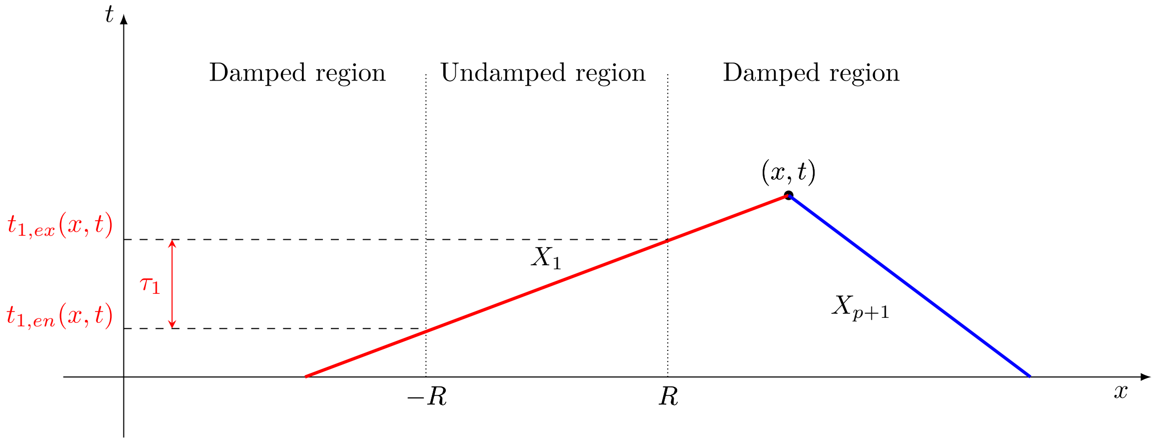

FIGURE 2. Illustration of the quantities $t_{i,en}(x,t)$ and $t_{i,ex}(x,t)$.

We also define $S_{c,i}$ the semigroups associated with each component such that

$$v_i(x,t) = S_{c,i}(t,0)V_0(x) = v_{i,0}(x-\lambda_1 t), \qquad (x,t) \in \mathbb{R} \times (0,\infty). \tag{2.5}$$

From standard energy estimates, as the matrix D is diagonal and therefore symmetric, we infer that, for $p \in [2,\infty]$,

$$\|V(\cdot,t)\|_{L^p(\mathbb{R})} = \|S_c(t,0)V_0\|_{L^p(\mathbb{R})} = \|V_0\|_{L^p(\mathbb{R})}, \qquad t \geq 0. \tag{2.6}$$

2.2.2. *Partially dissipative semigroup.* Inside ω, the solutions of (2.2) behave similarly to the solutions of

$$\partial_t V + \Lambda \partial_x V = -P^{-1}BPV, \qquad (t,x) \in (0,\infty) \times \mathbb{R}, \tag{2.7}$$

which undergoes dissipation thanks to the damping term active on $\mathbb{R}$. More precisely, as in the introduction, we define S_d as the semigroup associated to this system and note that, for a given initial data $V_0 \in L^2(\mathbb{R})$, the solution of (2.7) can be expressed as

$$\begin{aligned} V(x,t) &= S_d(t,0)V_0(x) \\ &= \begin{pmatrix} v_{1,0}(x-\lambda_1 t) - \displaystyle\int_0^t \sum_{j=1}^n \widetilde{b}_{1,j} V_j(s, x-\lambda_1(t-s))\,\mathrm{d}s \\ \vdots \\ v_{n,0}(x-\lambda_n t) - \displaystyle\int_0^t \sum_{j=1}^n \widetilde{b}_{n,j} V_j(s, x-\lambda_n(t-s))\,\mathrm{d}s \end{pmatrix}, \qquad (x,t) \in \mathbb{R} \times (0,\infty). \end{aligned} \tag{2.8}$$

We also define $S_{d,i}$ as the semigroup associated with each component

$$\begin{aligned} v_i(x,t) &= S_{d,i}(t,0)V_0(x) \\ &= v_{i,0}(x-\lambda_1 t) - \int_0^t \sum_{j=1}^n \widetilde{b}_{i,j} V_j(s, x-\lambda_i(t-s))\,\mathrm{d}s, \qquad (x,t) \in \mathbb{R} \times (0,\infty). \end{aligned} \tag{2.9}$$

2.2.3. *Method of characteristics.* We are now in a position to detail the construction of a solution for (2.2).

Proposition 2.3 (Representation of solutions to system (2.2))**.** *The solution of* (2.2) *is given by*

$$v_i(x,t) = [S_{d,i}(t, t_{i,ex}(x,t)) S_{c,i}(t_{i,ex}(x,t), t_{i,en}(x,t)) S_{d,i}(t_{i,en}(x,t), 0) v_{i,0}](x), \qquad i \in \{1,\ldots,n\}.$$

Specifically:

if $x \geq R$,

$$\begin{cases} v_i(x,t) = [S_{d,i}(t, t_{i,ex}(x,t)) S_{c,i}(t_{i,ex}(x,t), t_{i,en}(x,t)) S_{d,i}(t_{i,en}(x,t), 0) v_{i,0}](x), & i \in \{1,\ldots,p\}, \\ v_i(x,t) = [S_{d,i}(t,0)v_{i,0}](x), & i \in \{p+1,\ldots,n\}; \end{cases}$$

if $x \leq -R$,

$$\begin{cases} v_i(x,t) = [S_{d,i}(t,0)v_{i,0}](x), & i \in \{1,\dots,p\}, \\ v_i(x,t) = [S_{d,i}(t,t_{i,ex}(x,t))S_{c,i}(t_{i,ex},t_{i,en}(x,t))S_{d,i}(t_{i,en}(x,t),0)v_{i,0}](x), & i \in \{p+1,\dots,n\}; \end{cases}$$

if $x \in [-R,R]$,

$$\begin{cases} v_i(x,t) = [S_{c,i}(t,t_{i,en}(x,t))S_{d,i}(t_{i,en}(x,t),0)v_{i,0}](x), & i \in \{1,\dots,p\}, \\ v_i(x,t) = [S_{c,i}(t,t_{i,en}(x,t))S_{d,i}(t_{i,en}(x,t),0)v_{i,0}](x), & i \in \{p+1,\dots,n\}. \end{cases}$$

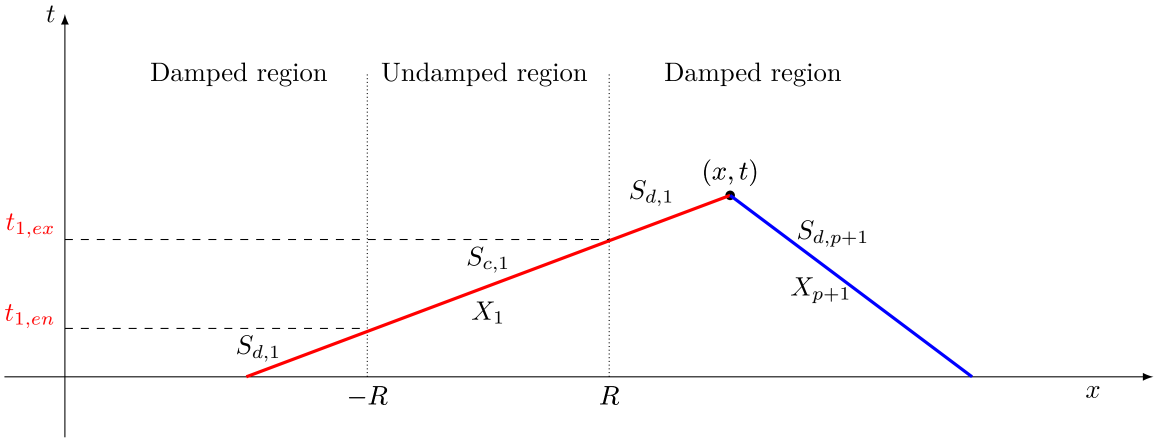

FIGURE 3. Illustration of the composition of the semigroups S_d and S_c.

Proof. We shall analyze two cases separately. Let us fix $(x,t) \in \mathbb{R} \times (0,\infty)$.
Case 1: $x \in \omega$. We distinguish two subcases.
Subcase 1a: $x > R$. For each component v_i associated to positive eigenvalues, i.e. $i \in \{1,\dots,p\}$, we can use the method of characteristics to write down the solution. Going back along the characteristic $X_i(\cdot,x,t)$, we can write

$$v_i(x,t) = S_{d,i}(t,t_{i,ex}(x,t))v_i(x,t_{i,ex}(x,t)), \qquad (x,t) \in (R,+\infty) \times (0,\infty).$$

Then, the characteristic lines enter ω^c and the conservative semigroup is active on the time-interval $[t_{i,ex}(x,t),t_{i,en}(x,t)]$,

$$v_i(x,t_{i,ex}(x,t)) = S_{c,i}(t_{i,ex}(x,t),t_{i,en}(x,t))v_1(x,t_{i,en}(x,t)), \qquad (x,t) \in (R,+\infty) \times (0,\infty).$$

After the characteristic line exits ω^c, the dissipative semigroup is active on the time-interval $[t_{i,en}(x,t),0]$ and we have

$$v_i(x,t_{i,en}(x,t)) = [S_{d,i}(t_{i,en}(x,t),0))v_{i,0}](x), \qquad (x,t) \in (R,+\infty) \times (0,\infty).$$

This leads, for every $(x,t) \in (R,+\infty) \times (0,\infty)$, to

$$v_i(x,t) = [S_{d,i}(t,t_{i,ex}(x,t))S_{c,i}(t_{i,ex}(x,t),t_{i,en}(x,t))S_{d,i}(t_{i,en}(x,t),0)v_{i,0}](x). \tag{2.10}$$

For the components v_i associated with negative eigenvalues, i.e. $i \in \{p+1,\dots,n\}$, only the dissipative semigroup is active and we have

$$v_i(x,t) = [S_{d,i}(t,0)v_{i,0}](x).$$

Subcase 1b: $x < -R$. This case is analogous to Subcase 1a by symmetry.
Case 2: $x \in \omega^c$. For each component v_i associated to positive eigenvalues, i.e. $i \in \{1,\dots,p\}$, going back along the characteristic $X_i(\cdot,x,t)$ we have

$$v_i(x,t) = S_{c,i}(t,t_{i,en}(x,t))v_1(x,t_{i,en}(x,t)), \qquad (x,t) \in (-R,R) \times (0,\infty).$$

After the characteristic line exits ω^c, the dissipative semigroup is active on the time-interval $[t_{i,en}(x,t),0]$ and we have

$$v_i(x,t_{i,en}(x,t)) = [S_{d,i}(t_{i,en}(x,t),0))v_{i,0}](x), \qquad (x,t) \in (-R,R) \times (0,\infty).$$

This leads to

$$v_i(x,t) = [S_{c,i}(t,t_{i,en}(x,t))S_{d,i}(t_{i,en}(x,t),0)v_{i,0}](x), \qquad (x,t) \in (-R,R)\times(0,\infty). \tag{2.11}$$

For the components v_i associated with negative eigenvalues, i.e. $i \in \{p+1,\dots,n\}$, symmetrically, we obtain the same formula as (2.11). □

2.3. **Strategy of the proof of Theorem 1.2.** The first difficulty encountered when trying to prove time-decay estimates stems from the fact that, *individually*, each semigroup $S_{d,i}$ may not be dissipative. The decay can only be achieved through the coupling between all the equations, as the (SK) condition guarantees that this coupling generates dissipation for all components (including those that are not directly damped).

