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Relaxation Approximation and Asymptotic Stability of Stratified Solutions to the IPM Equation

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Abstract

We prove the nonlinear asymptotic stability of stably stratified solutions to the Incompressible Porous Media equation (IPM) for initial perturbations in $\dot{H}^{1-\tau}(\mathbb{R}^2)\cap \dot{H}^s(\mathbb{R}^2)$ with $s > 3$ and for any $0 < \tau < 1$. Such a result improves upon the existing literature, where the asymptotic stability is proved for initial perturbations belonging at least to $H^{20}(\mathbb{R}^2)$. More precisely, the aim of the article is threefold. First, we provide a simplified and improved proof of global-in-time well-posedness of the Boussinesq equations with strongly damped vorticity in $H^{1-\tau}(\mathbb{R}^2)\cap\dot{H}^s(\mathbb{R}^2)$ with $s > 3$ and $0 < \tau < 1$. Next, we prove the strong convergence of the Boussinesq system with damped vorticity towards (IPM) under a suitable scaling. Lastly, the asymptotic stability of stratified solutions to (IPM) follows as a byproduct. A symmetrization of the approximating system and a careful study of the anisotropic properties of the equations via anisotropic Littlewood-Paley decomposition play key roles to obtain uniform energy estimates. Finally, one of the main new and crucial points is the integrable time decay of the vertical velocity $\|u_2(t)\|_{L^\infty(\mathbb{R}^2)}$ for initial data only in $\dot{H}^{1-\tau}(\mathbb{R}^2) \cap \dot{H}^s(\mathbb{R}^2)$ with $s > 3$.

1. Introduction

The Incompressible Porous Media (IPM) system in two space dimensions is an active scalar equation

$$\begin{cases} \partial_t \eta + \mathbf{u}\cdot\nabla\eta = 0, \\ \mathbf{u} = -\kappa\nabla P + \mathbf{g}\eta, \qquad \mathbf{g} = (0,-g)^T, \qquad \text{(Darcy law)} \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{1}$$

modeling the dynamics of a fluid of density $\eta = \eta(t,x,y) : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ through a porous medium according to the Darcy law, where $\kappa > 0$ and $g > 0$

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are the permeability coefficient and the gravity acceleration respectively, which hereafter are assumed to be $\kappa = g = 1$. We refer to [6] and references therein for further explanations on the physics and the applications of the model. The active scalar velocity $\mathbf{u} = (u_1, u_2)$ of system (1) can be reformulated in terms of a singular integral operator of degree 0

$$\mathbf{u} = (-\mathcal{R}_2, \mathcal{R}_1)\mathcal{R}_1\rho, \tag{2}$$

where $(\mathcal{R}_1, \mathcal{R}_2)$ is the two-dimensional Riesz transform, i.e.

$$\mathcal{R}_1 = (-\Delta)^{-1/2}\partial_x, \qquad \mathcal{R}_2 = (-\Delta)^{-1/2}\partial_y. \tag{3}$$

We are interested in the stability properties of the stratified steady state $\overline{\rho}_{\mathrm{eq}}(y) = \rho_0 - y$, where $\rho_0 > 0$ is a constant averaged density. Introducing the perturbed unknown $\rho = \rho(t, x, y)$ such that $\eta(t, x, y) = \overline{\rho}_{\mathrm{eq}}(y) + \rho(t, x, y)$, the perturbation ρ satisfies

$$\partial_t\rho - \mathcal{R}_1^2\rho = (\mathcal{R}_2\mathcal{R}_1\rho, -\mathcal{R}_1^2\rho)\cdot\nabla\rho. \tag{IPM-diss}$$

The nonlinear asymptotic stability of the stratified steady state $\overline{\rho}_{\mathrm{eq}}(y) = \rho_0 - y$ in the whole space $\mathbb{R}^2$ has been first established by Elgindi [16], for initial data at least in $H^{20}(\mathbb{R}^2)$. The analogous result but in the periodic finite channel $\mathbb{T}\times[-\pi, \pi]$ is due to Castro et al. [6] under slightly less restrictive regularity assumptions. We remark that the linear operator

$$\partial_t\rho - \mathcal{R}_1^2\rho = 0$$

in frequency space $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ is dissipative everywhere but the hyperplane $\xi_1 = 0$ thanks to the monotonicity $\bar{\rho}'_{\mathrm{eq}}(y) < 0$ (stable stratification, see [15,19]). Therefore, it appears natural to approximate (IPM-diss) by a system with a (partially) dissipative linear part. In this regard, it turns out that the two-dimensional Boussinesq equations in vorticity form with strongly damped vorticity is a good approximation to (IPM-diss) under a suitable scaling of time and unknowns. To the best of our knowledge, the approximation of (IPM-diss) via the Boussinesq system with damped vorticity is new and establishing its rigorous validity in the sense of strong convergence of solutions is in the scope of the present work.
The two-dimensional Boussinesq equations with damped velocity with gravity acceleration $g = 1$ and background density profile $\overline{\rho}_{\mathrm{eq}}(y) = \rho_0 - y$ read as follows:

$$\begin{cases} \partial_t b - u_2 = -(u_1\partial_x b + u_2\partial_y b), \\ \partial_t u_1 + \partial_x P = -\dfrac{u_1}{\varepsilon} - (u_1\partial_x u_1 + u_2\partial_y u_1), \\ \partial_t u_2 + \partial_y P = -b - \dfrac{u_2}{\varepsilon} - (u_1\partial_x u_2 + u_2\partial_y u_2), \\ \partial_x u_1 + \partial_y u_2 = 0. \end{cases} \tag{4}$$

Here $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ is the velocity field, b is the buoyancy term and P is the incompressible pressure. This system is obtained by a linearization of the

density-dependent incompressible Euler equations with damped velocity around the *hydrostatic stratified steady state*

$$(\overline{\rho}_{\mathrm{eq}}(y), (0,0), \overline{P}_{\mathrm{eq}}(y)) \quad \text{with} \quad \overline{P}'_{\mathrm{eq}}(y) = -\overline{\rho}_{\mathrm{eq}}(y). \tag{5}$$

Several mathematical works in the existing literature have been devoted to this system. Besides explaining the applications of (4) to electrocapillarity, Castro et al. [7] (see also references therein) establish the global-in-time well-posedness of system (4) in the bounded domain $\mathbb{T} \times [0, 1]$. In the whole space, global smooth solutions have been first provided by Wan [35] by means of Green function analysis and energy methods.

We refer to [3,7] for a (formal) derivation of (4) by a linearization of the density-dependent incompressible Euler equations around the hydrostatic stratified steady state under the strong Boussinesq approximation $\rho \sim \rho_0$ [15].
For our scopes, it is convenient to rewrite the system in vorticity ω and stream function ϕ formulation where $\mathbf{u} = \nabla^{\perp}\phi$ with the sign convention (as in [16]) $\nabla^{\perp} = (\partial_y, -\partial_x)$, which gives

$$\begin{cases} \partial_t b - (-\Delta)^{-1}\partial_x \omega = -\mathbf{u}\cdot\nabla b, \\ \partial_t \omega + \dfrac{\omega}{\varepsilon} - \partial_x b = -\mathbf{u}\cdot\nabla\omega, \\ \Delta\phi = \omega. \end{cases} \tag{2D-Bouss}$$

Our first goal is to establish a systematic and improved proof of the global-in-time existence of smooth solutions for small data to (2D-Bouss) (in Theorem 2.1), exploiting the anisotropic nature of the system by means of anisotropic Littlewood-Paley decomposition of the Fourier space $(\xi_1, \xi_2) \in \mathbb{R}^2$ as introduced in [8,21,33]. Since the *linear* and *undamped* approximation of the system supports the propagation of anisotropic waves of dispersion relation $\pm\xi_1/|\xi| = \mp i\widehat{\mathcal{R}_1}$ (see [2,15]), it is not surprising that the horizontal anisotropic decomposition of the phase space plays a key role in our refined analysis. In our case, the linear part of system (2D-Bouss) is dissipative provided that $\xi_1 \neq 0$, therefore it is natural to build an energy functional with an anisotropic Fourier multiplier as a weight (multiplier method). This idea allows us to prove that $\|u_2(t)\|_{L^\infty(\mathbb{R}^2)}$ is integrable in time without assuming any additional $L^1(\mathbb{R}^2)$ integrability of the initial data. More precisely, the control of $\|u_2\|_{L^1_T L^\infty(\mathbb{R}^2)}$ without L^1-integrability and high regularity assumptions is new and this is a striking point to provide a substantially improved global-in-time well-posedness of (2D-Bouss) for small data only in $\dot{H}^{1-\tau}(\mathbb{R}^2) \cap \dot{H}^s(\mathbb{R}^2)$ with $0 < \tau < 1$ and $s \geqq 3 + \tau$.

For convenience of the reader, we point out that our small-data global-in-time existence in Theorem 2.1 could be reformulated as a result of *nonlinear asymptotic stability* of the hydrostatic steady state (5) with $\overline{\rho}_{\mathrm{eq}}(y) = \rho_0 - y$, under the evolution of the Boussinesq system

$$\begin{cases} \partial_t \eta + \mathbf{u}\cdot\nabla\eta = 0, \\ \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \eta\mathbf{g}, \qquad \mathbf{g} = (0, -g), \\ \nabla\cdot\mathbf{u} = 0. \end{cases} \tag{E}$$

This reformulation holds for system (E) with initial density $\eta_{\text{in}}(x, y)$ such that $\|\eta_{\text{in}} - \overline{\rho}_{\text{eq}}\|_X \ll 1$ for a suitable functional space X, in the same spirit of [7].

Our second and main goal is to rigorously justify the relaxation limit of the two-dimensional Boussinesq equations with damped vorticity (2D-Bouss) towards (IPM-diss) under a suitable scaling that we introduce later on. To the best of our knowledge, this relaxation approximation is new.

Finally, as a byproduct of the relaxation limit and the global well-posedness of (2D-Bouss), we provide a new proof of existence of global smooth solutions in $\mathbb{R}^2$ to the equation (IPM-diss) for the perturbation ρ, with small initial data $\rho(0, x, y) = \rho_{\text{in}}(x, y) \in \dot{H}^{1-\tau}(\mathbb{R}^2) \cap \dot{H}^s(\mathbb{R}^2)$ with $0 < \tau < 1$ and $s \geqq 3 + \tau$. Again, this result can be reformulated in terms of the solution η to (1), for an initial datum $\eta_{\text{in}}(x, y) = \overline{\rho}_{\text{eq}}(y) + \rho_{\text{in}}(x, y)$ such that $\|\eta_{\text{in}}(x, y) - \overline{\rho}_{\text{eq}}(y)\|_X = \|\rho_{\text{in}}(x, y)\|_{\dot{H}^{1-\tau} \cap \dot{H}^s} \ll 1$. Such reformulation yields the nonlinear asymptotic stability of equation (1) around the stratified steady state $\overline{\rho}_{\text{eq}}(y)$, namely the setting of Elgindi [16] that proves the result in $H^{20}(\mathbb{R}^2)$, while in our Theorem 2.3 we only need that the initial perturbation $\rho_{\text{in}}(x, y) \in \dot{H}^{1-\tau}(\mathbb{R}^2) \cap \dot{H}^s$ with $s > 3$.

