Partially dissipative hyperbolic systems, results[1] and perspectives

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[1] in collaboration with Raphaël Danchin

We look at multi-dimensional first order n-component systems in $\mathbb{R}^d$:

$$\frac{\partial V}{\partial t} + \sum_{k=1}^{d} A^k(V)\frac{\partial V}{\partial x_k} + \frac{LV}{\varepsilon} = 0,$$

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Our main example of application is the compressible Euler equations with damping:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla P + \dfrac{\rho v}{\varepsilon} = 0. \end{cases} \tag{1}$$

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- $\rightarrow$ There exists a purely damped mode in the low frequencies regime associated to the eigenvalue $\frac{1}{\varepsilon}$.
- Indeed, defining $z = u + \varepsilon\nabla\rho$ we can recast the linear Euler system into the following *diagonal* form:

$$\begin{cases} \partial_t \rho - \varepsilon\Delta\rho = -\mathrm{div}\, z, \\ \partial_t z + \dfrac{z}{\varepsilon} = \varepsilon\Delta v. \end{cases} \qquad (2)$$

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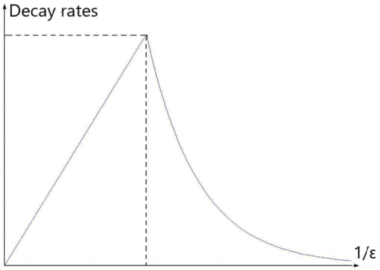

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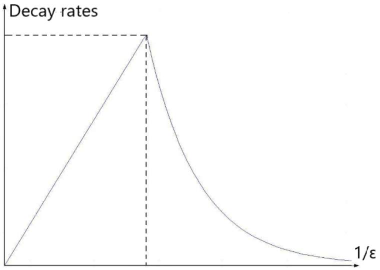

- $\to$ To capture this phenomenon more precisely, we set the threshold between low and high frequencies at $J_\varepsilon = \lfloor -\log_2 \varepsilon \rfloor$ in our homogeneous Besov norms.

Relaxation limit

Theorem (Danchin, C-B '22)

Let $d \geq 1$, $p \in [2,4]$, $\varepsilon > 0$ and $\bar{\rho}$ be a strictly positive constant.
Let $(\rho - \bar{\rho}, v)$ be the global small solution of the compressible Euler system with damping associated with the initial data (ρ_0, v_0) that we constructed.
And let $\mathcal{N} - \bar{\rho}$ be the global small solution associated to the porous media equation:

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0 \\ \mathcal{N}(0,x) = \mathcal{N}_0 \end{cases}.$$

Defining $(\widetilde{\rho}^\varepsilon, \widetilde{v}^\varepsilon)(t,x) \triangleq (\rho, \varepsilon^{-1} v)(\varepsilon^{-1} t, x)$ and assuming that

$$\|\widetilde{\rho}_0^\varepsilon - \mathcal{N}_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \leq C\varepsilon,$$

then

$$\|\widetilde{\rho}^\varepsilon - \mathcal{N}\|_{L^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|\widetilde{\rho}^\varepsilon - \mathcal{N}\|_{L^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+1}_{p,1})} + \left\| \frac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon} + \widetilde{v}^\varepsilon \right\|_{L^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})} \leq C\varepsilon.$$

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- Compressible multi-fluid system: Pressure-relaxation limit for a one-velocity Baer-Nunziato model to a Kapila model. Joint work with Cosmin Burtea and Jin Tan.
- Global existence and relaxation limit for the hyperbolic-parabolic chemotaxis system. Joint work with Qingyou He and Ling-Yun Shou.
- Anisotropic systems e.g. 2D Boussinesq with damping. Work in progress with Roberta Bianchini.

Merci pour votre attention !

Well-posedness result

Theorem (Danchin, C-B '21)

Let $d \geq 1$, $p \in [2,4]$ et $\varepsilon > 0$. There exists $k_p \in \mathbb{Z}$ et $c_0 = c_0(p) > 0$ such that for all $J_\varepsilon \triangleq \lfloor -\log_2 \varepsilon \rfloor + k_p$, if we assume

$$\|Z_0\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}} + \varepsilon \|Z_0\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \leq c_0,$$

then the system admits a unique solution Z satisfying

$$X_{p,\varepsilon}(t) \lesssim \|Z_0\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}} + \varepsilon \|Z_0\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \quad \text{for all } t \geq 0, \text{ and where}$$

$$X_{p,\varepsilon}(t) \triangleq \varepsilon \|Z\|^{h}_{L^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|Z\|^{h}_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \varepsilon^{-\frac{1}{2}} \|Z_2\|_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}$$
$$+ \|Z\|^{\ell}_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \varepsilon \|Z_1\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})} + \|Z_2\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})} + \|W\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}.$$

Relaxation limit

Theorem (Danchin, C-B '22)

Let $d \geq 1$, $p \in [2,4]$ and $\varepsilon > 0$. Let $\bar{\rho}$ be a strictly positive constant and $(\rho - \bar{\rho}, v)$ be the solution constructed in our global well-posedness result.

Let the positive function $\mathcal{N}_0$ such that $\mathcal{N}_0 - \bar{\rho}$ is small enough in $\dot{B}^{\frac{d}{p}}_{p,1}$, and let $\mathcal{N} \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{d}{p}+2}_{p,1})$ be the unique solution associated to the Cauchy problem:

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0 \\ \mathcal{N}(0,x) = \mathcal{N}_0 \end{cases}$$

If we assume that

$$\|\widetilde{\rho}^{\varepsilon}_0 - \mathcal{N}_0\|_{B^{\frac{d}{p}-1}_{p,1}} \leq C\varepsilon,$$

then

$$\|\widetilde{\rho}^{\varepsilon} - \mathcal{N}\|_{L^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|\widetilde{\rho}^{\varepsilon} - \mathcal{N}\|_{L^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+1}_{p,1})} + \left\| \frac{\nabla P(\widetilde{\rho}^{\varepsilon})}{\widetilde{\rho}^{\varepsilon}} + \widetilde{v}^{\varepsilon} \right\|_{L^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})} \leq C\varepsilon.$$