In other words, it is only possible to establish dissipation for the solution V if all the semigroups $S_{d,i}$ are active over the same time interval. For example, examining the effect of $S_{d,1}$ on the first component does not generally imply any time-decay properties for the solution V or the component v_1. This implies that if one of the $S_{c,i}$ is active over a time-interval, the entire solution experiences no decay during that interval. The key observation that enables us to prove Theorem 1.2 is that the conservative semigroups $S_{c,i}$ are only active over a finite union of bounded time intervals. Roughly speaking, the duration that each component spends in the undamped region produces a delay in the decay for all other components.

Let us have a more precise look at the case $x \geq R$. Proposition 2.3 yields

$$\begin{cases} v_i(x,t) = S_{d,i}(t,t_{1,ex}(x,t))S_{c,i}(t_{i,ex}(x,t),t_{1,en}(x,t))S_{d,i}(t_{i,en}(x,t),0)v_{i,0}(x), & i \in \{1,\dots,p\}, \\ v_i(x,t) = S_{d,i}(t,0)v_{i,0}(x), & i \in \{p+1,\dots,n\}. \end{cases}$$

This means that, for $i \in \{1,\dots,p\}$, the dissipative semigroup $S_{d,i}$ is active on the interval $[0,t_{i,en}(x,t)] \cup [t_{i,ex}(x,t),t]$ and the conservative one on $[t_{i,en}(x,t),t_{i,ex}(x,t)]$; on the other hand, for $i \in \{p+1,\dots,n\}$, the dissipative semigroup $S_{d,i}$ is active on the whole interval $[0,t]$.

Consequently, at least one component $S_{c,i}$ of S_c will be active in the union of intervals

$$\mathcal{I}(x,t) = \bigcup_{i=1}^{p} [t_{i,en}(x,t), t_{i,ex}(x,t)]. \tag{2.12}$$

Meanwhile, all components of S_d will be active in the complement of $\mathcal{I}$. We can rigorously establish that the delay is directly proportional to the length of $\mathcal{I}$. Since time each characteristic spends in ω^c cannot more than $2R/|\lambda_i|$, we have the bound

$$\sup_{x\geq R,\, t>0} |\mathcal{I}(x,t)| \leq \sum_{i=1}^{p} \frac{2R}{|\lambda_i|}. \tag{2.13}$$

This inequality is a key step in the proof of Theorem 1.2.

3. Case studies

Before tackling the proof of the main theorem, we will study the following simpler cases:

- $(n,p) = (1,1)$: scalar equations;
- $(n,p) = (2,1)$: 2×2 systems where the components have different velocity signs;
- $(n,p) = (2,2)$: 2×2 systems where both components have the same velocity sign.

3.1. **Analysis of the scalar case.** In this section, we look at the scalar equation

$$\partial_t v_1 + \lambda_1 \partial_x v_1 = -v_1 \mathbb{1}_\omega, \qquad (x,t) \in \mathbb{R}\times(0,\infty). \tag{3.1}$$

While we could determine the result by explicitly computing the solution of the system, we opt to present a method that will be applicable to scenarios with multiple components.

Let us fix a time $t > 0$. Proposition 2.3 allows us to make the following observations (see Figure 4):

- for $x \geq R$, the dissipative semigroup is active over the time interval $[0,t_{1,en}(x,t)] \cup [t_{1,ex}(x,t),t]$, while the conservative semigroup is active over $[t_{1,en}(x,t),t_{1,ex}(x,t)]$;
- for $-R \leq x \leq R$, the conservative semigroup is active over $[t_{en}(x,t),t]$, while the dissipative semigroup is active over $[0,t_{en}(x,t)]$;
- for $x \leq -R$, the dissipative semigroup is active over $[0,t]$.

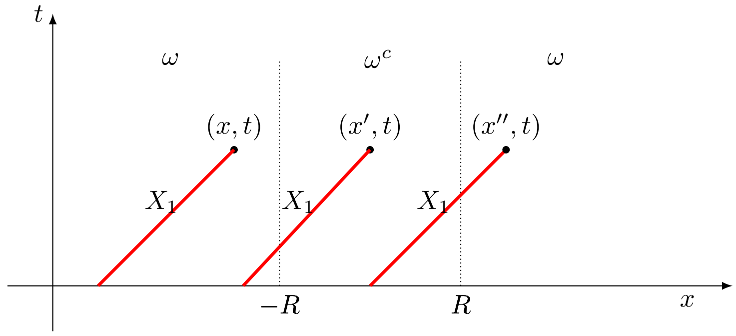

FIGURE 4. Examples of characteristics crossing ω or ω^c.

To recover the Shizuta-Kawashima decay estimates for the component v_1 with a time-delay of $\tau = 2R/|\lambda_1|$, we divide the spatial region into three parts and bound the time each characteristic line spends in ω^c. The main difficulty is that the entering and exiting times depend on x, and thus, we are not able to apply the semigroup properties (decay or conservation) directly. To solve this issue, the following lemma studies the cases $x \geq R$, $x \leq -R$, and $-R \leq x \leq R$ separately and refines the analysis for each of these subcases.

Proposition 3.1 (Decay estimate, scalar case)**.** *Let v_1 be the solution of (3.1) associated to the initial data $v_{1,0} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. For $t > \tau := \frac{2R}{|\lambda_1|}$, the following estimates hold:*

$$\|v_1(\cdot,t)\|_{L^2(\mathbb{R})} \leq 4C_K e^{-\gamma(t-\tau)}\|v_{1,0}\|_{L^2(\mathbb{R})} + 4C_K(t-\tau)^{-\frac{1}{4}}\|v_{1,0}\|_{L^2(\mathbb{R})}, \tag{3.2}$$

$$\|v_1(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq 4C_K(t-\tau)^{-\frac{1}{2}}\|v_{1,0}\|_{L^1(\mathbb{R})} \tag{3.3}$$

where the positive constants C_K and γ are defined in Theorem 1.2.

Remark 3.2. Compared to the result of Theorem 1.2, the multiplicative coefficient in front of the time-decay estimates can be lowered in this simple setting. This is due to the fact that when there are only positive (or negative) characteristic speed, the domain decomposition we use can be simplified.

Proof. We focus first on the case $x \geq R$, the cases $x \leq -R$ and $-R \leq x \leq R$ are treated later.
Case 1: $x \geq R$.

Step 1: *Representation of the solution in term of semigroups.* Thanks to Proposition 2.3, we have the following representation of the solution

$$v_1(x,t) = [S_{d,1}(t, t_{1,ex}(x,t))S_{c,1}(t_{1,ex}(x,t), t_{1,en}(x,t))S_{d,1}(t_{1,en}(x,t), 0)v_{1,0}](x).$$

Step 2: *Splitting of the space-domain.* We divide the half-line $[R, +\infty)$ into two parts: for the rest of the paper, we define $\mathcal{D}_k \cup \mathcal{F}_k = [R, +\infty)$ such that

$$\mathcal{D}_k := \{R \leq x \leq t\lambda_k + R\} \quad \text{and} \quad \mathcal{F}_k := \{x \geq t\lambda_k + R\} \tag{3.4}$$

(see Figure 5), where the natural number k is chosen so that λ_k is the smallest of the negative eigenvalues in modulus. An analogous definition will be adopted later for the half-line $(-\infty, -R]$. In this analysis of the scalar case, we have $k = 1$. As we observe in Figure 5, the domain $\mathcal{F}_1$ corresponds to the case where the characteristic lines do not pass through the undamped region, ω^c; on the other hand, in $\mathcal{D}_1$, the characteristics cross ω^c and the time-decay rates will be impacted.

Step 3: *Analysis of $\mathcal{F}_1$.* For every $x \in \mathcal{F}_1$, by definition we have $t_{1,en}(x,t) = t_{1,ex}(x,t) = 0$, meaning that the solution does not cross the undamped region ω^c and thus decays thanks to the (SK) condition. More precisely,

$$v_1(x,t) = S_{d,1}(t,0)v_{i,0}(x);$$

therefore,

$$\|v_1(\cdot,t)\|_{L^2(\mathcal{F}_1)} \leq C_K e^{-\gamma t}\|v_{1,0}\|_{L^2(\mathbb{R})} + C_K t^{-1/4}\|v_{1,0}\|_{L^2(\mathbb{R})} \tag{3.5}$$

and

$$\|v_1(\cdot,t)\|_{L^\infty(\mathcal{F}_1)} \leq C_K t^{-1/2}\|v_{1,0}\|_{L^1(\mathbb{R})}. \tag{3.6}$$

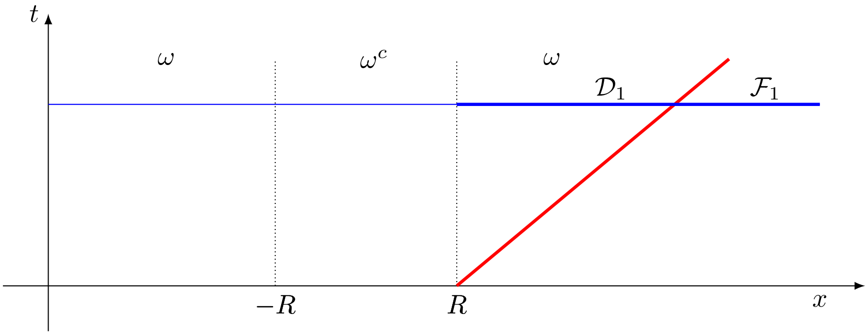

FIGURE 5. Space regions $\mathcal{D}_1$ and $\mathcal{F}_1$

Step 4: *Analysis of the interval* $\mathcal{D}_1$. For every $x \in \mathcal{D}_1$, the difficulty is that we cannot use directly the decaying/conservative properties of the semigroups since the $t_{1,en}$ and $t_{1,ex}$ depend on x. To solve this issue, we decompose $\mathcal{D}_1$ into small intervals:

$$\mathcal{D}_1 = \bigcup_{i=1}^{N} [a_i, a_{i+1}] \quad \text{s.t. } a_1 = R,\ a_N = t\lambda_1 + R, \text{ and } a_{i+1} - a_i \leq \frac{C}{N} \text{ for } N \in \mathbb{N}^* \text{ and } C > 0. \tag{3.7}$$