1.1. A New Formulation

A first key element of our approach is the use of the symmetrized variables introduced in [3]. With the notation

$$\Lambda = (-\Delta)^{1/2} = \sqrt{\xi_1^2 + \xi_2^2},$$

where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ denotes the frequency coordinate in Fourier space, we introduce the new unknown

$$\Omega := \Lambda^{-1}\omega, \tag{6}$$

so that system (2D-Bouss) can be as rewritten

$$\begin{cases} \partial_t b - \mathcal{R}_1\Omega = (\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega) \cdot (\nabla b), \\ \partial_t \Omega + \dfrac{\Omega}{\varepsilon} - \mathcal{R}_1 b = \Lambda^{-1}[(\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega) \cdot (\nabla\Lambda\Omega)], \end{cases} \tag{2D-B}$$

where $\mathcal{R}_j$, $j \in \{1, 2\}$ are the components of the Riesz transform as in (3). Now, we introduce the auxiliary variable

$$z := \Omega - \varepsilon\mathcal{R}_1 b. \tag{7}$$

The system in (b, z) then reads as follows:

$$\begin{cases} \partial_t b - \varepsilon\mathcal{R}_1^2 b = \mathcal{R}_1 z + (\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega) \cdot \nabla b, \\ \partial_t z + \dfrac{z}{\varepsilon} = -\varepsilon\mathcal{R}_1^2\Omega - \varepsilon\mathcal{R}_1[(\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega) \cdot \nabla b] + \Lambda^{-1}[(\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega) \cdot (\nabla\Lambda\Omega)]. \end{cases} \tag{8}$$

Such formulation partially diagonalizes the linear part of the system, except two linear terms in the right-hand side of (8) that in the energy estimates are absorbed by the left-hand side. This allows us to avoid hypocoercivity techniques [1,12],

obtaining a priori estimates simply based on the energy method. We point out that these a priori estimates are *uniform* in the vanishing parameter ε, which is a key point to justify the relaxation towards (IPM-diss).

The use of the good unknown $z = \Omega - \varepsilon \mathcal{R}_1 b$ is inspired both by the work of Hoff and Haspot in [22,23], where the authors introduce the effective velocity for the compressible Navier–Stokes equations and the results of Crin-Barat and Danchin in [10,11,13] on partially dissipative hyperbolic systems. In the aforementioned works, this reformulation is only possible in some specific frequency regime (high frequencies for Navier–Stokes and low frequencies for partially dissipative systems) where the eigenvalues of the linear operator are real. In the present paper, the particular form of the Riesz transform (an operator of degree 0) allows us to use such diagonalization in all frequencies as the eigenvalues $\lambda_\pm$ of the linear part of system (2D-B)

$$\lambda_\pm = \frac{1}{2\varepsilon} \pm \frac{1}{2\varepsilon}\sqrt{1 - \frac{4\varepsilon^2 \xi_1^2}{|\xi|^2}}$$

are real in the whole frequency space for any $\varepsilon \leqq 1/2$, so that the variable z in (7) is a good unknown in all frequency regimes.

1.2. Formal Justification of the Relaxation Limit as $\varepsilon \to 0$

Taking inspiration from the theory of partially dissipative systems (see for instance [4,9,24,25,28,31,32,36] and references therein), one can expect to rigorously justify the relaxation limit from (2D-B) towards (IPM-diss) as $\varepsilon \to 0$, by applying the following scaling:

$$(\widetilde{b}^\varepsilon, \widetilde{\Omega}^\varepsilon)(\tau, x) \triangleq (b, \frac{\Omega}{\varepsilon})(t, x) \quad \text{with} \quad \tau = \varepsilon t. \tag{9}$$

The system (2D-B) in the scaled unknowns $(\widetilde{b}^\varepsilon, \widetilde{\Omega}^\varepsilon)$ reads as follows:

$$\begin{cases} \partial_t \widetilde{b}^\varepsilon - \mathcal{R}_1 \widetilde{\Omega}^\varepsilon = (\mathcal{R}_2 \widetilde{\Omega}^\varepsilon)\partial_x \widetilde{b}^\varepsilon - (\mathcal{R}_1 \widetilde{\Omega}^\varepsilon)\partial_y \widetilde{b}^\varepsilon, \\ \varepsilon^2 \partial_t \widetilde{\Omega}^\varepsilon - \mathcal{R}_1 \widetilde{b}^\varepsilon + \widetilde{\Omega}^\varepsilon = \varepsilon^2 \Lambda^{-1}[(\mathcal{R}_2 \widetilde{\Omega}^\varepsilon, -\mathcal{R}_1 \widetilde{\Omega}^\varepsilon) \cdot (\nabla \Lambda \widetilde{\Omega}^\varepsilon)]. \end{cases} \tag{10}$$

• Our first result is a systematic and improved proof of the global-in-time well-posedness, with uniform estimates in the relaxation parameter $\varepsilon \leqq \frac{1}{2}$, of the above system with initial data $(b_{\text{in}}, \Omega_{\text{in}}) \in \dot{H}^{1-\tau} \cap \dot{H}^s$ for any $0 < \tau < 1$ and $s \geqq 3 + \tau$. The approach is based on the use of the anisotropic Littlewood-Paley decomposition that allows to capture the (anisotropic) nature of the equation in a nearly optimal way. For references on the use of anisotropic Besov spaces in the analysis of incompressible flows we refer to [21] and we mention [17] for a study of the effect of anisotropy (the Riesz transform) in low regularity.

Sending $\varepsilon \to 0$ in system (10), one formally obtains that $\widetilde{\Omega}^\varepsilon \to \Omega$ and $\widetilde{b}^\varepsilon \to \rho$, where ρ satisfies the Incompressible Porous Media equation (IPM-diss) and the Darcy law

$$\Omega - \mathcal{R}_1 \rho = 0. \tag{11}$$

We improperly call *diffusive scaling* the change of coordinates (9). Of course it is not the usual diffusive scaling $(t/\varepsilon^2, x/\varepsilon, y/\varepsilon)$ under which the heat equation is invariant, but inserting (11) into the linear part of the equation for $\widetilde{b}^\varepsilon$ in (10) yields

$$\partial_t \rho - \mathcal{R}_1^2 \rho = 0,$$

where $\mathcal{R}_1^2 = (-\Delta)^{-1}\partial_{xx}$ is a partially *dissipative operator*.

• Our next result is a mathematical proof of the relaxation limit of (10) towards (IPM-diss) as $\varepsilon \to 0$. To the best of our knowledge, the relaxation approximation of (IPM-diss) provided by (2D-B) is new as well as its rigorous justification.

On the limit system (IPM-diss), the existence of global-in-time smooth solutions to (IPM-diss) with small data (or, equivalently, the asymptotic stability of (1) around the stratified steady state) has been first proved by Elgindi in [16], both in the full space $\mathbb{R}^2$ and in $\mathbb{T}^2$. In particular, Elgindi shows that the stratified steady state $\overline{\rho}_{\text{eq}}(y) = \rho_0 - y$ is asymptotically stable in $H^s(\mathbb{R}^2)$, for $s \geqq 20$, with the additional integrability assumption that initial perturbations $\rho_{\text{in}}(x, y)$ belong to the L^1-based Sobolev space $W^{4,1}$ (this is also a key hypothesis to use dispersion effects in [18]). Although groundbreaking, the result in [16] requires very high regularity of solutions.

• As a byproduct of the relaxation limit, in this article we provide a new proof of existence of global-in-time smooth solutions to (IPM-diss), only assuming that the initial datum $\rho_{\text{in}}(x, y) \in \dot{H}^{1-\tau} \cap \dot{H}^s$ for any $0 < \tau < 1$ and $s \geqq \tau + 3$.

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2. Main Results

Recalling the notation $\Lambda = (-\Delta)^{\frac{1}{2}}$, we introduce the homogeneous Sobolev space for $s \in \mathbb{R}$

$$\|f\|_{\dot{H}^s} = \|\Lambda^s f\|_{L^2(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |\xi|^{2s} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi\right)^{\frac{1}{2}}. \tag{12}$$

We also use the notation, for any $s, s' \in \mathbb{R}$,

$$\|f\|_{\dot{H}^s \cap \dot{H}^{s'}} = \|f\|_{\dot{H}^s} + \|f\|_{\dot{H}^{s'}}. \tag{13}$$

Our main results hold in Sobolev spaces, however we will rely on the properties of *anisotropic* Besov spaces to obtain some estimates that play a crucial role in our proof. Such anisotropic spaces allow to perfectly capture the anisotropic nature of the 2D Boussinesq system. Similar approaches have been applied to the MHD system by Lin and Zhang in [29,30] and to the incompressible Navier–Stokes equations by Chemin and Zhang [8] and Paicu [33]. More recently, an approach based on anisotropic Besov spaces has been developed for 3D rotating incompressible fluids by Guo, Pausader and Widmayer [21].

Our first result concerns the uniform global well-posedness of system (2D-Bouss). For any $r > 0$, we define the following functional

$$\mathcal{M}_r(T) := \|(b, \Omega, z)\|_{L^\infty_T(\dot{H}^r)} + \sqrt{\varepsilon}\|\mathcal{R}_1 b\|_{L^2_T(\dot{H}^r)} + \frac{1}{\sqrt{\varepsilon}}\|\Omega\|_{L^2_T(\dot{H}^r)} + \frac{1}{\sqrt{\varepsilon}}\|z\|_{L^2_T(\dot{H}^r)}. \tag{14}$$

We obtain the following.

Theorem 2.1. *(Global existence for (2D-B)) For any $0 < \varepsilon \leqq 1/2$ and any $0 < \tau < 1$, let $s \geqq 3+\tau$. For any couple of initial data $(b_{\text{in}}, \Omega_{\text{in}}) \in \dot{H}^{1-\tau}(\mathbb{R}^2) \cap \dot{H}^s(\mathbb{R}^2)$, there exists a constant value $0 < \delta_0 \ll 1$ such that, under the following assumption that*

$$\mathcal{M}(0) = \|(b_{\text{in}}, \Omega_{\text{in}})\|_{\dot{H}^{1-\tau}\cap\dot{H}^s} \leqq \delta_0, \tag{15}$$

there exists a unique global-in-time smooth solution (b, Ω) to system (2D-B) satisfying the following inequality for all times $T > 0$

$$X(T) := \mathcal{M}(T) + \|\nabla u_2\|_{L^1_T(L^\infty)} + \|\Lambda u_2\|_{L^1_T(L^\infty)} \leqq \mathcal{M}(0), \tag{16}$$

where

$$u_2 = (-\Delta)^{-1/2}\partial_x\Omega = \mathcal{R}_1\Omega, \tag{17}$$

and

$$\mathcal{M}(T) := \mathcal{M}_{1-\tau}(T) + \mathcal{M}_s(T). \tag{18}$$

Remark 2.1. (On the expression of $\mathcal{M}(t)$) The functional $\mathcal{M}(t)$ in (18) is the sum of two terms, i.e. $\mathcal{M}_{1-\tau}(t)$ and $\mathcal{M}_s(t)$, defined in (14), which are both crucial for the embedding into anisotropic Besov spaces $\dot{H}^s \cap \dot{H}^{1-\tau} \subset B^{s_1,s_2}$ in Lemma 3.2. This is a key point: in fact, although the core of our analysis will be developed in anisotropic Besov spaces B^{s_1,s_2} (introduced in Section 3.1), however the final result is stated in Sobolev regularity precisely thanks to the embedding $\dot{H}^s \cap \dot{H}^{1-\tau} \subset B^{s_1,s_2}$.

Remark 2.2. (Our setting and comparison with the result of Wan [35]) The global well-posedness of the 2D Boussinesq system with damping (4) in $\mathbb{R}^2$ was first established by Wan in [35]. Besides providing a more systematic and shorter proof (of Theorem 2.1) that exploits the anisotropic nature of the system, the present work improves several points.

- The regularity assumptions are lowered: [35] requires $b \in \dot{H}^{-1} \cap \dot{H}^{s_0}$, $\omega \in \dot{H}^{-2} \cap \dot{H}^{s_0-1}$ with $s_0 \geqq 6$, while here we only need $b \in \dot{H}^{1-\tau} \cap \dot{H}^s$, $\omega \in \dot{H}^{-\tau} \cap \dot{H}^{s-1}$ for any $0 < \tau < 1$ and $s \geqq 3 + \tau$.
- While [35] only bounds $\|u_2\|_{L^{\frac{4}{3}}_T L^\infty}$ relying on spectral analysis and sophisticated estimates, here we provide a control of $\|u_2\|_{L^1_T L^\infty}$ thanks to the anisotropic Littlewood-Paley approach; notice that the L^1_T control of $\|u_2\|_{L^\infty}$ with $u_2 = \mathcal{R}_1\Omega$ is natural as $\|\mathcal{R}_1\Omega\|_{L^\infty}$ is expected to decay at integrable rate like ∇u_2 (see Proposition 3.4 in [16], [3], and Remark 4.2 in [35]).

- In stark contrast with [35] where $b \in \dot{H}^{-1}$ (see Remark 4.1 in [35]), we stress that here we do not need any unnatural assumption on b, which simply belongs to $\dot{H}^{1-\tau} \cap \dot{H}^s$, $s \geqq 3+\tau$, $0 < \tau < 1$.
- Finally, it is interesting to point out that in our framework $b \in \dot{H}^{1-\tau} \cap \dot{H}^s$ (homogeneous spaces with positive indexes) is not required to be square integrable, which is natural as both the Boussinesq equations (2D-Bouss) and the incompressible porous media equation (IPM-diss) are invariant by the transformation $b \to b + C$, for any constant $C \in \mathbb{R}$.

Remark 2.3. (On the $\dot{H}^{1-\tau}$ estimate) As just remarked before, we only take $(b_{\text{in}}, \Omega_{\text{in}}) \in \dot{H}^{1-\tau} \cap \dot{H}^s$ without involving any negative Sobolev space (in [35] $b_{\text{in}} \in \dot{H}^{-1} \cap \dot{H}^{s_0}$ and $\omega_{\text{in}} \in \dot{H}^{-2} \cap H^{s_0-1}$ with $s_0 \geqq 6$). However, we point out that there is no need of $\omega \in \dot{H}^{-\tau}$ (i.e. $\Omega \in \dot{H}^{1-\tau}$) in the first part, namely the proof of Proposition 4.1. The assumption $\Omega_{\text{in}} \in \dot{H}^{1-\tau}$ is only used in the anisotropic Besov part (more precisely in the control of the $Y(t)$ functional in (35)) to bound $\|\Omega\|_{L^\infty_T(B^{\frac{1}{2},\frac{1}{2}})}$ by means of the embedding (Lemma 3.2) $\dot{H}^s \cap \dot{H}^{1-\tau} \subset B^{\frac{1}{2},\frac{1}{2}}$, and therefore to control $\mathcal{M}_{1-\tau}$.

The result below concerns the justification of the relaxation limit.

Theorem 2.2. (Relaxation limit) *Let the hypotheses of Theorem 2.1 be fulfilled and let $(\widetilde{b}^\varepsilon, \widetilde{\Omega}^\varepsilon)$ be the unique solution, scaled with (9), associated to the initial data $(b_{\text{in}}, \Omega_{in})$ as in Theorem 2.1.*

Then, for any $0 < s' < s$ and $0 < \tau < \tau' < 1$, one has the limit as $\varepsilon \to 0$,

$$\widetilde{b}^\varepsilon \to \rho \text{ strongly in } C([0,T], \dot{H}^{1-\tau'}_{loc} \cap \dot{H}^{s-s'}_{loc}),$$

where ρ is the unique solution of (IPM-diss) associated to the initial data b_{in}. Moreover, it holds that

$$\|\rho(\cdot,t)\|_{\dot{H}^{1-\tau}\cap\dot{H}^s} \leqq C\|(b_{\text{in}}, \Omega_{\text{in}})\|_{\dot{H}^{1-\tau}\cap\dot{H}^s},$$

where the constant C is independent of ε.
Finally, we recover the Darcy law in the following sense:

$$\|\widetilde{\Omega}^\varepsilon - \mathcal{R}_1\widetilde{b}^\varepsilon\|_{L^1_T(B^{\frac{3}{2},\frac{1}{2}}\cap B^{\frac{1}{2},\frac{1}{2}})} \leqq \varepsilon\mathcal{M}(0). \tag{19}$$

Remark 2.4. (On the Darcy law) Note that applying the operator $\nabla^\perp\cdot$ to the velocity $\mathbf{u}$ in (1) with $\kappa = g = 1$ (and replacing the notation η by ρ) yields

$$\omega = \nabla^\perp \cdot \mathbf{u} = \partial_x \rho,$$

which in terms of the variables (Ω, ρ) reads exactly $\Omega = \mathcal{R}_1\rho$.

Our analysis also provides a new proof of existence of global-in-time smooth solutions to the incompressible porous media equation (IPM-diss) for small data.

Theorem 2.3. *(Existence for (IPM-diss)) For any $0 < \tau < 1$, let $s \geqq 3+\tau$. For any initial datum $\rho_{\text{in}} \in \dot{H}^{1-\tau}(\mathbb{R}^2) \cap \dot{H}^s(\mathbb{R}^2)$, there exists a constant value $0 < \delta_0 \ll 1$ such that, under the assumption*

$$\|\rho_{\text{in}}\|_{\dot{H}^{1-\tau}\cap\dot{H}^s} \leqq \delta_0,$$

there exists a unique global-in-time smooth solution ρ to system (IPM-diss) satisfying the following inequality for all times $t > 0$

$$\begin{aligned} X(t) := \|\rho\|_{L^\infty_T(\dot{H}^{1-\tau}\cap\dot{H}^s)} + \|\mathcal{R}_1\rho\|_{L^2_T(\dot{H}^{1-\tau}\cap\dot{H}^s)} \\ + \|(\nabla\mathcal{R}_1^2\rho, \Lambda\mathcal{R}_1^2\rho)\|_{L^1_T(L^\infty)} \lesssim \|\rho_{\text{in}}\|_{\dot{H}^{1-\tau}\cap\dot{H}^s}. \end{aligned}$$

Remark 2.5. (Comparison with the result of Elgindi [16]) The global well-posedness of (IPM-diss) for small data, namely the asymptotic stability of (1) around the stratified steady state $\overline{\rho}_{\text{eq}}(y) = \rho_0 - y$, was first proved by Elgindi in [16] taking $b \in W^{4,1} \cap H^{s_0}$, $s_0 \geqq 20$.

Our Theorem 2.3 provides a new proof of Elgindi's result, which allows to substantially lower the regularity assumption of [16], taking only $b \in \dot{H}^{1-\tau} \cap \dot{H}^s$ with $0 < \tau < 1$ and $s \geqq 3 + \tau$. Once again, we take advantage of the anisotropic Littlewood-Paley decomposition and anisotropic Besov spaces that capture the time-integrability of the solution without relying on Green function estimates of the linearized problem. We also mention the asymptotic stability result by Castro-Córdoba-Lear [6] of (1) around the stratified steady state $\overline{\rho}_{\text{eq}} = \rho_0 - y$ in the domain $\mathbb{T} \times [-\pi, \pi]$.

Remark 2.6. (Asymptotic stability on the torus $\mathbb{T}^2$ and near more general density profiles) Besides the whole space $\mathbb{R}^2$, Elgindi [16] provides asymptotic stability of (IPM-diss) on the 2D torus $\mathbb{T}^2$ and near more general density profiles. We will discuss the adaptations of our proof to these cases in Section 7.

Remark 2.7. (Instability results from Kiselev-Yao) A consequence of the recent result [26, Theorem 1.5] by Kiselev and Yao is that there exists an initial perturbation $\rho_{\text{in}}(x, y)$ satisfying $\|\rho_{\text{in}}\|_{H^{2-\gamma}(\mathbb{T}\times[-\pi,\pi])} \ll 1$ for any $\gamma > 0$, such that the solution $\rho(t, x, y)$ to (IPM-diss) (provided it remains smooth for all times) displays the time growth

$$\limsup_{t\to\infty} t^{-\frac{s}{2}} \|\rho(t)\|_{\dot{H}^{s+1}(\mathbb{T}\times[-\pi,\pi])} = +\infty$$

for all $s > 0$. In [26, Remark 1.6], the authors ask whether $\rho_{\text{in}}(x, y)$ can be made small in higher Sobolev spaces, while $\rho(t, x, y)$ still displays time growth. Even though in the present work we study the case of the full space $\mathbb{R}^2$ rather than the bounded periodic channel, we underline that our Theorem 2.3 states that all perturbations ρ_{in} that are small in $H^s(\mathbb{R}^2)$ with $s > 3$ generate solutions $\rho(t, x, y)$ that remain small for all times. Thus, if a blow-up in finite time or a time-growth happens for solutions in the whole space $\mathbb{R}^2$, the initial perturbation must have less regularity than $H^s(\mathbb{R}^2)$, $s > 3$.