Computing at the L^2 norm of the solution on each interval $[a_i, a_{i+1}]$, using the explicit expression of the semigroups $S_{c,1}$ and $S_{d,1}$, we have

$$\begin{aligned}
&\int_{a_i}^{a_i+1} |v_1(x,t)|^2 \,\mathrm{d}x \\
&= \int_{a_i}^{a_i+1} \left| v_{1,0}(x - \lambda_1 t) + \int_{t_{1,ex}(x,t)}^{t} v_1(s, x - \lambda_1 t + \lambda_1 s)\,\mathrm{d}s + \int_0^{t_{1,en}(x,t)} v_1(s, x - \lambda_1 t + \lambda_1 s)\,\mathrm{d}s \right|^2 \mathrm{d}x \\
&= \int_{a_i}^{a_i+1} \left| v_{1,0}(x - \lambda_1 t) + \int_0^{t} v_1(s, x - \lambda_1 t + \lambda_1 s)\,\mathrm{d}s - \int_{t_{1,en}(x,t)}^{t_{1,ex}(x,t)} v_1(s, x - \lambda_1 t + \lambda_1 s)\,\mathrm{d}s \right|^2 \mathrm{d}x.
\end{aligned}$$

We define the quantity

$$\mathcal{Q} := \int_0^t v_1(s, x - \lambda_1 t + \lambda_1 s)\,\mathrm{d}s - \int_{t_{1,en}(x,t)}^{t_{1,ex}(x,t)} v_1(s, x - \lambda_1 t + \lambda_1 s)\,\mathrm{d}s$$

and assume that $v_{1,0}(x - \lambda_1 t) + \mathcal{Q}$ is positive (the case where it is negative can be treated in a similar manner by reversing the bounds below).

Let us have a closer look at the second term of $\mathcal{Q}$. Defining the positive and negative parts of the quantity inside the integral by $v_1^+ := \max(0, v_1(s, x-\lambda_1 t+\lambda_1 s))$ and $v_1^- := \max(0, -v_1(s, x-\lambda_1 t+\lambda_1 s))$, we have

$$\int_{a_i}^{a_i+1} \int_{t_{1,en}(x,t)}^{t_{1,ex}(x,t)} v_1(s, x - \lambda_1 t + \lambda_1 s)\,\mathrm{d}s\,\mathrm{d}x = \int_{a_i}^{a_i+1} \int_{t_{1,en}(x,t)}^{t_{1,ex}(x,t)} \left(v_1^+ - v_1^-\right) \mathrm{d}s\,\mathrm{d}x.$$

Then, thanks to the inequality

$$t_{1,en}(a_i, t) \leq t_{1,en}(x,t) \leq t_{1,en}(a_{i+1}, t) \leq t_{1,ex}(a_i, t) \leq t_{1,ex}(x,t) \leq t_{1,ex}(a_{i+1}, t), \tag{3.8}$$

we infer that

$$[t_{1,en}(a_{i+1}, t), t_{1,ex}(a_i, t)] \subset [t_{1,en}(x,t), t_{1,ex}(x,t)] \subset [t_{1,en}(a_i, t), t_{1,ex}(a_{i+1}, t)].$$

This allows us to bound the integral as follows (see Figure 6):

$$\int_{a_i}^{a_i+1} \int_{t_{1,en}(a_{i+1},t)}^{t_{1,ex}(a_i,t)} v_1^+ \leq \int_{a_i}^{a_i+1} \int_{t_{1,en}(x,t)}^{t_{1,ex}(x,t)} v_1^+ \leq \int_{a_i}^{a_i+1} \int_{t_{1,en}(a_{i+1},t)}^{t_{1,ex}(a_i,t)} v_1^+$$

and

$$-\int_{a_i}^{a_i+1} \int_{t_{1,en}(a_i,t)}^{t_{1,ex}(a_{i+1},t)} v_1^- \leq -\int_{a_i}^{a_i+1} \int_{t_{1,en}(x,t)}^{t_{1,ex}(x,t)} v_1^- \leq -\int_{a_i}^{a_i+1} \int_{t_{1,en}(a_i,t)}^{t_{1,ex}(a_{i+1},t)} v_1^-.$$

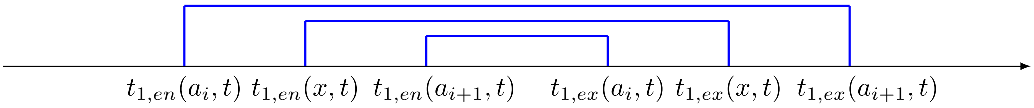

FIGURE 6. Decomposition of the time-interval used in the proof of Proposition 3.2.

We have that, for $x \in [a_i, a_{i+1}]$,

$$\int_0^t v_1^+ - \int_{t_{1,en}(a_{i+1},t)}^{t_{1,ex}(a_i,t)} v_1^+ - \int_0^t v_1^- + \int_{t_{1,en}(a_i,t)}^{t_{1,ex}(a_{i+1},t)} v_1^- \le \mathcal{Q};$$

thus

$$\int_{t_{1,ex}(a_i,t)}^{t} v_1^+ + \int_0^{t_{1,en}(a_{i+1},t)} v_1^+ - \int_{t_{1,ex}(a_{i+1},t)}^{t} v_1^- - \int_0^{t_{1,en}(a_i,t)} v_1^- \le \mathcal{Q},$$

which can be rewritten as

$$\int_{t_{1,ex}(a_{i+1},t)}^{t} v_1 + \int_{t_{1,ex}(a_i,t)}^{t_{1,ex}(a_{i+1},t)} v_1^+ + \int_0^{t_{1,en}(a_i,t)} v_1 - \int_{t_{1,en}(a_i,t)}^{t_{1,en}(a_{i+1},t)} v_1^- \le \mathcal{Q}.$$

Similarly, using the other inequalities in (3.8) leads to the upper bound

$$\mathcal{Q} \le \int_{t_{1,ex}(a_i,t)}^{t} v_1 + \int_{t_{1,ex}(a_{i+1},t)}^{t_{1,ex}(a_i,t)} v_1^+ + \int_0^{t_{1,en}(a_{i+1},t)} v_1 - \int_{t_{1,en}(a_i,t)}^{t_{1,en}(a_{i+1},t)} v_1^-.$$

Therefore, by gathering the above estimates, we obtain

$$\begin{aligned}
&\int_{a_i}^{a_i+1} |v_1(x,t)|^2 \,\mathrm{d}x \\
&\le \int_{a_i}^{a_i+1} \left| v_{1,0}(x-\lambda_1 t) + \int_{t_{1,ex}(a_i,t)}^{t} v_1 + \int_{t_{1,ex}(a_{i+1},t)}^{t_{1,ex}(a_i,t)} v_1^+ + \int_0^{t_{1,en}(a_{i+1},t)} v_1 + \int_{t_{1,en}(a_i,t)}^{t_{1,en}(a_{i+1},t)} v_1^- \right|^2 \\
&\le \int_{a_i}^{a_i+1} |S_{d,1}(t, t_{1,ex}(a_i,t)) S_{c,1}(t_{1,ex}(a_i,t), t_{1,en}(a_{i+1},t))) S_{d,1}(t_{1,en}(a_{i+1},t), 0) v_{1,0}(x) + \mathcal{R}_i|^2
\end{aligned}$$

where

$$\mathcal{R}_i(t) := \int_{t_{1,ex}(a_{i+1},t)}^{t_{1,ex}(a_i,t)} v_1^+ + \int_{t_{1,en}(a_i,t)}^{t_{1,en}(a_{i+1},t)} v_1^- \qquad \text{for } i \in \{1,\dots,N\}.$$

Then, applying the square root and Minkowski's inequality yields

$$\begin{aligned}
&\|v_1(\cdot,t)\|_{L^2([a_i,a_{i+1}])} \\
&\quad \le \|S_{d,1}(t, t_{1,ex}(a_i,t)) S_{c,1}(t_{1,ex}(a_i,t), t_{1,en}(a_{i+1},t))) S_{d,1}(t_{1,en}(a_{i+1},t), 0) v_{1,0}\|_{L^2([a_i,a_{i+1}])} \\
&\qquad + \|\mathcal{R}_i\|_{L^2([a_i,a_{i+1}])]}
\end{aligned}$$

Summing over $i = 1, \dots, N$, we obtain

$$\begin{aligned}
&\|v_1(\cdot,t)\|_{L^2(\mathcal{D}_1)} \\
&\quad \le \|S_{d,1}(t, t_{1,ex}(a_i,t)) S_{c,1}(t_{1,ex}(a_i,t), t_{1,en}(a_{i+1},t))) S_{d,1}(t_{1,en}(a_{i+1},t), 0) v_{1,0}\|_{L^2(\mathcal{D}_1)} \\
&\qquad + \sum_{i=1}^{N} \|\mathcal{R}_i\|_{L^2([a_i,a_{i+1})]}.
\end{aligned} \tag{3.9}$$

Step 5: *Estimates for $\mathcal{R}_i$.* By definition, there exists a $C > 0$ such that

$$t_{1,en}(a_{i+1},t) - t_{1,en}(a_i,t) \le \frac{C}{N} \quad \text{and} \quad t_{1,ex}(a_{i+1},t) - t_{1,ex}(a_i,t) \le \frac{C}{N}. \tag{3.10}$$

Since the quantities v_1^+ and v_1^- are bounded independently of N, we may infer that

$$\int_{t_{1,ex}(a_{i+1},t)}^{t_{1,ex}(a_i,t)} v_1^+ \to 0 \quad \text{as } N \to \infty$$

and $\sum_{i=1}^N \|\mathcal{R}_i\|_{L^2([a_i,a_{i+1})]} \to 0$ when $N\to\infty$. Indeed,

$$\begin{aligned}\sum_{i=1}^N \|\mathcal{R}_i\|_{L^2([a_i,a_{i+1})]} &\le \sum_{i=1}^N \left(\int_{a_i}^{a_{i+1}} |t_{1,en}(a_{i+1},t)-t_{1,en}(a_i,t)|^2\|v_1(\cdot,t)\|^2_{L^\infty(\mathbb{R})}\right)^{1/2}\\ &+\sum_{i=1}^N \left(\int_{a_i}^{a_{i+1}} |t_{1,ex}(a_i,t)-t_{1,en}(a_{i+1},t)|^2\|v_1(\cdot,t)\|^2_{L^\infty(\mathbb{R})}\right)^{1/2}\\ &\le 2\sum_{i=1}^N \left(\int_{a_i}^{a_{i+1}} \frac{1}{N^2}\|v_1(\cdot,t)\|^2_{L^\infty(\mathbb{R})}\right)^{1/2}\\ &\le 2\sum_{i=1}^N \left(\frac{1}{N^3}\|v_1(\cdot,t)\|^2_{L^\infty(\mathbb{R})}\right)^{1/2}\\ &\le \sum_{i=1}^N \frac{C}{N^{3/2}}\|v_1(\cdot,t)\|_{L^\infty(\mathbb{R})}\\ &\le \frac{C}{N^{1/2}}\|v(\cdot,t)\|_{L^\infty(\mathbb{R})} \underset{N\to\infty}{\longrightarrow} 0,\end{aligned}\tag{3.11}$$

where $C>0$ is a constant independent of N.