Remark 2.8. (On the regularity assumption) As mentioned previously, our regularity setting is $b \in \dot{H}^{1-\tau} \cap \dot{H}^{s}$, $\omega \in \dot{H}^{-\tau} \cap \dot{H}^{s-1}$ for any $0 < \tau < 1$ and $s \geqq 3 + \tau$. While our results are significantly sharper than the previous ones, an interesting problem that remains open is the proof of asymptotic stability of stratified solutions to the IPM equation for $s > 2$ (i.e. $H^2(\mathbb{R}^2)$ is a critical space for the IPM equation). The primary obstacle preventing us from reaching critical regularity lies in controlling the nonlinearity within anisotropic Besov spaces. For instance, the first term of the right-hand side of the estimate (51) requires a control of $\|\partial_y b\|_{B^{\frac{3}{2},\frac{1}{2}}} \lesssim \|\partial_y b\|_{\dot{H}^{2-\tau}\cap\dot{H}^{2+\tau}} \lesssim \|b\|_{\dot{H}^{3-\tau}\cap\dot{H}^{3+\tau}}$. To overcome this difficulty, the use of product rules in the proof of Proposition 4.2 should be replaced by suitable commutator estimates. However, as far as our knowledge extends, commutator estimates within anisotropic Besov spaces have yet to be presented in existing literature. Providing such estimates would entail a considerable amount of technical effort and would stand as an independent achievement.

3. Anisotropic Besov Spaces

3.1. Anisotropic Littlewood–Paley Decomposition

We introduce the following anisotropic Littlewood-Paley decompositions: for $j, q, k \in \mathbb{Z}$, we denote that

- $\dot{\Delta}_j$ are the blocks associated to the Littlewood-Paley decomposition in $|\xi|$;
- $\dot{\Delta}_q^h$ are the blocks associated to the Littlewood-Paley decomposition in the direction ξ_1,
- $\dot{\Delta}_k^v$ are the blocks associated to the Littlewood-Paley decomposition in the direction ξ_2,

such that, denoting by $\mathcal{F}$ the Fourier transform,

$$\dot{\Delta}_j u = \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{u})\dot{\Delta}_q^h u = \mathcal{F}^{-1}(\varphi(2^{-q}\xi_1)\widehat{u}) \quad \text{and} \quad \dot{\Delta}_k^v u = \mathcal{F}^{-1}(\varphi(2^{-k}\xi_2)\widehat{u}),$$

where $\varphi(\xi) = \phi(\xi/2) - \phi(\xi)$ and $\phi \in C_c^\infty$ is such that $\phi = 1$ for $|\xi| \leqq 1/2$ and $\phi(\xi) = 0$ for $|\xi| \geqq 1$. We define the following *homogeneous* and *anisotropic* Besov semi-norms:

$$\|f\|_{\dot{B}^s_{p,r}} \triangleq \left\|2^{js}\|\dot{\Delta}_j f\|_{L^p(\mathbb{R}^d)}\right\|_{\ell^r(j\in\mathbb{Z})},$$
$$\|f\|_{\dot{B}^{s_1,s_2}_{p,r}} \triangleq \left\|2^{js_1}2^{qs_2}\|\dot{\Delta}_j\dot{\Delta}_q^h f\|_{L^p(\mathbb{R}^d)}\right\|_{\ell^r(j\in\mathbb{Z},k\in\mathbb{Z})}.$$

Hereafter we will omit the *dot* (standing for *homogeneous* spaces) and the second and third Besov indexes will be dropped as well for lightening the notation

$$\|f\|_{B^{s_1,s_2}} \triangleq \|f\|_{\dot{B}^{s_1,s_2}_{2,1}}.$$

3.2. Technical Results in Anisotropic Besov Spaces

We now state an anisotropic version of Bernstein's lemma, the proof of which can be found in [30,33].

Lemma 3.1. (Bernstein-type inequalities) *For $x = (x_1, x_2) \in \mathbb{R}^2$, let B_1 be a ball of $\mathbb{R}_{x_1}$, B_2 be a ball of $\mathbb{R}_{x_2}$, C_1 be an annulus of $\mathbb{R}_{x_1}$ and C_2 an annulus of $\mathbb{R}_{x_2}$. Let $1 \leqq p_1 \leqq p_2 \leqq \infty$ and $1 \leqq q_1 \leqq q_2 \leqq \infty$. Then, we have that*

- *If the support of $\widehat{a}$ is included in $2^q B_1$, then*

$$\|\partial_{x_1}^s a\|_{L_{x_1}^{p_2}(L_{x_2}^{q_1})} \lesssim 2^{q(|s|+(\frac{1}{p_1}-\frac{1}{p_2}))}\|a\|_{L_{x_1}^{p_1}(L_{x_2}^{q_1})}.$$

- *If the support of $\widehat{a}$ is included in $2^k B_2$, then*

$$\|\partial_{x_2}^s a\|_{L_{x_1}^{p_1}(L_{x_2}^{q_2})} \lesssim 2^{k(|s|+(\frac{1}{q_1}-\frac{1}{q_2}))}\|a\|_{L_{x_1}^{p_1}(L_{x_2}^{q_1})}.$$

- *If the support of $\widehat{a}$ is included in $2^q C_1$, then*

$$\|a\|_{L_{x_1}^{p_1}(L_{x_2}^{q_1})} \lesssim 2^{-q|s|}\|\partial_{x_1}^s a\|_{L_{x_1}^{p_1}(L_{x_2}^{q_1})}.$$

- *If the support of $\widehat{a}$ is included in $2^k C_2$, then*

$$\|a\|_{L_{x_1}^{p_1}(L_{x_2}^{q_1})} \lesssim 2^{-k|s|}\|\partial_{x_2}^s a\|_{L_{x_1}^{p_1}(L_{x_2}^{q_1})}.$$

Embeddings of Sobolev spaces into anisotropic Besov spaces are provided by the result below.

Lemma 3.2. (Embedding in Sobolev space, [29, Lemma 3.2]) *Let $s_1, s_2, \tau_1, \tau_2 \in \mathbb{R}$ such that $\tau_1 < s_1 + s_2 < \tau_2$ and $s_2 > 0$. If $a \in \dot{H}^{\tau_1}(\mathbb{R}^2) \cap \dot{H}^{\tau_2}(\mathbb{R}^2)$ and $a \in B^{s_1,s_2}$, then*

$$\|a\|_{B^{s_1,s_2}} \lesssim \|a\|_{B^{s_1+s_2}} \lesssim \|a\|_{\dot{H}^{\tau_1}} + \|a\|_{\dot{H}^{\tau_2}}.$$

We will rely on the embeddings $B^{\frac{3}{2},\frac{1}{2}} \hookrightarrow \mathrm{Lip}$, $B^{\frac{1}{2},\frac{3}{2}} \hookrightarrow \mathrm{Lip}(\mathcal{R}_1\cdot)$ and $B^{-\frac{1}{2},\frac{5}{2}} \hookrightarrow \mathrm{Lip}(\mathcal{R}_1^2\cdot)$, where for $n = 1, 2$, the notation $\mathrm{Lip}(\mathcal{R}_1^n\cdot)$ denotes the space of functions whose Riesz transform of order n is Lipschitz, cf. the left-hand sides of (20) for the associated norms.

Lemma 3.3. (Embeddings in *Lip*) *Let $a \in B^{\frac{3}{2},\frac{1}{2}} \cap B^{\frac{1}{2},\frac{1}{2}} \cap B^{\frac{1}{2},\frac{3}{2}} \cap B^{-\frac{1}{2},\frac{5}{2}}$. The following inequalities hold:*

$$\|a\|_{L^\infty} \lesssim \|a\|_{B^{\frac{1}{2},\frac{1}{2}}}, \quad \|\nabla a\|_{L^\infty} \lesssim \|a\|_{B^{\frac{3}{2},\frac{1}{2}}} \quad \text{and} \quad \|\Lambda a\|_{L^\infty} \lesssim \|a\|_{B^{\frac{3}{2},\frac{1}{2}}}.$$

When the Riesz operator in the direction x is involved, one has

$$\|\nabla\mathcal{R}_1 a\|_{L^\infty} \lesssim \|a\|_{B^{\frac{1}{2},\frac{3}{2}}}, \quad \|\Lambda\mathcal{R}_1 a\|_{L^\infty} \lesssim \|a\|_{B^{\frac{1}{2},\frac{3}{2}}} \quad \text{and} \quad \|\nabla\mathcal{R}_1^2 a\|_{L^\infty} \lesssim \|a\|_{B^{-\frac{1}{2},\frac{5}{2}}}. \tag{20}$$

Proof. Using the anisotropic Bernstein Lemma 3.1, one obtains

$$\begin{aligned}\|\nabla a\|_{L^\infty} &\lesssim \sum_{j,q\in\mathbb{Z}^2} 2^j \|\dot\Delta_j \dot\Delta_q^h a\|_{L^\infty} \lesssim \sum_{j,q\in\mathbb{Z}^2, j\geqq k} 2^j \|\dot\Delta_j \dot\Delta_q^h \dot\Delta_k^v a\|_{L^\infty}\\ &\lesssim \sum_{j,q,k\in\mathbb{Z}^3, j\geqq k} 2^j 2^{\frac{q}{2}} 2^{\frac{k}{2}} \|\dot\Delta_j \dot\Delta_q^h \dot\Delta_k^v a\|_{L^2}\\ &\lesssim \sum_{j,q\in\mathbb{Z}^2} 2^{\frac{3j}{2}} 2^{\frac{q}{2}} \|\dot\Delta_j \dot\Delta_q^h a\|_{L^2}\\ &\lesssim \|a\|_{B^{\frac{3}{2},\frac{1}{2}}_{2,1}}.\end{aligned}$$

When replacing the operator ∇ by Λ, the proof follows exactly the same lines. The estimates involving the Riesz operator $\mathcal{R}_1$ can be obtained in a similar manner noticing that, for $s \in \{1, 2\}$

$$\|\nabla \mathcal{R}_1^s a\|_{L^\infty} \lesssim \sum_{j,q\in\mathbb{Z}^2} 2^j 2^{-sj} 2^{sq} \|\dot\Delta_j \dot\Delta_q^h a\|_{L^\infty}.$$

□

4. Proof of Theorem 2.1

The proof of Theorem 2.1 is divided in two main steps. We first provide the estimates in Sobolev spaces, namely we control the functional $\mathcal{M}(t)$ in (18). The outcome of those estimates in Proposition 4.1 involves the quantities $\|\nabla u_2\|_{L^1_T(L^\infty)}$, $\|\Lambda u_2\|_{L^1_T(L^\infty)}$, whose control is not provided by the energy functional $\mathcal{M}(t)$. Thus, in a second step in Section 4.2, we rely on *anisotropic* Besov spaces to estimate the aforementioned quantities and to conclude the proof.