Step 6: *Use of the semigroups' properties.* We are now in a position to use the dissipative properties of the semigroup $S_{d,1}$ as the entering and exiting time do not depend on x anymore. Bounding the right-hand side integral by the integral on $\mathbb{R}$ and using the properties (1.5)-(1.8) of the semigroups $S_{d,1}$ and $S_{c,1}$ (2.6), from (3.9), we get

$$\begin{aligned}\|v_1(\cdot,t)\|_{L^2(\mathcal{D}_1)} &\le \|S_{d,1}(t,t_{1,ex}(a_i,t))S_{c,1}(t_{1,ex}(a_i,t),t_{1,en}(a_{i+1},t)))S_{d,1}(t_{1,en}(a_{i+1},t),0)v_{1,0}\|_{L^2(\mathcal{D}_1^1)}\\ &\quad+\sum_{i=1}^N \|\mathcal{R}_i\|_{L^2([a_i,a_{i+1})]}\\ &\le \|S_{d,1}(t,t_{1,ex}(a_i,t))S_{c,1}(t_{1,ex}(a_i,t),t_{1,en}(a_{i+1},t)))S_{d,1}(t_{1,en}(a_{i+1},t),0)v_{1,0}\|_{L^2(\mathbb{R})}\\ &\quad+\sum_{i=1}^N \|\mathcal{R}_i\|_{L^2([a_i,a_{i+1})]}\\ &\le \|e^{-\gamma\min(1,|\cdot|^2)(t-t_{1,ex}(a_i,t))}e^{-\gamma\min(1,|\cdot|^2)t_{1,en}(a_{i+1},t)}\widehat{v}_{1,0}\|_{L^2(\mathbb{R})}\\ &\quad+\sum_{i=1}^N \|\mathcal{R}_i\|_{L^2([a_i,a_{i+1}])}.\\ &\le C_K e^{-\gamma(t-(t_{1,ex}(a_i,t)-t_{1,en}(a_{i+1},t))}\|v_{1,0}\|_{L^2(\mathbb{R})}\\ &\quad+C_K(t-(t_{1,ex}(a_i,t)-t_{1,en}(a_{i+1},t))^{-\frac14}\|v_{1,0}\|_{L^2(\mathbb{R})}\\ &\quad+\sum_{i=1}^N \|\mathcal{R}_i\|_{L^2([a_i,a_{i+1}])}.\end{aligned}$$

Now, we have $t_{1,ex}(a_i,t)-t_{1,en}(a_{i+1},t)\le \tau+\dfrac{1}{N}$ and by summing on i and taking the limit as $N\to\infty$, we obtain, for $t\ge\tau=\frac{2R}{\lambda_1}$,

$$\|v_1(\cdot,t)\|_{L^2(\mathcal{D}_1)} \le C_K e^{-\gamma(t-\tau)}\|v_{1,0}\|_{L^2(\mathbb{R})}+C_K(t-\tau)^{-\frac14}\|v_{1,0}\|_{L^2(\mathbb{R})}.\tag{3.12}$$

Step 7: *Time-decay in L^∞.* The analysis performed above is also applicable to L^∞. By utilizing the fact that the Fourier transform is bounded from L^1 to L^∞, we deduce

$$\begin{aligned}\|v_1(\cdot,t)\|_{L^\infty(\mathcal{D}_1)} &\le \|S_{d,1}(t,t_{1,ex}(a_i,t))S_{c,1}(t_{1,ex}(a_i,t),t_{1,en}(a_{i+1},t)))S_{d,1}(t_{1,en}(a_{i+1},t),0)v_{1,0}\|_{L^\infty(\mathbb{R})}\\ &\quad+\sum_{i=1}^N \|\mathcal{R}_i\|_{L^\infty([a_i,a_{i+1})]}\\ &\le \|e^{-\gamma\min(1,|\cdot|^2)(t-t_{1,ex}(a_i,t))}e^{-\gamma\min(1,|\cdot|^2)t_{1,en}(a_{i+1},t)}\widehat{v}_{1,0}\|_{L^1(\mathbb{R})}\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i=1}^{N} \|\mathcal{R}_i\|_{L^\infty([a_i,a_{i+1})]} \\
&\le C_K e^{-\gamma(t-(t_{1,ex}(a_i,t)-t_{1,en}(a_{i+1},t)))}\|v_{1,0}\|_{L^1(\mathbb{R})} \\
&\quad + C_K(t-(t_{1,ex}(a_i,t)-t_{1,en}(a_{i+1},t))^{-\frac{1}{2}}\|v_{1,0}\|_{L^1(\mathbb{R})} \\
&\quad + \sum_{i=1}^{N} \|\mathcal{R}_i\|_{L^\infty([a_i,a_{i+1})]}.
\end{aligned}
$$

Since we can show that the remainder term tends to 0 as $N \to \infty$ as in (3.11), we have

$$\|v_1(\cdot,t)\|_{L^\infty(\mathcal{D}_1)} \le C_K(t-\tau)^{-\frac{1}{2}}\|v_{1,0}\|_{L^1(\mathbb{R})}.$$

Step 8: *Conclusion of Case* 1. Gathering the inequalities (3.5) and (3.12), we obtain

$$
\begin{aligned}
&\|v_1(\cdot,t)\|_{L^2(\mathcal{D}_1\cup\mathcal{F}_1)} \\
&\le \left(C_K e^{-\gamma(t-\tau)} + C_K e^{-\gamma t}\right)\|v_{1,0}\|_{L^2(\mathbb{R})} + \left(C_K(t-\tau)^{-\frac{1}{4}} + C_K t^{-\frac{1}{4}}\right)\|v_{1,0}\|_{L^2(\mathbb{R})}. \\
&\le \left(e^{\tau}+1\right)C_K e^{-\gamma t}\|v_{1,0}\|_{L^2(\mathbb{R})} + \left((t-\tau)^{-\frac{1}{4}} + t^{-\frac{1}{4}}\right)C_K\|v_{1,0}\|_{L^2(\mathbb{R})} \\
&\le 2C_K e^{-\gamma(t-\tau)}\|v_{1,0}\|_{L^2(\mathbb{R})} + 2C_K(t-\tau)^{-\frac{1}{4}}\|v_{1,0}\|_{L^2(\mathbb{R})}
\end{aligned}
\tag{3.13}
$$

Case 2: $x \le -R$. The analysis is similar to the domain $\mathcal{F}_1$ as the characteristics never touch ω^c. More precisely, we always have

$$v_1(x,t) = S_{d,1}(t,0)v_{1,0}(x).$$

Therefore we can deduce the usual time-decay rates without delay:

$$\|v_1(\cdot,t)\|_{L^2((-\infty,-R])} \le C_K e^{-\gamma t}\|v_{1,0}\|_{L^2(\mathbb{R})} + C_K t^{-\frac{1}{4}}\|v_{1,0}\|_{L^2(\mathbb{R})} \tag{3.14}$$

Case 3: $-R \le x \le R$. For every $x \in [-R,R]$, we have

$$v_1(x,t) = [S_{c,1}(t,t_{1,en}(x,t))S_{d,1}(t_{1,en}(x,t),0)v_{1,0}](x)$$

and

$$t - t_{1,en}(x,t) \le \tau = \frac{2R}{|\lambda_1|}.$$

Thus, a decomposition of the interval $[-R,R]$ similar to the decomposition we did for $\mathcal{D}_1$ leads to a delay of τ. We omit the details as the computations are similar. We have

$$\|v_1(\cdot,t)\|_{L^2([-R,R])} \le C_K e^{-\gamma(t-\tau)}\|v_{1,0}\|_{L^2(\mathbb{R})} + C_K(t-\tau)^{-\frac{1}{4}}\|v_{1,0}\|_{L^2(\mathbb{R})}. \tag{3.15}$$

Conclusion. Combining the results of the three cases, we obtain

$$
\begin{aligned}
\|v_1(\cdot,t)\|_{L^2(\mathbb{R})} &= \|v_1(\cdot,t)\|_{L^2((-\infty,-R])} + \|v_1(\cdot,t)\|_{L^2([-R,R])} + \|v_1(\cdot,t)\|_{L^2([R,+\infty))} \\
&\le C_K e^{-\gamma t}\|v_{1,0}\|_{L^2(\mathbb{R})} + C_K t^{-\frac{1}{4}}\|v_{1,0}\|_{L^2(\mathbb{R})} \\
&\quad + C_K e^{-\gamma(t-\tau)}\|v_{1,0}\|_{L^2(\mathbb{R})} + C_K(t-\tau)^{-\frac{1}{4}}\|v_{1,0}\|_{L^2(\mathbb{R})} \\
&\quad + 2C_K e^{-\gamma(t-\tau)}\|v_{1,0}\|_{L^2(\mathbb{R})} + 2C_K(t-\tau)^{-\frac{1}{4}}\|v_{1,0}\|_{L^2(\mathbb{R})} \\
&\le 4C_K e^{-\gamma(t-\tau)}\|v_{1,0}\|_{L^2(\mathbb{R})} + 4C_K(t-\tau)^{-\frac{1}{4}}\|v_{1,0}\|_{L^2(\mathbb{R})}.
\end{aligned}
$$

□

3.2. **Analysis of the 2×2 system.** The analysis of 2×2 systems needs to be carried out in two cases: (i) the eigenvalues have the same sign; (ii) the eigenvalues have the opposite sign.

3.2.1. *Analysis of the case $n = 2$ and $p = 1$: eigenvalues with different signs.* It is possible to study the negative and positive eigenvalues separately when decomposing the space into three regions. Indeed, when looking at the space $\{x \geq R\}$, the components associated with negative eigenvalues do not play a role in the time-delay (and the ones associated with positive eigenvalues for the region $\{x \leq -R\}$). In the region $\{-R \leq x \leq R\}$, the total delay that the solution could undergo is always smaller than in the other two regions. Note that such decomposition is only possible thanks to the superposition principle allowing us to decompose the support of the initial data and that holds since the system we are investigating is linear.

Proposition 3.3 (Decay estimate, 2×2 system with speed of different signs)**.** *Let $n = 2$, $p = 1$, and V be the solution of (2.1) associated to the initial data $V_0 \in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Then, for $t > \tau := \max\left(\frac{2R}{|\lambda_1|}, \frac{2R}{|\lambda_2|}\right)$, the following estimates hold:*

$$\|V(\cdot,t)\|_{L^2(\mathbb{R})} \leq \widetilde{C}_K e^{\gamma(t-\tau)}\|V_0\|_{L^2(\mathbb{R})} + \widetilde{C}_K (t-\tau)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}, \tag{3.16}$$

$$\|V(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \widetilde{C}_K (t-\tau)^{-\frac{1}{2}}\|V_0\|_{L^1(\mathbb{R})}, \tag{3.17}$$

where $\widetilde{C}_K = 6C_K$, and the positive constants C_K and γ are defined in Theorem 1.2.

Proof. Recalling Proposition 2.3 (cf. Figure 3), we distinguish three cases.

Case 1: $x \geq R$.

Step 1: *Representation of the solution in terms of semigroups.* For every $(x,t) \in (R,+\infty)\times(0,\infty)$, we have

$$\begin{cases} v_1(x,t) = [S_{d,1}(t, t_{1,ex}(x,t))S_{c,1}(t_{1,ex}(x,t), t_{1,en}(x,t))S_{d,1}(t_{1,en}(x,t),0)v_{1,0}](x), \\ v_2(x,t) = [S_{d,2}(t,0)v_{2,0}](x). \end{cases}$$

The semigroup $S_{d,2}$ associated with the component v_2 is always active and the component v_1 stays for the time $2R/|\lambda_1|$ in the undamped region. Therefore,

$$v_1(x,t) = v_{1,0}(x-\lambda_1 t) + \int_0^t \sum_{i=1}^2 \widetilde{b}_{1,i} v_i(s, x-\lambda_1 t+\lambda_1 s)\,\mathrm{d}s - \int_{t_{1,en}(x,t)}^{t_{1,ex}(x,t)} \sum_{i=1}^2 \widetilde{b}_{1,i} v_i(s, x-\lambda_1 t+\lambda_1 s)\,\mathrm{d}s,$$

$$v_2(x,t) = v_{2,0}(x-\lambda_2 t) + \int_0^t \sum_{i=1}^2 \widetilde{b}_{2,i} v_i(s, x-\lambda_2 t+\lambda_2 s)\,\mathrm{d}s.$$

The whole semigroup S_d is always active is the time interval $[t, t_{1,ex}(x,t)]\cap[t_{1,en}(x,t),0]$. The component v_2 does not enter the undamped region ($S_{d,2}$ is active on $[0,t]$); thus it does not increase the time-delay of the decay estimates.

Step 2: *Splitting of the space-domain.* We decompose the space $\{x > 0\}$ into $\mathcal{D}_1 \cup \mathcal{F}_1$ as in (3.4). Looking at the region $\mathcal{D}_1$ and defining the a_i as in Section 3.1, we get

$$\begin{aligned} &\int_{a_i}^{a_{i+1}} |V(x,t)|^2 \\ &\quad\leq \int_{a_i}^{a_{i+1}} \left|\begin{pmatrix} [S_{d,1}(t, t_{1,ex}(a_i,t))S_c(t_{1,ex}(a_i,t), t_{1,en}(a_{i+1},t)S_d(t_{1,en}(a_{i+1},t),0)v_{1,0}](x) + \mathcal{R}_{1,i} \\ [S_{d,2}(t,0)v_{2,0}](x) \end{pmatrix}\right|^2. \end{aligned}$$

Since the second line can obviously be rewritten as

$$[S_{d,2}(t,0)v_{2,0}](x) = [S_{d,2}(t, t_{2,ex}(a_i,t))S_d(t_{2,ex}(a_i,t), t_{2,en}(a_{i+1})S_d(t_{2,en}(a_{i+1}),0)v_{2,0}](x),$$

we deduce

$$\begin{aligned} &\int_{a_i}^{a_{i+1}} |V(x,t)|^2 \\ &\quad\leq \int_{a_i}^{a_{i+1}} \left|[S_d(t, t_{1,ex}(a_i,t)) \begin{pmatrix} S_{c,1}(t_{1,ex}(a_i,t), t_{1,en}(a_{i+1},t)) \\ S_{d,2}(t_{1,ex}(a_i,t), t_{1,en}(a_{i+1},t)) \end{pmatrix} S_d(t_{1,en}(a_{i+1},t),0)\begin{pmatrix} v_{1,0} \\ v_{2,0}\end{pmatrix}\right|^2 \\ &\qquad + \int_{a_i}^{a_{i+1}} \left|\begin{pmatrix} \mathcal{R}_1 \\ 0 \end{pmatrix}\right|^2. \end{aligned} \tag{3.18}$$

where the remainder term $\mathcal{R}_1$ is defined and handled as in Section 3.1; see (3.11).

Step 3: *Use of the semigroups' properties.* Recalling that the mixed operator $\begin{pmatrix} S_{c,1} \\ S_{d,2} \end{pmatrix}$ preserves the L^2 norm of the solution – since $S_{c,1}$ is conservative and $S_{d,2}$ associated with a positive definite matrix –, we obtain that the classical decay is delayed by the time $t_{1,ex}(a_i,t) - t_{1,en}(a_{i+1},t)$. Thanks to the uniform bound $t_{1,ex}(a_i,t) - t_{1,en}(a_{i+1},t) \le 2R/|\lambda_1|$, we get

$$\|V(\cdot,t)\|_{L^2(\mathcal{D}_1)} \le e^{-\gamma\left(t-\frac{2R}{|\lambda_1|}\right)}\|V_0\|_{L^2(\mathbb{R})} + \left(t - \frac{2R}{|\lambda_1|}\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}.$$

Step 4: *Analysis of $\mathcal{F}_1$.* As none of the characteristics lines cross the undamped region ω^c in this case, we have

$$\|V(\cdot,t)\|_{L^2(\mathcal{F}_1)} \le C_K e^{-\gamma t}\|V_0\|_{L^2(\mathbb{R})} + C_K t^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}.$$

Step 5: *Conclusion of Case 1.* Gathering the estimates from Steps 3 and 4, we get

$$\|V(\cdot,t)\|_{L^2([R,+\infty))} \le 2C_K e^{-\gamma\left(t-\frac{2R}{|\lambda_1|}\right)}\|V_0\|_{L^2(\mathbb{R})} + 2C_K\left(t - \frac{2R}{|\lambda_1|}\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}.$$

Case 2: $x \le -R$. Symmetrically, we obtain

$$\|V(\cdot,t)\|_{L^2((-\infty,-R])} \le 2C_K e^{-\gamma\left(t-\frac{2R}{|\lambda_2|}\right)}\|V_0\|_{L^2(\mathbb{R})} + 2C_K\left(t - \frac{2R}{|\lambda_2|}\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})},$$

where the delay depends on R and λ_2.
Case 3: $-R \le x \le R$. For every $(x,t) \in [-R,R]\times(0,\infty)$, we have

$$\begin{cases} v_1(x,t) = [S_{c,1}(t,t_{1,en}(x,t))S_{d,1}(t_{1,en}(x,t),0)v_{1,0}](x), \\ v_2(x,t) = [S_{c,2}(t,t_{2,en}(x,t))S_{d,2}(t_{2,en}(x,t),0)v_{2,0}](x). \end{cases}$$

Compared to the scalar case, here we need to decompose the domain into two parts. Let us look at the domain $\mathcal{H}_1 = \{(x,t) \in [-R,R]\times(0,\infty) : t_{1,en}(x,t) \le t_{2,en}(x,t)\}$, the other domain $\mathcal{H}_2 = \{(x,t) \in [-R,R]\times(0,\infty) : t_{1,en}(x,t) \ge t_{2,en}(x,t)\}$ can be treated symmetrically. For a sequence $(a_i)_{i=1}^N$ decomposing $\mathcal{H}_1$ as in (3.7), we have

$$\begin{aligned} &\int_{a_i}^{a_{i+1}} |V(x,t)|^2 \\ &\le \int_{a_i}^{a_{i+1}} \left| [S_c(t,t_{2,ex}(a_i,t)) \begin{pmatrix} S_{c,1}(t_{2,ex}(a_i,t),t_{1,en}(a_{i+1},t)) \\ S_{d,2}(t_{2,ex}(a_i,t),t_{1,en}(a_{i+1},t)) \end{pmatrix} S_d(t_{1,en}(a_{i+1},t),0) \begin{pmatrix} v_{1,0} \\ v_{2,0} \end{pmatrix} \right|^2 \\ &\quad + \int_{a_i}^{a_{i+1}} \left| \begin{pmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{pmatrix} \right|^2. \end{aligned} \tag{3.19}$$

Reasoning as in the Section 3.1 to deal with the remainder terms and using the semigroup properties, since, for $x \in \mathcal{H}_1$,

$$t - t_{1,en}(x,t) \le \frac{2R}{\lambda_1},$$

we have

$$\|V(\cdot,t)\|_{L^2(\mathcal{H}_1)} \le C_K e^{-\gamma\left(t-\frac{2R}{|\lambda_1|}\right)}\|V_0\|_{L^2(\mathbb{R})} + C_K\left(t - \frac{2R}{|\lambda_1|}\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}.$$

Symmetrically, for $\mathcal{H}_2$, we have

$$\|V(\cdot,t)\|_{L^2(\mathcal{H}_2)} \le C_K e^{-\gamma\left(t-\frac{2R}{|\lambda_2|}\right)}\|V_0\|_{L^2(\mathbb{R})} + C_K\left(t - \frac{2R}{|\lambda_2|}\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}.$$