4.1. I. Control of $\mathcal{M}(t)$

The estimates in homogeneous Sobolev spaces are provided by the result below.

Proposition 4.1. *Let (b, Ω) be a smooth solution to system (2D-B). Then the following holds*

$$\mathcal{M}(T) \lesssim X(0) + X^3(T).$$

Proof. The estimate of $\mathcal{M}(t)$ is divided in two parts.

i) Control of $\mathcal{M}_s(t)$, with $s \geqq 3 + \tau$. We consider the equations for the unknowns (b, z) in system (8),

applying the operator $\Lambda^s = (-\Delta)^{\frac{s}{2}}$ to each term with the notation $(\dot b, \dot z, \dot\Omega) = (\Lambda^s b, \Lambda^s z, \Lambda^s \Omega)$.

Noticing that $\Lambda^s\mathcal{R}_1 = \Lambda^{s-1}\partial_x$, the quasi-linearized system reads as

$$\begin{cases} \partial_t\dot{b} - \varepsilon\mathcal{R}_1^2\dot{b} = \mathcal{R}_1\dot{z} + (\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot\nabla\dot{b} + [\Lambda^s, \mathcal{R}_2\Omega]\partial_x b - [\Lambda^s, \mathcal{R}_1\Omega]\partial_y b \\ \qquad := \mathcal{R}_1\dot{z} + \mathcal{I}_1 + \mathcal{C}_{1,1} - \mathcal{C}_{1,2}, \\ \partial_t\dot{z} + \dfrac{\dot{z}}{\varepsilon} = -\varepsilon\mathcal{R}_1^2\dot{\Omega} - \varepsilon((\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot\nabla\mathcal{R}_1\dot{b}) + (\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot\nabla\dot{\Omega} \\ \qquad + [\Lambda^{s-1}, \mathcal{R}_2\Omega]\partial_x\Lambda\Omega - [\Lambda^{s-1}, \mathcal{R}_1\Omega]\partial_y\Lambda\Omega \\ \qquad - \varepsilon[\Lambda^{s-1}\partial_x, \mathcal{R}_2\Omega]\partial_x b + \varepsilon[\Lambda^{s-1}\partial_x, \mathcal{R}_1\Omega]\partial_y b \\ \qquad =: -\varepsilon\mathcal{R}_1^2\dot{\Omega} - \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{C}_{2,1} - \mathcal{C}_{2,2} - \mathcal{C}_{3,1} + \mathcal{C}_{3,2}. \end{cases} \tag{21}$$

Similarly, the quasi-linearized equation for Ω (from (2D-B)) reads as

$$\begin{aligned} \partial_t\dot{\Omega} + \frac{\dot{\Omega}}{\varepsilon} &= \mathcal{R}_1\dot{b} + ((\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot(\nabla\dot{\Omega})) \\ &\quad + [\Lambda^{s-1}, \mathcal{R}_2\Omega]\partial_x\Lambda\Omega - [\Lambda^{s-1}, \mathcal{R}_1\Omega]\partial_y\Lambda\Omega. \end{aligned} \tag{22}$$

Let us provide the desired estimate.

The linear evolution

We first look at the linear terms, neglecting the nonlinearity. Using the skew-symmetry of the Riesz transform $(\mathcal{R}_1\dot{z}, \dot{b})_{L^2} = -(\dot{z}, \mathcal{R}_1\dot{b})_{L^2}$ and $(\mathcal{R}_1^2\dot{\Omega}, \dot{z})_{L^2} = -(\mathcal{R}_1\dot{\Omega}, \mathcal{R}_1\dot{z})_{L^2}$,

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|b\|_{\dot{H}^s}^2 + \|\Omega\|_{\dot{H}^s}^2 + \|z\|_{\dot{H}^s}^2) + \varepsilon\|\mathcal{R}_1 b\|_{\dot{H}^s}^2 \qquad + \frac{1}{\varepsilon}\|z\|_{\dot{H}^s}^2 + \frac{1}{\varepsilon}\|\Omega\|_{\dot{H}^s}^2 \\ &\quad = -(\mathcal{R}_1\dot{b}, \dot{z})_{L^2} + \varepsilon(\mathcal{R}_1\dot{\Omega}, \mathcal{R}_1\dot{z})_{L^2}. \end{aligned}$$

Using the Cauchy-Schwarz inequality

$$\begin{aligned} |(\mathcal{R}_1\dot{b}, \dot{z})_{L^2}| &\leqq \frac{\varepsilon}{2}\|\mathcal{R}_1 b\|_{\dot{H}^s}^2 + \frac{1}{2\varepsilon}\|z\|_{\dot{H}^s}^2, \\ \varepsilon|(\mathcal{R}_1\dot{\Omega}, \mathcal{R}_1\dot{z})_{L^2}| &\leqq \frac{\varepsilon^2}{2}\|\mathcal{R}_1 b\|_{\dot{H}^s}^2 + \frac{1}{2}\|\mathcal{R}_1 z\|_{\dot{H}^s}^2 \end{aligned}$$

and the continuity of the Riesz transform $\|\mathcal{R}_j a\|_{L^2} \leqq \|a\|_{L^2}$ for any $a \in L^2$, the last two terms are absorbed by the linear dissipation, yielding the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|b\|_{\dot{H}^s}^2 + \|\Omega\|_{\dot{H}^s}^2 + \|z\|_{\dot{H}^s}^2) + \frac{\varepsilon}{4}\|\mathcal{R}_1 b\|_{\dot{H}^s}^2 + \frac{1}{4\varepsilon}\|z\|_{\dot{H}^s}^2 + \frac{1}{\varepsilon}\|\Omega\|_{\dot{H}^s}^2 \leqq 0.$$

Now we deal with the nonlinearity of system (21).

Estimates of the nonlinear term Consider the quasi-linearized terms $\mathcal{I}_j$ for $j \in \{1, 2, 3\}$. The divergence-free condition and integration by parts yield

$$(\mathcal{I}_1, \dot{b})_{L^2} = ((\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot\nabla\dot{b}, \dot{b})_{L^2} = 0.$$

Furthermore, it is easy to see that

$$-\mathcal{I}_2 + \mathcal{I}_3 = (\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot\nabla\dot{z},$$

readily implying, using again the divergence free condition, that $(-\mathcal{I}_2+\mathcal{I}_3, \dot{z}) = 0$. Next, let us focus on the commutators.

Commutator estimates We begin with the terms of type $(\mathcal{C}_{2,j}, \dot{z})_{L^2}$, $(\mathcal{C}_{3,j}, \dot{z})_{L^2}$ for $j \in \{1, 2\}$, which are easier being quadratic in (some norm of) the dissipative variable z. Notice indeed that as the products $(\mathcal{C}_{1,j}, \dot{b})_{L^2}$ are quadratic in b, they require a more careful treatment since the L^2_T control of the (spatial norm of the) variable b is not provided by the energy functional (one only controls $\mathcal{R}_1 b$, see the definition of $\mathcal{M}_\tau(T)$ in (14)).

Let us first look at $\mathcal{C}_{2,1} = [\Lambda^{s-1}, \mathcal{R}_2\Omega]\partial_x\Lambda\Omega$. By applying the commutator estimates in Lemma B.3 (in the "Appendix") with $s \geqq 3+\tau > \frac{d}{2}+2$, one has

$$\|\mathcal{C}_{2,1}\|_{L^2} \lesssim \|\nabla\mathcal{R}_2\Omega\|_{L^\infty}\|\mathcal{R}_1\Omega\|_{\dot{H}^s} + \|\partial_x\Lambda\Omega\|_{L^\infty}\|\mathcal{R}_2\Omega\|_{\dot{H}^{s-1}}.$$

Now, applying first Lemma 3.3 and after Lemma 3.2 with $1-\tau = \tau_1 < 2 < s = \tau_2$, we have

$$\|\nabla\mathcal{R}_2\Omega\|_{L^\infty} \lesssim \|\mathcal{R}_2\Omega\|_{B^{\frac{3}{2},\frac{1}{2}}} \lesssim \|\mathcal{R}_2\Omega\|_{\dot{H}^{1-\tau}\cap\dot{H}^s}.$$

[1]Similarly, using again Lemma 3.3 and Lemma 3.2

$$\begin{aligned}\|\partial_x\Lambda\Omega\|_{L^\infty} &\lesssim \|\Lambda^2\mathcal{R}_1\Omega\|_{L^\infty} \lesssim \|\Lambda\Omega\|_{B^{\frac{1}{2},\frac{3}{2}}} \lesssim \|\Lambda\Omega\|_{\dot{H}^{2-\tau}\cap\dot{H}^{2+\tau}}\\ &\lesssim \|\Omega\|_{\dot{H}^{3-\tau}\cap\dot{H}^s}, \quad (s \geqq 3+\tau),\end{aligned}$$

where now, using the interpolation Lemma B.4

$$\|\Omega\|_{\dot{H}^{3-\tau}} \lesssim \|\Omega\|^{\theta}_{\dot{H}^{1-\tau}}\|\Omega\|^{1-\theta}_{\dot{H}^s}, \quad \text{with} \quad \theta = \frac{s+\tau-3}{s+\tau-1}.$$

Finally, the Young inequality $ab \leqq \frac{a^p}{p} + \frac{b^q}{q}$ with $p = \frac{1}{\theta}$, $q = \frac{1}{1-\theta}$ gives

$$\|\Omega\|_{\dot{H}^{3-\tau}} \lesssim \|\Omega\|_{\dot{H}^{1-\tau}} + \|\Omega\|_{\dot{H}^s}.$$

In the same way, using that $\|\mathcal{R}_1 a\|_{\dot{H}^s} \leqq \|a\|_{\dot{H}^s}$ for any $s \geqq 0$ and $a \in \dot{H}^s$, by interpolation

$$\begin{aligned}\|\mathcal{R}_2\Omega\|_{\dot{H}^{s-1}} &\lesssim \|\Omega\|_{\dot{H}^{s-1}} \lesssim \|\Omega\|^{\tilde{\theta}}_{\dot{H}^{1-\tau}}\|\Omega\|^{\tilde{\theta}}_{\dot{H}^s}\\ &\lesssim \|\Omega\|_{\dot{H}^{1-\tau}} + \|\Omega\|_{\dot{H}^s}, \quad \text{with} \quad \tilde{\theta} = \frac{1}{s+\tau-1}.\end{aligned}$$

[1] Note that the controls $\|\Omega\|_{\dot{H}^{2-\tau}}$, $\|b\|_{\dot{H}^{2-\tau}}$ would be enough and there is no need of $\|\Omega\|_{\dot{H}^{1-\tau}}$, $\|b\|_{\dot{H}^{1-\tau}}$ at this stage.