Therefore,

$$\|V(\cdot,t)\|_{L^2([-R,R])} \le 2C_K e^{-\gamma\left(t-\max\left(\frac{2R}{|\lambda_1|},\frac{2R}{|\lambda_2|}\right)\right)}\|V_0\|_{L^2(\mathbb{R})} + 2C_K\left(t - \max\left(\frac{2R}{|\lambda_1|},\frac{2R}{|\lambda_2|}\right)\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}.$$

Conclusion. Combining the results of the three cases, we obtain

$$\|V(\cdot,t)\|_{L^2(\mathbb{R})} = \|V(\cdot,t)\|_{L^2((-\infty,-R])} + \|V(\cdot,t)\|_{L^2([-R,R])} + \|V(\cdot,t)\|_{L^2([R,+\infty))}$$

$$
\begin{aligned}
&\leq 2C_K e^{-\gamma(t-\frac{2R}{|\lambda_2|})}\|V_0\|_{L^2(\mathbb{R})} + 2C_K(t-\frac{2R}{|\lambda_2|})^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}\\
&+ 2C_K e^{-\gamma\left(t-\max\left(\frac{2R}{|\lambda_1|},\frac{2R}{|\lambda_2|}\right)\right)}\|V_0\|_{L^2(\mathbb{R})} + 2C_K\left(t-\max\left(\frac{2R}{|\lambda_1|},\frac{2R}{|\lambda_2|}\right)\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}\\
&+ 2C_K e^{-\gamma(t-\frac{2R}{|\lambda_1|})}\|V_0\|_{L^2(\mathbb{R})} + 2C_K(t-\frac{2R}{|\lambda_1|})^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}\\
&\leq 6C_K e^{-\gamma(t-\tau)}\|V_0\|_{L^2(\mathbb{R})} + 4C_K(t-\tau)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}.
\end{aligned}
$$

where $\tau = \max\left(\frac{2R}{|\lambda_1|},\frac{2R}{|\lambda_2|}\right)$. □

3.2.2. *Analysis of the case $n = p = 2$: eigenvalues with the same signs.* Studying both components separately as in the previous case is not possible because the characteristics of both components are going in the same direction. And, as we shall see, this will increase the time-delay of the decay rates.

Proposition 3.4 (Decay estimate, 2×2 system with speed of the same sign)**.** *Let $n = 2$, $p = 2$, and V be the solution of (2.1) associated to the initial data $V_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then, for $t > \tau := \frac{2R}{|\lambda_1|} + \frac{2R}{|\lambda_2|}$, the following estimates hold:*

$$\|V(\cdot,t)\|_{L^2(\mathbb{R})} \leq 4C_K e^{-\gamma(t-\tau)}\|V_0\|_{L^2(\mathbb{R})} + 4C_K(t-\tau)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}, \tag{3.20}$$

$$\|V(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq 4C_K(t-\tau)^{-\frac{1}{2}}\|V_0\|_{L^1(\mathbb{R})}. \tag{3.21}$$

where C_K and γ are defined in Theorem 1.2.

Proof. Recalling Proposition 2.3 (cf. Figure 3), we distinguish three cases.
Case 1: $x \geq R$.

Step 1: *Representation of the solution.* For every $(x,t) \in [R,\infty)\times(0,\infty)$, the solutions of the two transport equations are given by

$$
\begin{aligned}
v_1(x,t) &= v_{1,0}(x-\lambda_1 t) + \int_0^t \sum_{i=1}^2 \widetilde{b}_{1,i} v_i(s, x-\lambda_1 t+\lambda_1 s)\,\mathrm{d}s - \int_{t_{1,en}(x,t)}^{t_{1,ex}(x,t)} \sum_{i=1}^2 \widetilde{b}_{1,i} v_i(s, x-\lambda_1 t+\lambda_1 s)\,\mathrm{d}s,\\
v_2(x,t) &= v_{2,0}(x-\lambda_2 t) + \int_0^t \sum_{i=1}^2 \widetilde{b}_{2,i} v_i(s, x-\lambda_2 t+\lambda_2 s)\,\mathrm{d}s - \int_{t_{2,en}(x,t)}^{t_{2,ex}(x,t)} \sum_{i=1}^2 \widetilde{b}_{2,i} v_i(s, x-\lambda_2 t+\lambda_2 s)\,\mathrm{d}s.
\end{aligned}
$$

Denoting $\mathcal{B}_1 := \sum_{i=1}^2 \widetilde{b}_{1,i} v_i(s, x-\lambda_1 t+\lambda_1 s)$ and $\mathcal{B}_2 := \sum_{i=1}^2 \widetilde{b}_{2,i} v_i(s, x-\lambda_2 t+\lambda_2 s)$, we have

$$|V(x,t)|^2 = \left|\begin{pmatrix} v_{1,0}(x-\lambda_1 t)\\ v_{2,0}(x-\lambda_2 t)\end{pmatrix} + \begin{pmatrix} \int_0^t \mathcal{B}_1 - \int_{t_{1,en}(x,t)}^{t_{1,ex}(x,t)} \mathcal{B}_1\\ \int_0^t \mathcal{B}_2 - \int_{t_{2,en}(x,t)}^{t_{2,ex}(x,t)} \mathcal{B}_2\end{pmatrix}\right|^2. \tag{3.22}$$

Let us assume that the quantities in the two rows of (3.22) are positive, the other three scenarios being treatable in a similar fashion as we always have upper and lower bounds at hand.

Step 2: *Splitting of the space-domain.* We decompose the interval $[R,\infty]$ into $\mathcal{D}_2 \cup \mathcal{F}_2$ as defined in (3.4). The choice of $k = 2$ in (3.4) ensures that we decompose the interval with respect to the slowest eigenvalue.

Step 3: *Analysis of the interval $\mathcal{D}_2$.* We define the sequence $(a_i)_{i\in\{1,\dots,N\}}$ in $\mathcal{D}_2$ similarly as in Section 3.1 as:

$$\mathcal{D}_2 = \bigcup_{i=1}^N [a_i, a_{i+1}] \quad \text{s.t. } a_1 = R,\ a_N = t\lambda_2 + R,\ \text{and } a_{i+1} - a_i \leq \frac{C}{N} \text{ for } N \in \mathbb{N}^* \text{ and } C > 0. \tag{3.23}$$

Proceeding as in the previous cases, we obtain

$$
\begin{aligned}
&\int_{a_i}^{a_{i+1}} |V(x,t)|^2\\
&\leq \int_{a_i}^{a_{i+1}} \left|\begin{pmatrix} [S_{d,1}(t, t_{1,ex}(a_i,t))S_c(t_{1,ex}(a_i,t), t_{1,en}(a_{i+1},t))S_d(t_{1,en}(a_{i+1},t),0)v_{1,0}](x) + \mathcal{R}_{1,i}\\ [S_{d,2}(t, t_{2,ex}(a_i,t))S_c(t_{2,ex}(a_i,t), t_{2,en}(a_{i+1})))S_d(t_{2,en}(a_{i+1}),0)v_{2,0}](x) + \mathcal{R}_{2,i}\end{pmatrix}\right|^2,
\end{aligned}
$$

where

$$\mathcal{R}_{j,i}(t) := \int_{t_{j,ex}(a_{i+1})}^{t_{j,ex}(a_i)} \mathcal{B}_j^+ + \int_{t_{j,en}(a_i)}^{t_{j,en}(a_{i+1})} \mathcal{B}_j^- \qquad \text{for } j \in \{1,2\} \text{ and } i \in \{1,\dots,N\}.$$

Then, applying the square root and Minkowski's inequality yields

$$\begin{aligned}
&\|V(\cdot,t)\|_{L^2([a_i,a_{i+1}])} \\
&\le \Bigg\| \Bigg[S_d(t, t_{1,ex}(a_i,t)) \begin{pmatrix} S_{c,1}(t_{1,ex}(a_i,t), t_{2,ex}(a_i)) \\ S_{d,2}(t_{1,ex}(a_i,t), t_{2,ex}(a_i)) \end{pmatrix} \begin{pmatrix} S_{c,1}(t_{2,ex}(a_i,t), t_{1,en}(a_{i+1},t)) \\ S_{c,2}(t_{2,ex}(a_i,t), t_{1,en}(a_{i+1},t)) \end{pmatrix} \\
&\qquad \begin{pmatrix} S_{d,1}(t_{1,en}(a_{i+1},t)), t_{2,en}(a_{i+1})) \\ S_{c,2}(t_{1,en}(a_{i+1},t)), t_{2,en}(a_{i+1})) \end{pmatrix} S_d(t_{2,en}(a_{i+1}),0) \begin{pmatrix} v_{1,0} \\ v_{2,0} \end{pmatrix} \Bigg\|_{L^2([a_i,a_{i+1}])} \\
&\quad + \left\| \begin{pmatrix} \mathcal{R}_{1,i} \\ \mathcal{R}_{2,i} \end{pmatrix} \right\|_{L^2([a_i,a_{i+1}])}.
\end{aligned}$$

Step 4: *Use of the semigroups' properties.* Since it is only possible to recover dissipation when $S_{d,1}$ and $S_{d,2}$ are active on the same time-interval, bounding the right-hand side integral by the integral on $\mathbb{R}$ and using the properties of the semigroups $S_{d,1}$ (1.5)-(1.8) and $S_{c,1}$ (2.6), from (3.9), we get

$$\|V(\cdot,t)\|_{L^2([a_i,a_{i+1}])} \le \|e^{-\gamma \min(1,|\cdot|^2)(t-|\mathcal{I}_i|)} V_0\|_{L^2(\mathbb{R})} + \left\| \begin{pmatrix} \mathcal{R}_{1,i} \\ \mathcal{R}_{2,i} \end{pmatrix} \right\|_{L^2([a_i,a_{i+1}])},$$

where $|\mathcal{I}_i| = t_{1,ex}(a_i,t) - t_{1,en}(a_{i+1},t) + t_{2,ex}(a_i) - t_{2,en}(a_{i+1})$. Summing over i and taking the limit as $N \to \infty$, we obtain

$$\|V(\cdot,t)\|_{L^2(\mathcal{D}_2)} \le C_K e^{-\gamma(t-\sup_{x\in\mathbb{R}} |\mathcal{I}(x,t)|)} \|V_0\|_{L^2(\mathbb{R})} + C_K (t - \sup_{x\in\mathbb{R}} |\mathcal{I}(x,t)|)^{-\frac{1}{4}} \|V_0\|_{L^2(\mathbb{R})},$$

where $\mathcal{I}(x,t) = [t_{1,en}(x,t), t_{1,ex}(x,t)] \cup [t_{2,en}(x,t), t_{2,ex}(x,t)]$ and the remainder terms $\mathcal{R}_{j,i}$ were handled exactly as in Section 3.1 thanks to the fact that

$$|t_{j,ex}(a_{i+1}) - t_{j,ex}(a_i)| \le \frac{C}{N} \quad \text{and} \quad |t_{j,en}(a_i) - t_{j,en}(a_{i+1})| \le \frac{C}{N} \quad \text{for } j \in \{1,2\}.$$