Altogether, this it yields that

$$\begin{aligned}\int_0^T |(\mathcal{C}_{2,1}, \dot{z})_{L^2}|\,\mathrm{d}t &\leqq \int_0^T \|\mathcal{C}_{2,1}\|_{L^2}\|z\|_{\dot{H}^s}\,\mathrm{d}t\\ &\lesssim \int_0^T \|\Omega\|_{\dot{H}^{1-\tau}\cap\dot{H}^s}\|\Omega\|_{\dot{H}^{1-\tau}\cap\dot{H}^s}\|z\|_{\dot{H}^s}\,\mathrm{d}t\\ &\lesssim (\|\Omega\|_{L^\infty_T(\dot{H}^{1-\tau})} + \|\Omega\|_{L^\infty_T(\dot{H}^s)})(\|\Omega\|_{L^2_T(\dot{H}^{1-\tau})}\\ &\quad + \|\Omega\|_{L^2_T(\dot{H}^s)})\|z\|_{L^2_T(\dot{H}^s)}\\ &\lesssim \mathcal{M}^3(T).\end{aligned}$$

The commutator $\mathcal{C}_{2,2}$ is completely analogous, we omit it. Let us consider $\mathcal{C}_{3,1} = \varepsilon[\Lambda^{s-1}\partial_x, \mathcal{R}_2\Omega]\partial_x b$, which gives

$$\|\mathcal{C}_{3,1}\|_{L^2} \lesssim \varepsilon(\|\nabla\mathcal{R}_2\Omega\|_{L^\infty}\|\mathcal{R}_1 b\|_{\dot{H}^s} + \|\partial_x b\|_{L^\infty}\|\mathcal{R}_1\mathcal{R}_2\Omega\|_{\dot{H}^s}).$$

Now note that, applying the same reasoning as before (Lemmas 3.3, 3.2),

$$\|\partial_x b\|_{L^\infty} \lesssim \|\Lambda\mathcal{R}_1 b\|_{B^{\frac{1}{2},\frac{1}{2}}} \lesssim \|\Lambda\mathcal{R}_1 b\|_{\dot{H}^{1-\tau}\cap\dot{H}^{s-1}} \lesssim \|\mathcal{R}_1 b\|_{\dot{H}^{2-\tau}\cap\dot{H}^s},$$

where, again,

$$\|\mathcal{R}_1 b\|_{\dot{H}^{2-\tau}} \lesssim \|\mathcal{R}_1 b\|^{\theta}_{\dot{H}^{1-\tau}}\|\mathcal{R}_1 b\|^{1-\theta}_{\dot{H}^s} \lesssim \|\mathcal{R}_1 b\|_{\dot{H}^{1-\tau}} + \|\mathcal{R}_1 b\|_{\dot{H}^s}, \quad \theta = \frac{s+\tau-2}{s+\tau-1}.$$

This way,

$$\begin{aligned}\int_0^T |(\mathcal{C}_{3,1}, \dot{\Omega})_{L^2}|\,\mathrm{d}t &\lesssim \varepsilon\int_0^T (\|\nabla\mathcal{R}_2\Omega\|_{L^\infty}\|\mathcal{R}_1 b\|_{\dot{H}^s} + \|\partial_x b\|_{L^\infty}\|\mathcal{R}_2\Omega\|_{\dot{H}^s})\|\Omega\|_{\dot{H}^s}\,\mathrm{d}t\\ &\lesssim \varepsilon\int_0^T \|\Omega\|^2_{\dot{H}^{1-\tau}\cap\dot{H}^s}\|b\|_{\dot{H}^s} + \|\mathcal{R}_1 b\|_{\dot{H}^{1-\tau}\cap\dot{H}^s}\|\Omega\|^2_{\dot{H}^s}\,\mathrm{d}t\\ &\lesssim \varepsilon^2\|b\|_{L^\infty_T(\dot{H}^{1-\tau}\cap\dot{H}^s)} \times \frac{\|\Omega\|^2_{L^2_T(\dot{H}^{1-\tau}\cap\dot{H}^s)}}{\varepsilon}\\ &\lesssim \mathcal{M}^3(T).\end{aligned}$$

The commutator $\mathcal{C}_{3,2}$ is bounded similarly.

Now we deal with the more delicate estimates involving the commutators $\mathcal{C}_{1,j}$ for $j \in \{1,2\}$ in the equation of $\dot{b}$. Let us start with the term with $\mathcal{C}_{1,1} = [\Lambda^s, \mathcal{R}_2\Omega]\partial_x b$, where it is easy to reconstruct the term $\mathcal{R}_1\dot{b}$, which is controlled in L^2_T by the energy functional $\mathcal{M}(T)$. We have, applying the above reasoning, that

$$\begin{aligned}\|\mathcal{C}_{1,1}\|_{L^2} &\lesssim \|\nabla\mathcal{R}_2\Omega\|_{L^\infty}\|\partial_x b\|_{\dot{H}^{s-1}} + \|\mathcal{R}_2\Omega\|_{\dot{H}^s}\|\partial_x b\|_{L^\infty}\\ &\lesssim \|\Omega\|_{\dot{H}^{1-\tau}\cap\dot{H}^s}\|\mathcal{R}_1 b\|_{\dot{H}^s} + \|\mathcal{R}_1 b\|_{\dot{H}^{1-\tau}\cap\dot{H}^s}\|\Omega\|_{\dot{H}^s},\end{aligned}$$

so that

$$\begin{aligned}\int_0^T |(\mathcal{C}_{1,1}, \dot{b})_{L^2}|\,\mathrm{d}t &\leqq \int_0^T \|\mathcal{C}_{1,1}\|_{L^2}\|b\|_{\dot{H}^s}\,dt\\ &\lesssim \int_0^T \frac{1}{\sqrt{\varepsilon}}\|\Omega\|_{\dot{H}^{1-\tau}\cap\dot{H}^s}\|\mathcal{R}_1 b\|_{\dot{H}^{1-\tau}\cap\dot{H}^s}\sqrt{\varepsilon}\|b\|_{\dot{H}^s}\,\mathrm{d}t\\ &\lesssim \frac{1}{\sqrt{\varepsilon}}\|\Omega\|_{L^2_T(\dot{H}^{1-\tau}\cap\dot{H}^s)}\|b\|_{L^\infty_T(\dot{H}^s)}\sqrt{\varepsilon}\|\mathcal{R}_1 b\|_{L^2_T(\dot{H}^{1-\tau}\cap\dot{H}^s)}\\ &\lesssim \mathcal{M}^3(T).\end{aligned}$$

The next commutator $\mathcal{C}_{1,2} = [\Lambda^s, \mathcal{R}_1\Omega]\partial_y b$ requires a more careful treatment. We rely on the fractional Leibniz rule, which is an extension, due to Li [27] (see also D'Ancona [14]) of the Kenig–Ponce–Vega inequality to the case $s \geqq 1$.

We introduce the notation $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ ($\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$) and $\nabla^\alpha = (\partial_x^{\alpha_1}, \partial_y^{\alpha_2})$, while the operator $\Lambda^{s,\alpha}$ is defined via Fourier transform as

$$\widehat{\Lambda^{s,\alpha} f}(\xi) = \widehat{\Lambda^{s,\alpha}}(\xi)\widehat{f}(\xi), \qquad \widehat{\Lambda^{s,\alpha}}(\xi) = i^{-|\alpha|}\partial_\xi^\alpha(|\xi|^s), \tag{23}$$

where $|\alpha| = \alpha_1 + \alpha_2$ (resp. $|\beta| = \beta_1 + \beta_2$). Notice that $\Lambda^{s,\alpha}$ is a pseudo-differential operator of order $s - |\alpha|$. Now, we apply Lemma B.6, with $s_1 = 1, s_2 = s - 1$, which gives

$$\begin{aligned}&\left\|\mathcal{C}_{1,2} - \sum_{|\alpha|=1}\frac{1}{\alpha!}\nabla^\alpha\mathcal{R}_1\Omega\Lambda^{s,\alpha}\partial_y b - \sum_{|\beta|\leqq s-2}\frac{1}{\beta!}\nabla^\beta\partial_y b\Lambda^{s,\beta}\mathcal{R}_1\Omega\right\|_{L^2}\\ &\lesssim \|\Lambda\mathcal{R}_1\Omega\|_{\mathrm{BMO}}\|\partial_y b\|_{\dot{H}^{s-1}}.\end{aligned} \tag{24}$$

Then we write the scalar product adding and subtracting the above right-hand side as

$$\begin{aligned}(\mathcal{C}_{1,2}, \dot{b})_{L^2} &= \left(\mathcal{C}_{1,2} - \sum_{|\alpha|=1}\frac{1}{\alpha!}\nabla^\alpha\mathcal{R}_1\Omega\Lambda^{s,\alpha}\partial_y b - \sum_{|\beta|\leqq s-2}\frac{1}{\beta!}\nabla^\beta\partial_y b\Lambda^{s,\beta}\mathcal{R}_1\Omega, \dot{b}\right)_{L^2}\\ &\quad + \sum_{|\alpha|=1}\frac{1}{\alpha!}(\nabla^\alpha\mathcal{R}_1\Omega\Lambda^{s,\alpha}\partial_y b, \dot{b})_{L^2} + \sum_{|\beta|\leqq s-2}\frac{1}{\beta!}(\nabla^\beta\partial_y b\Lambda^{s,\beta}\mathcal{R}_1\Omega, \dot{b})_{L^2},\end{aligned}$$

yielding, thanks to (24),

$$|(\mathcal{C}_{1,2}, \dot{b})_{L^2}| \lesssim \|\Lambda\mathcal{R}_1\Omega\|_{\mathrm{BMO}}\|\partial_y b\|_{\dot{H}^{s-1}}\|b\|_{\dot{H}^s} + |\mathcal{J}_1| + |\mathcal{J}_2|, \tag{25}$$

where we denote that

$$\begin{aligned}\mathcal{J}_1 &:= \sum_{|\alpha|=1}\frac{1}{\alpha!}(\nabla^\alpha\mathcal{R}_1\Omega\Lambda^{s,\alpha}\partial_y b, \dot{b})_{L^2} \quad\text{and}\quad \mathcal{J}_2\\ &:= \sum_{|\beta|\leqq s-2}\frac{1}{\beta!}(\nabla^\beta\partial_y b\Lambda^{s,\beta}\mathcal{R}_1\Omega, \dot{b})_{L^2}.\end{aligned}$$

Now note that $\mathcal{J}_1$ can be rewritten as

$$\sum_{|\alpha|=1} \nabla^\alpha \mathcal{R}_1 \Omega \Lambda^{s,\alpha} \partial_y b = -s(\nabla \mathcal{R}_1 \Omega)(\Lambda^{s-2} \nabla \partial_y b) = -s(\nabla \mathcal{R}_1 \Omega)(\Lambda^{s-1} \nabla \mathcal{R}_2 b),$$

so that using the continuity of $\mathcal{R}_2$ in L^2,

$$|\mathcal{J}_1| \lesssim \|\nabla \mathcal{R}_1 \Omega\|_{L^\infty} \|\Lambda^s \mathcal{R}_2 b\|_{L^2} \|b\|_{\dot{H}^s} \lesssim \|\nabla \mathcal{R}_1 \Omega\|_{L^\infty} \|b\|_{\dot{H}^s}^2, \tag{26}$$

and integrating in time, recalling that $u_2 = -\mathcal{R}_1 \Omega$,

$$\int_0^T |\mathcal{J}_1| \, \mathrm{d}t \lesssim \|\nabla \mathcal{R}_1 \Omega\|_{L^1_T(L^\infty)} \|b\|^2_{L^\infty_T(\dot{H}^s)} \lesssim \|\nabla u_2\|_{L^1_T(L^\infty)} \mathcal{M}_s(T) \lesssim X(T)^3.$$

Now let us deal with $\mathcal{J}_2$. As $\mathcal{R}_1 = \Lambda^{-1} \partial_x$, we integrate by parts in the horizontal direction

$$\begin{aligned} \mathcal{J}_2 = \sum_{|\beta| \leqq s-2} ((\nabla^\beta \partial_y b) \Lambda^{s-1,\beta} \partial_x \Omega, \dot{b})_{L^2} &= - \sum_{|\beta| \leqq s-2} ((\nabla^\beta \partial^2_{xy} b) \Lambda^{s-1,\beta} \Omega, \dot{b})_{L^2} \\ &\quad - \sum_{|\beta| \leqq s-2} ((\nabla^\beta \partial_y b) \Lambda^{s-1,\beta} \Omega, \partial_x \dot{b})_{L^2}. \end{aligned} \tag{27}$$

Now, we deal with the last term in (27). Using that $\partial_x \dot{b} = \Lambda(\partial_x \Lambda^{s-1} b)$ and the symmetry of Λ, it can be written as

$$\begin{aligned} \sum_{|\beta| \leqq s-2} ((\nabla^\beta \partial_y b) \Lambda^{s-1,\beta} \Omega, \partial_x \dot{b})_{L^2} &= \sum_{|\beta| \leqq s-2} (\Lambda(\nabla^\beta \partial_y b \Lambda^{s-1,\beta} \Omega), \partial_x \Lambda^{s-1} b)_{L^2} \\ &= \sum_{|\beta| \leqq s-2} (\Lambda(\nabla^\beta \partial_y b \Lambda^{s-1,\beta} \Omega), \mathcal{R}_1 \dot{b})_{L^2}. \end{aligned}$$

Now, we deal with the last term in (27). Using that $\partial_x \dot{b} = \Lambda(\partial_x \Lambda^{s-1} b)$ and the symmetry of Λ, it can be written as

$$\begin{aligned} \sum_{|\beta| \leqq s-2} ((\nabla^\beta \partial_y b) \Lambda^{s-1,\beta} \Omega, \partial_x \dot{b})_{L^2} &= \sum_{|\beta| \leqq s-2} (\Lambda(\nabla^\beta \partial_y b \Lambda^{s-1,\beta} \Omega), \partial_x \Lambda^{s-1} b)_{L^2} \\ &= \sum_{|\beta| \leqq s-2} (\Lambda(\nabla^\beta \partial_y b \Lambda^{s-1,\beta} \Omega), \mathcal{R}_1 \dot{b})_{L^2}. \end{aligned}$$

We use the following decomposition:

$$\begin{aligned} &\sum_{|\beta| \leqq s-2} (\Lambda(\nabla^\beta \partial_y b \Lambda^{s-1,\beta} \Omega), \mathcal{R}_1 \dot{b})_{L^2} \\ &= (\Lambda(\partial_y b \Lambda^{s-1} \Omega), \mathcal{R}_1 \dot{b})_{L^2} + \sum_{|\beta|=1 \text{ and } \beta=(1,1)} (\Lambda(\nabla^\beta \partial_y b \Lambda^{s-1,\beta} \Omega), \mathcal{R}_1 \dot{b})_{L^2} \\ &\quad + \sum_{2 \leqq |\beta| \leqq s-2 \text{ and } \beta \neq (1,1)} (\Lambda(\nabla^\beta \partial_y b \Lambda^{s-1,\beta} \Omega), \mathcal{R}_1 \dot{b})_{L^2} \\ &= \mathcal{J}_2^a + \mathcal{J}_2^b + \mathcal{J}_2^c. \end{aligned}$$

By virtue of the product Lemma B.2, one obtains

$$\begin{aligned}
\mathcal{J}_2^a &\lesssim \|\mathcal{R}_1 b\|_{\dot{H}^s}(\|\Lambda\partial_y b\|_{L^\infty}\|\Lambda^{s-1}\Omega\|_{L^2} + \|\partial_y b\|_{L^\infty}\|\Lambda^s\Omega\|_{L^2}) \qquad (\beta = (0,0))\\
\mathcal{J}_2^b &\lesssim \|\mathcal{R}_1 b\|_{\dot{H}^s}(\|\Lambda\nabla\partial_y b\|_{L^2}\|\Lambda^{s-3}\nabla\Omega\|_{L^\infty}\\
&\quad + \|\nabla\partial_y b\|_{L^\infty}\|\Lambda^{s-2}\nabla\Omega\|_{L^2}) \qquad (|\beta| = 1 \text{and} \beta = (1,1))\\
\mathcal{J}_2^c &\lesssim \|\mathcal{R}_1 b\|_{\dot{H}^s} \sum_{2\leqq|\beta|\leqq s-2} (\|\Lambda\nabla^\beta\partial_y b\|_{L^2}\|\Lambda^{s-1,\beta}\Omega\|_{L^\infty}\\
&\quad + \|\nabla^\beta\partial_y b\|_{L^2}\|\Lambda^{s,\beta}\Omega\|_{L^\infty}) \qquad (|\beta| \geqq 2).
\end{aligned}$$

Concerning the first term of $\mathcal{J}_2$ in (27), one has

$$\sum_{|\beta|\leqq s-2} |((\nabla^\beta\partial_{xy}^2 b)\Lambda^{s-1,\beta}\Omega, \dot{b})_{L^2}| \lesssim \sum_{|\beta|\leqq s-2} \|(\nabla^\beta\partial_{xy}^2 b)\Lambda^{s-1,\beta}\Omega\|_{L^2}\|b\|_{\dot{H}^s}.$$

For $\beta = (0,0)$, we have

$$\|(\partial_{xy}^2 b)\Lambda^{s-1}\Omega\|_{L^2} \lesssim \|\partial_{xy} b\|_{L^\infty}\|\Omega\|_{\dot{H}^{s-1}} \lesssim \|\mathcal{R}_1 b\|_{\dot{H}^{3-\tau}\cap\dot{H}^{3+\tau}}\|\Omega\|_{\dot{H}^{s-1}}.$$

The remaining terms yield

$$\begin{aligned}
&\sum_{|\beta|=1 \text{ and } \beta=(1,1)} |((\nabla^\beta\partial_{xy}^2 b)\Lambda^{s-1,\beta}\Omega, \dot{b})_{L^2}|\\
&\lesssim \sum_{|\beta|=1 \text{ and } \beta=(1,1)} \|\nabla^\beta\partial_{xy}^2 b\|_{L^2}\|\Lambda^{s-1,\beta}\Omega\|_{L^\infty}\|b\|_{\dot{H}^s}\\
&\lesssim \|\Omega\|_{\dot{H}^{3-\tau}\cap\dot{H}^{3+\tau}}\|b\|_{\dot{H}^s}^2
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{2\leqq|\beta|\leqq s-2} |((\nabla^\beta\partial_{xy}^2 b)\Lambda^{s-1,\beta}\Omega, \dot{b})_{L^2}|\\
&\lesssim \sum_{2\leqq|\beta|\leqq s-2} \|\nabla^\beta\partial_{xy}^2 b\|_{L^2}\|\Lambda^{s-1,\beta}\Omega\|_{L^\infty}\|b\|_{\dot{H}^s}.
\end{aligned}$$

Altogether, appealing to the embeddings of Lemma 3.3 and Lemma 3.2 ($s \geqq 3+\tau$), one obtains

$$\mathcal{J}_2 \lesssim (\|\mathcal{R}_1 b\|_{\dot{H}^{1-\tau}} + \|\mathcal{R}_1 b\|_{\dot{H}^s})(\|b\|_{\dot{H}^{1-\tau}} + \|b\|_{\dot{H}^s})(\|\Omega\|_{\dot{H}^{1-\tau}} + \|\Omega\|_{\dot{H}^s}),$$

so that

$$\begin{aligned}
\int_0^T |\mathcal{J}_2|\,\mathrm{d}t &\lesssim (\|\mathcal{R}_1 b\|_{L_T^2(\dot{H}^{1-\tau})} + \|\mathcal{R}_1 b\|_{L_T^2(\dot{H}^s)})(\|b\|_{L_T^\infty(\dot{H}^{1-\tau})} + \|b\|_{L_T^\infty(\dot{H}^s)})\\
&\quad \times (\|\Omega\|_{L_T^2(\dot{H}^{1-\tau})} + \|\Omega\|_{L_T^2(\dot{H}^s)})\\
&\lesssim \mathcal{M}(T)^3.
\end{aligned}$$

Inserting the latter in (25) together with (26) and using the embedding $L^\infty \hookrightarrow$ BMO yields

$$\int_0^T |(\mathcal{C}_{1,2}, \dot{b})_{L^2}| \, \mathrm{d}t \lesssim (\|\Lambda \mathcal{R}_1 \Omega\|_{L^1_T(L^\infty)} + \|\nabla \mathcal{R}_1 \Omega\|_{L^1_T(L^\infty)}) \mathcal{M}^2(T) + \mathcal{M}^3(T).$$

To control the terms $\|\nabla \mathcal{R}_1 \Omega\|_{L^1_T(L^\infty)}$, $\|\Lambda \mathcal{R}_1 \Omega\|_{L^1_T(L^\infty)}$, we shall rely on anisotropic Besov spaces in Step II (Section 4.2).