Since

$$\sup_{x\ge R,\, t>0} |\mathcal{I}(x,t)| = \frac{2R}{|\lambda_1|} + \frac{2R}{|\lambda_2|} = \tau, \tag{3.24}$$

we recover

$$\|V(\cdot,t)\|_{L^2(\mathcal{D}_2)} \le C_K e^{-\gamma(t-\tau)} \|V_0\|_{L^2(\mathbb{R})} + C_K (t-\tau)^{-\frac{1}{4}} \|V_0\|_{L^2(\mathbb{R})}.$$

Step 5: *Analysis of $\mathcal{F}_2$.* In this domain, we have $t_{1,en} = t_{2,en} = t_{1,ex} = t_{2,ex} = 0$ and therefore we recover

$$\|V(\cdot,t)\|_{L^2(\mathcal{F}_2)} \le C_K e^{-\gamma t} \|V_0\|_{L^2(\mathbb{R})} + C_K t^{-\frac{1}{4}} \|V_0\|_{L^2(\mathbb{R})}.$$

Step 6: *Conclusion of Case 1.* Putting the previous steps together, we obtain the claimed delay in the region $x \ge R$:

$$\|V(\cdot,t)\|_{L^2([R,+\infty))} \le 2C_K e^{-\gamma(t-\tau)} \|V_0\|_{L^2(\mathbb{R})} + 2C_K (t-\tau)^{-\frac{1}{4}} \|V_0\|_{L^2(\mathbb{R})}.$$

Case 2: $x \le -R$. The characteristics lines never cross the undamped region ω^c and thus, as in $\mathcal{F}_2$, we obtain

$$\|V(\cdot,t)\|_{L^2((-\infty,-R])} \le C_K e^{-\gamma t} \|V_0\|_{L^2(\mathbb{R})} + C_K t^{-\frac{1}{4}} \|V_0\|_{L^2(\mathbb{R})}.$$

Case 3: $-R \le x \le R$. Performing a decomposition similar to the one we used for $\mathcal{D}_2$ is necessary. Since the characteristics lines all start inside the undamped region ω^c and $|\lambda_1| \ge |\lambda_2|$, we recover a delay of $\max\left(\frac{2R}{|\lambda_1|}, \frac{2R}{|\lambda_2|}\right) = \frac{2R}{|\lambda_2|}$. Compared to the proof of Proposition 3.3, here we do not need to split the space domain furthermore as, for all $(x,t) \in [-R,R] \times (0,\infty)$, we have

$$t_{1,en}(x,t) \ge t_{2,en}(x,t).$$

We obtain

$$\|V(\cdot,t)\|_{L^2([-R,R])} \le C_K e^{-\gamma\left(t-\frac{2R}{|\lambda_2|}\right)} \|V_0\|_{L^2(\mathbb{R})} + C_K \left(t - \frac{2R}{|\lambda_2|}\right)^{-\frac{1}{4}} \|V_0\|_{L^2(\mathbb{R})}.$$

Conclusion. Combining the results of the three cases yields

$$\begin{aligned}\|V(\cdot,t)\|_{L^2(\mathbb{R})} &= \|V(\cdot,t)\|_{L^2((-\infty,-R])} + \|V(\cdot,t)\|_{L^2([-R,R])} + \|V(\cdot,t)\|_{L^2([R,+\infty))} \\ &\leq C_K e^{-\gamma t}\|V_0\|_{L^2(\mathbb{R})} + C_K t^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})} \\ &+ C_K e^{-\gamma\left(t-\frac{2R}{|\lambda_2|}\right)}\|V_0\|_{L^2(\mathbb{R})} + C_K\left(t-\frac{2R}{|\lambda_2|}\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})} \\ &+ 2C_K e^{-\gamma(t-\tau)}\|V_0\|_{L^2(\mathbb{R})} + 2C_K(t-\tau))^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})} \\ &\leq 4C_K e^{-\gamma(t-\tau)}\|V_0\|_{L^2(\mathbb{R})} + 4C_K(t-\tau)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})},\end{aligned}$$

where $\tau = \frac{2R}{|\lambda_1|} + \frac{2R}{|\lambda_2|}$.

□

4. Proof of Theorem 1.2

The preceding analysis in the general setting leads to the following result.

Corollary 4.1 (Upper bound on the time-delay). *Let V be the solution of (2.1) associated with the initial data $V_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.*

If $x \geq R$, at least one component of the conservative semigroup S_c is active in

$$\mathcal{I}(x,t) = \bigcup_{i=1}^{p}[t_{i,en}(x,t), t_{i,ex}(x,t)]. \tag{4.1}$$

If $x \leq -R$, at least one component of the conservative semigroup S_c is active in

$$\mathcal{I}(x,t) = \bigcup_{i=p+1}^{n}[t_{i,en}(x,t), t_{i,ex}(x,t)]. \tag{4.2}$$

If $x \in [-R,R]$, at least one component of the conservative semigroup S_c is active in

$$\mathcal{I}(x,t) = \bigcup_{i=1}^{n}[t_{i,en}(x,t), t]. \tag{4.3}$$

Moreover,

$$\sup_{x\in\mathbb{R},\, t>0} |\mathcal{I}(x,t)| \leq \tau = \max\left(\sum_{i=1}^{p}\frac{2R}{|\lambda_i|}, \sum_{i=p+1}^{n}\frac{2R}{|\lambda_i|}\right). \tag{4.4}$$

Proof. The identities (4.1), (4.2), and (4.3) follow directly from Proposition 2.3. The upper bound for $|\mathcal{I}(x,(t)|$ can be obtained by estimating each space region separately:

$$\sup_{x>R,\, t>0} |\mathcal{I}(x,t)| \leq \sum_{i=1}^{p}\frac{2R}{|\lambda_i|}; \tag{4.5}$$

$$\sup_{x<-R,\, t>0} |\mathcal{I}(x,t)| \leq \sum_{i=p+1}^{n}\frac{2R}{|\lambda_i|}; \tag{4.6}$$

$$\sup_{-R<x<R,\, t>0} |\mathcal{I}(x,t)| \leq \max_{i\in\{1,\dots,n\}}\frac{2R}{|\lambda_i|}. \tag{4.7}$$

Gathering these estimates leads to (4.4). □

With this proposition in hand, we are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. As in the previous cases, the proof relies on a precise decomposition of the space-domain $\mathbb{R}$.

Case 1: $x \geq R$. **Step 1:** *Representation of the solution.* Owing to Proposition 2.3, for all $(x,t) \in [R,+\infty) \times (0,\infty)$, we have

$$v_i(x,t) = [S_{d,i}(t, t_{i,ex}(x,t))S_{c,i}(t_{i,ex}(x,t), t_{i,en}(x,t))S_{d,i}(t_{i,en}(x,t), 0)v_{i,0}](x), \qquad i \in \{1,\dots,p\}.$$

and

$$v_i(x,t) = [S_{d,i}(t,0)v_{i,0}](x), \qquad i \in \{p+1,\dots,n\}.$$

In the proof of Propositions 3.1 and 3.4, breaking the analysis into multiple cases allowed us to obtain an explicit representation of the solution in terms of $S_{d,i}$ and $S_{c,i}$. However, for the general case, the vast number of potential configurations[1] makes it infeasible to provide an explicit representation for each case. We shall focus on studying the scenarios that generate the largest time-delay in the time-decay rates.

Corollary 4.1 yields that the largest time-delay possible is given by

$$\sum_{i=1}^{p} \frac{2R}{|\lambda_i|} = \sup_{x\geq R,\, t>0} |\mathcal{I}(x,t)|.$$

This corresponds to cases where the characteristic lines of each component spends the maximum amount of time in the undamped region without overlapping with other components. In other words, when one component's characteristic line is in the undamped region, none of the other components' characteristic lines are inside it. This condition minimizes the time during which all the 0-th order couplings are active.

Concretely, this configuration occurs when

$$t_{i,en}(x,t) \leq t_{i+1,ex}(x,t), \qquad i \in \{1,\dots,p\}. \tag{4.8}$$

We define, for $t>0$, $\mathcal{J}_t = \{x \in [R,+\infty] : \text{(4.8) is satisfied}\}$.

Step 2: *Splitting of the space-domain.* We decompose the domain $[R,+\infty)$ into $\mathcal{D}_p \cup \mathcal{F}_p$ where $\mathcal{D}_p$ and $\mathcal{F}_p$ are defined in (3.4). The specific choice of $k=p$ in (3.4) ensures that we decompose the interval with respect to the slowest eigenvalue.

Step 3: *Analysis of $\mathcal{D}_p \cap \mathcal{J}_t$.* Introducing a sequence $(a_i)_{i=1}^N$ to decompose $\mathcal{D}_p \cap \mathcal{J}_t$ as in (3.7), if (4.8) is satisfied, we have

$$\begin{aligned}\int_{a_i}^{a_{i+1}} |V(x,t)|^2 \leq \int_{a_i}^{a_{i+1}} \left| \left[S_d \begin{pmatrix} S_{c,1} \\ S_{d,2} \\ S_{d,3} \\ \vdots \\ S_{d,p} \end{pmatrix} S_d \begin{pmatrix} S_{d,1} \\ S_{c,2} \\ S_{d,3} \\ \vdots \\ S_{d,n} \end{pmatrix} \cdots \begin{pmatrix} S_{d,1} \\ S_{d,2} \\ \vdots \\ S_{c,p-1} \\ S_{d,n} \end{pmatrix} S_d \begin{pmatrix} S_{d,1} \\ S_{d,2} \\ \vdots \\ S_{d,p-1} \\ S_{c,p} \\ S_{d,p+1} \\ \vdots \\ S_{d,n} \end{pmatrix} S_d \begin{pmatrix} v_{1,0}(x,t) \\ \vdots \\ v_{n,0}(x,t) \end{pmatrix} \right](x) \right|^2 \\ + \int_{a_i}^{a_{i+1}} \left| \begin{pmatrix} \mathcal{R}_{1,i} \\ \vdots \\ \mathcal{R}_{p,i} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right|^2,\end{aligned} \tag{4.9}$$

where, to simplify the notation, we omitted the parameters a_i, a_{i+1}, and the explicit entering and existing time on the right-hand side.

[1]There are $\frac{p(p+1)}{2}+1$. Computing the number of possible configurations of the solution at a fixed time t is equivalent to the following problem: Let $I \subset \mathbb{R}$ be a compact interval and $(x_i)_{i=1}^p \subset \mathbb{R}$ be an increasing sequence of $p>0$ points. How many different configurations are there, depending on whether or not the points are in I? To answer this question, we only need to know how many configurations exist where exactly $q \in \{0,\dots,p+1\}$ points are in I. Summing over all these values, we can determine the total number of possible configurations. Since the points x_i are ordered, for $q \in \{1,\dots,p\}$, there are $p-q+1$ possible configurations such that exactly q points belong to I; additionally, there is one possibility for $q=0$. Summing up all these, we find that the total number of possible configurations is $\sum_{q=1}^p (p-q+1)+1 = \sum_{k=1}^p k + 1 = \frac{p(p+1)}{2}+1$.

Step 4: *Use of the semigroup properties.* Recalling that the mixed operators of the type

$$\begin{pmatrix} S_{c,1} \\ S_{d,2} \\ \vdots \\ S_{d,n} \end{pmatrix}$$

preserves the L^2 norm of the solution – since, for $i = 1, .., n$, the $S_{c,i}$ are conservative and $S_{d,i}$ associated with a positive definite matrix –, we obtain that the classical time-decay estimates delayed by the time $\sup_{x\geq R} |\mathcal{I}|$, where $\mathcal{I}$ is defined in (4.1). Using the properties of the semigroups and estimating the remainder terms as in (3.11), we obtain

$$\|V(\cdot,t)\|_{L^2(\mathcal{D}_p\cap\mathcal{J}_t)} \leq C_K e^{-\gamma(t-\sup_{x\geq R}|\mathcal{I}(x,t)|)}\|V_0\|_{L^2(\mathbb{R})} + C_K(t - \sup_{x\geq R}|\mathcal{I}(x,t)|)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}.$$

Step 5: *Analysis of the inteval $\mathcal{D}_p$.* We have seen how to recover decay in the region $\mathcal{D}_p\cap\mathcal{J}_t$. For the remaining space-domain $\mathcal{D}_p\backslash\mathcal{J}_t$, the representation of the solution will always be a combination of the semigroups $S_{d,i}$ and $S_{c,i}$ with the conservative semigroup $S_{c,i}$ being active only for a finite time bounded by τ. Although an explicit representation of the solution is cumbersome to formulate, the arguments for analyzing each case follow the same pattern. We have

$$\|V(\cdot,t)\|_{L^2(\mathcal{D}_p)} \leq C_K e^{-\gamma(t-\sup_{x\geq R}|\mathcal{I}(x,t)|)}\|V_0\|_{L^2(\mathbb{R})} + C_K(t - \sup_{x\geq R}|\mathcal{I}(x,t)|)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}.$$

Step 6: *Analysis of $\mathcal{F}_p$.* As none of the characteristics lines cross the undamped region ω^c in this case, we have

$$\|V(\cdot,t)\|_{L^2(\mathcal{F}_p)} \leq C_K e^{-\gamma t}\|V_0\|_{L^2(\mathbb{R})} + C_K t^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}.$$

Step 7: *Conclusion of Case 1.* Putting the previous steps together and using (4.5), we obtain the claimed delay in the region $x \geq R$:

$$\|V(\cdot,t)\|_{L^2([R,+\infty))} \leq 2C_K e^{-\gamma\left(t-\sum_{i=1}^{p}\frac{2R}{|\lambda_i|}\right)}\|V_0\|_{L^2(\mathbb{R})} + 2C_K\left(t - \sum_{i=1}^{p}\frac{2R}{|\lambda_i|}\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}.$$

Case 2: $x \leq -R$. The analysis of this space-domain can be done symmetrically with respect to Case 1. We have

$$\|V(\cdot,t)\|_{L^2((-\infty,R])} \leq 2C_K e^{-\gamma\left(t-\sum_{i=p+1}^{n}\frac{2R}{|\lambda_i|}\right)}\|V_0\|_{L^2(\mathbb{R})} + 2C_K\left(t - \sum_{i=p+1}^{n}\frac{2R}{|\lambda_i|}\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}.$$

Case 3: $-R \leq x \leq R$. We decompose the space-domain into $\mathcal{H}_1^p = \{(x,t) \in [-R,R]\times(0,\infty) : t_{p,en}(x,t) \leq t_{p+1,en}(x,t)\}$ and $\mathcal{H}_2^p = \{(x,t) \in [-R,R]\times(0,\infty) : t_{p,en}(x,t) \geq t_{p+1,en}(x,t)\}$. Reasoning as in the proof of Proposition 3.3, we have

$$\|V(\cdot,t)\|_{L^2(\mathcal{H}_1^p)} \leq C_K e^{-\gamma\left(t-\frac{2R}{|\lambda_p|}\right)}\|V_0\|_{L^2(\mathbb{R})} + C_K\left(t - \frac{2R}{|\lambda_p|}\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})},$$

and symmetrically, for $\mathcal{H}_2^p$, we have

$$\|V(\cdot,t)\|_{L^2(\mathcal{H}_2^p)} \leq C_K e^{-\gamma\left(t-\frac{2R}{|\lambda_{p+1}|}\right)}\|V_0\|_{L^2(\mathbb{R})} + C_K\left(t - \frac{2R}{|\lambda_{p+1}|}\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})},$$

Therefore,

$$\begin{aligned}\|V(\cdot,t)\|_{L^2([-R,R])} \leq{}& 2C_K e^{-\gamma\left(t-\max\left(\frac{2R}{|\lambda_p|},\frac{2R}{|\lambda_{p+1}|}\right)\right)}\|V_0\|_{L^2(\mathbb{R})} \\ &+ 2C_K\left(t-\max\left(\frac{2R}{|\lambda_p|},\frac{2R}{|\lambda_{p+1}|}\right)\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}.\end{aligned}$$

Conclusion. Combining the results of the three cases yields

$$\|V(\cdot,t)\|_{L^2(\mathbb{R})} = \|V(\cdot,t)\|_{L^2((-\infty,-R])} + \|V(\cdot,t)\|_{L^2([-R,R])} + \|V(\cdot,t)\|_{L^2([R,+\infty))}$$

$$
\begin{aligned}
&\leq 2C_K e^{-\gamma(t-\sum_{i=p+1}^{n}\frac{2R}{|\lambda_i|})}\|V_0\|_{L^2(\mathbb{R})} + 2C_K\left(t-\sum_{i=p+1}^{n}\frac{2R}{|\lambda_i|}\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}.\\
&+2C_K e^{-\gamma\left(t-\max\left(\frac{2R}{|\lambda_p|},\frac{2R}{|\lambda_{p+1}|}\right)\right)}\|V_0\|_{L^2(\mathbb{R})}\\
&\quad+2C_K\left(t-\max\left(\frac{2R}{|\lambda_p|},\frac{2R}{|\lambda_{p+1}|}\right)\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}\\
&+2C_K e^{-\gamma(t-\sum_{i=1}^{p}\frac{2R}{|\lambda_i|})}\|V_0\|_{L^2(\mathbb{R})} + 2C_K\left(t-\sum_{i=1}^{p}\frac{2R}{|\lambda_i|}\right)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})}.\\
&\leq 6C_K e^{-\gamma(t-\tau)}\|V_0\|_{L^2(\mathbb{R})} + 6C_K(t-\tau)^{-\frac{1}{4}}\|V_0\|_{L^2(\mathbb{R})},
\end{aligned}
$$

where $\tau = \max\left(\sum_{i=1}^{p}\frac{2R}{|\lambda_i|}, \sum_{i=p+1}^{n}\frac{2R}{|\lambda_i|}\right)$.

Similarly to the computations done in Section 3.1 (see Step 6 in the aforementioned section), we are also able to recover the desired L^∞ time-decay. The proof of the theorem 1.2 is completed. $\square$

5. Extensions and open problems

In this contribution, we studied the time-asymptotic behavior of linear hyperbolic systems under partial dissipation localized in suitable subsets of the domain. Our work opens up several possible extensions and open problems. We list some of them below.

1. *More general undamped domains.* Results similar to ours can be obtained whenever ω^c is a domain of finite measure. For example, if we consider ω^c as a finite union of bounded stripes, we can directly retrieve similar decay estimates with a delay depending on the time spent by each characteristic in each stripe.
2. *Problem posed on the half-line.* With the method developed in the present paper, we can also consider the case when the x-space $\mathbb{R}$ is replaced by the half-line $(-\infty, 0]$ and the undamped region is localized near the boundary. Specifically, consider the following linear hyperbolic system:

$$
\begin{cases}
\partial_t U + A\partial_x U = -BU\mathbb{1}_{\omega^*}, & (x,t)\in(-\infty,0]\times(0,\infty),\\
U(0,x) = U_0(x), & x\in(-\infty,0],\\
C_0U(0,t) = 0, & t\in(0,\infty).
\end{cases}
\tag{5.1}
$$

   where C_0 is a given matrix, A and B satisfy the same assumptions as before, and $\omega^* := \mathbb{R}\setminus[-R,0] = (-\infty,-R)$. If the matrix C_0 in (5.1) ensures that the characteristics are reflected at the boundary $x = 0$ (see [19, p. 649] for further details), then the time spent by the characteristics in the undamped region ω^c is finite. Thus, the asymptotic result would follow from similar arguments as in our previous analysis. We remark that the right-hand side of the system in (5.1) vanishes near the boundary $x = 0$; thus, system (5.1) reduces to uncoupled transport equations for which we can find suitable boundary conditions (see [12, 19]).
3. *Multidimensional problems.* For multi-dimensional systems, i.e.

$$
\partial_t U + \sum_{j=1}^{d} A^j(U)\partial_{x_j}U = -BU\mathbb{1}_\omega, \qquad (x,t)\in\mathbb{R}^d\times(0,\infty),
$$

   where the $A^j(U)$ and B are symmetric matrices, there is a direct obstruction to the use of our arguments. Indeed, the flux matrices A^j may not all be diagonalizable in the same basis and therefore we may not be able to rewrite the system as coupled transport equations. Therefore, different approaches are needed.

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