ii) Control of $\mathcal{M}_{1-\tau}(t)$, $0 < \tau < 1$. We apply $\Lambda^{1-\tau}$ to system (8), yielding

$$\begin{cases} \partial_t \dot{b} - \varepsilon \mathcal{R}_1^2 \dot{b} = \mathcal{R}_1 \dot{z} + \Lambda^{1-\tau}((\mathcal{R}_2 \Omega, -\mathcal{R}_1 \Omega) \cdot \nabla b), \\ \partial_t \dot{z} + \dfrac{\dot{z}}{\varepsilon} = -\varepsilon \mathcal{R}_1^2 \dot{\Omega} - \varepsilon \Lambda^{1-\tau} \mathcal{R}_1((\mathcal{R}_2 \Omega, -\mathcal{R}_1 \Omega) \cdot \nabla b) \\ \qquad\qquad + \Lambda^{-\tau}((\mathcal{R}_2 \Omega, -\mathcal{R}_1 \Omega) \cdot (\nabla \Lambda \Omega)). \end{cases} \tag{28}$$

Moreover, the equation for $\dot{\Omega}$ reads as

$$\partial_t \dot{\Omega} + \frac{\dot{\Omega}}{\varepsilon} = \Lambda^{-\tau}((\mathcal{R}_2 \Omega, -\mathcal{R}_1 \Omega) \cdot (\nabla \Lambda \Omega)).$$

The linear terms work exactly the same way as before, thus we focus on the non-linearities. We begin with

$$(\Lambda^{1-\tau}((\mathcal{R}_2 \Omega, -\mathcal{R}_1 \Omega) \cdot \nabla b), \dot{b})_{L^2} =: \mathcal{I}_1 + \mathcal{I}_2.$$

Let us start with $\mathcal{I}_1$. Appealing to the product estimate (Lemma B.2) with $p_1 = r_1 = \infty$ yields

$$\begin{aligned} \mathcal{I}_1 &= (\Lambda^{1-\tau}(\mathcal{R}_2 \Omega \partial_x b), \dot{b})_{L^2} \\ &\lesssim (\|\mathcal{R}_2 \Omega\|_{L^\infty} \|\partial_x b\|_{\dot{H}^{1-\tau}} + \|\Lambda^{1-\tau} \mathcal{R}_2 \Omega\|_{L^2} \|\partial_x b\|_{L^\infty}) \|b\|_{\dot{H}^{1-\tau}}, \end{aligned}$$

where interpolation (Lemma B.4) and Young inequality give

$$\|\partial_x b\|_{\dot{H}^{1-\tau}} \leqq \|\mathcal{R}_1 b\|_{\dot{H}^{2-\tau}} \lesssim \|\mathcal{R}_1 b\|_{\dot{H}^{1-\tau}}^{\theta} \|\mathcal{R}_1 b\|_{\dot{H}^s}^{1-\theta}, \quad \theta = \frac{s+\tau-2}{s+\tau-1}.$$

This yields

$$\begin{aligned} \int_0^T \mathcal{I}_1 \, \mathrm{d}t &\lesssim \sqrt{\varepsilon} (\|\mathcal{R}_1 b\|_{L^2_T(\dot{H}^{1-\tau})} + \|\mathcal{R}_1 b\|_{L^2_T(\dot{H}^s)}) \|b\|_{L^\infty_T(\dot{H}^{1-\tau})} \frac{1}{\sqrt{\varepsilon}} (\|\Omega\|_{L^2_T(\dot{H}^{1-\tau})} \\ &\quad + \|\Omega\|_{L^2_T(\dot{H}^s)}) \\ &\lesssim \mathcal{M}^3(T). \end{aligned}$$

We cannot use the same trick for the next term, which gives

$$\begin{aligned} \int_0^T \mathcal{I}_2 \, \mathrm{d}t &\lesssim \|b\|_{L^\infty_T(\dot{H}^{1-\tau})} (\|b\|_{L^\infty_T(\dot{H}^s)} + \|b\|_{L^\infty_T(\dot{H}^{1-\tau})}) \|\mathcal{R}_1 \Omega\|_{L^1_T(W^{1,\infty})} \\ &\lesssim \|u_2\|_{L^1_T(W^{1,\infty})} \mathcal{M}^2(T) \lesssim X^3(T). \end{aligned}$$

Next, it is easy to see that $\varepsilon\mathcal{R}_1(\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot\nabla b$ is very similar to the terms treated before, then it is omitted.

It remains to deal with the last term in the equation of $\dot{z}$ (which, in the energy estimate, is multiplied both by $\dot{z}$ and by $\dot{\Omega}$, but the computations are identical, so that we only detail one case). Using the symmetry of the multiplier $\Lambda^{-\tau}$, one obtains

$$\begin{aligned}\mathcal{I}_3 :&= (\Lambda^{-\tau}((\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot(\nabla\Lambda\Omega)), \dot{z})_{L^2}\\ &= ((\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot(\nabla\Lambda\Omega), \Lambda^{1-2\tau}z)_{L^2}.\end{aligned}$$

Now, we integrate by parts in x the first addend and in y the second one. As the term $\mathcal{R}_2\partial_x\Omega - \mathcal{R}_1\partial_y\Omega = 0$ (by definition of the Riesz Transform), it remains

$$\begin{aligned}\mathcal{I}_3 &= -(\mathcal{R}_2\Omega\,\Lambda\Omega, \Lambda^{1-2\tau}\partial_x z)_{L^2} + (\mathcal{R}_1\Omega\,\Lambda\Omega, \Lambda^{1-2\tau}\partial_y z)_{L^2}\\ &= -(\mathcal{R}_2\Omega\,\Lambda\Omega, \Lambda^{2(1-\tau)}\mathcal{R}_1 z)_{L^2} + (\mathcal{R}_1\Omega\,\Lambda\Omega, \Lambda^{2(1-\tau)}\mathcal{R}_2 z)_{L^2}\\ &= -(\Lambda^{1-\tau}(\mathcal{R}_2\Omega\,\Lambda\Omega), \Lambda^{1-\tau}\mathcal{R}_1 z)_{L^2} + (\Lambda^{1-\tau}(\mathcal{R}_1\Omega\,\Lambda\Omega), \Lambda^{1-\tau}\mathcal{R}_2 z)_{L^2}\\ &=: \mathcal{I}_3^a + \mathcal{I}_3^b.\end{aligned}$$

This way

$$|\mathcal{I}_3^a| \leqq \|\Lambda^{1-\tau}(\mathcal{R}_2\Omega\,\Lambda\Omega)\|_{L^2}\|z\|_{\dot{H}^{1-\tau}},$$

and using the product Lemma B.2, Lemma 3.3 and Lemma 3.2,

$$\begin{aligned}\|\Lambda^{1-\tau}(\mathcal{R}_2\Omega\,\Lambda\Omega)\|_{L^2} &\lesssim \|\mathcal{R}_2\Omega\|_{\dot{H}^{1-\tau}}\|\Lambda\Omega\|_{L^\infty} + \|\mathcal{R}_2\Omega\|_{L^\infty}\|\Lambda\Omega\|_{\dot{H}^{1-\tau}}\\ &\lesssim (\|\Omega\|_{\dot{H}^{1-\tau}} + \|\Omega\|_{\dot{H}^s})^2.\end{aligned}$$

The estimate of $\mathcal{I}_3^b$ is identical. Finally,

$$\begin{aligned}\int_0^T \mathcal{I}_3\,\mathrm{d}t &\lesssim \varepsilon(\|\Omega\|_{L^\infty_T(\dot{H}^{1-\tau})} + \|\Omega\|_{L^\infty_T(\dot{H}^s)})\frac{1}{\sqrt{\varepsilon}}(\|\Omega\|_{L^2_T(\dot{H}^{1-\tau})}\\ &\quad + \|\Omega\|_{L^2_T(\dot{H}^s)})\frac{1}{\sqrt{\varepsilon}}\|z\|_{L^2_T(\dot{H}^{1-\tau})}\\ &\lesssim \varepsilon\mathcal{M}^3(T).\end{aligned}$$

The proof of Proposition 4.1 is concluded. □

4.2. II. L^∞ and Lipschitz bounds for $u_2 = \mathcal{R}_1\Omega$

The purpose of this section is to prove the following proposition:

Proposition 4.2. *For $\varepsilon > 0$, let (b, z) be a smooth solution of (2D-B). One has*

$$\|(\nabla\mathcal{R}_1\Omega, \Lambda\mathcal{R}_1\Omega)\|_{L^1_T(L^\infty)} + \|\mathcal{R}_1\Omega\|_{L^1_T(L^\infty)} \lesssim X(0) + X(t)^2.$$

4.3. Linear a Priori Estimates in Anisotropic Spaces

Applying $\dot{\Delta}_j \dot{\Delta}_q^h$ to the linear part of (8), we obtain

$$\begin{cases} \partial_t b_{j,q} - \varepsilon \mathcal{R}_1^2 b_{j,q} = \mathcal{R}_1 z_{j,q} + h_{j,q}, \\ \partial_t z_{j,q} + \dfrac{z_{j,q}}{\varepsilon} = -\varepsilon \mathcal{R}_1^2 \Omega_{j,q} + g_{j,q}, \end{cases} \tag{29}$$

where $h = (\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot\nabla b$ and $g = -\varepsilon\mathcal{R}_1(\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot\nabla b + \Lambda^{-1}((\mathcal{R}_2\Omega, -\mathcal{R}_2\Omega)\cdot(\nabla\Lambda\Omega))$. Performing standard energy estimates, one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|b_{j,q}\|_{L^2}^2 + \varepsilon\|\mathcal{R}_1 b_{j,q}\|_{L^2}^2 \leqq \|\mathcal{R}_1 z_{j,q}\|_{L^2}\|b_{j,q}\|_{L^2} + \|h_{j,q}\|_{L^2}\|b_{j,q}\|_{L^2}. \tag{30}$$

Using Fourier-Plancherel theorem and the anisotropic Bernstein inequality in Lemma 3.1 yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|b_{j,q}\|_{L^2}^2 + 2^{-2j}2^{2q}\varepsilon\|b_{j,q}\|_{L^2}^2 \leqq \|\mathcal{R}_1 z_{j,q}\|_{L^2}\|b_{j,q}\|_{L^2} + \|h_{j,q}\|_{L^2}\|b_{j,q}\|_{L^2}. \tag{31}$$

Now, applying Lemma B.1, multiplying by $2^{js_1}2^{qs_2}$ and summing on $j, k \in \mathbb{Z}$ give, that

$$\|b\|_{L_T^\infty(B^{s_1,s_2})} + \varepsilon\|b\|_{L_T^1(B^{s_1-2,s_2+2})} \lesssim \|b_0\|_{B^{s_1,s_2}} + \|z\|_{B^{s_1-1,s_2+1}} + \|h\|_{L_T^1(B^{s_1,s_2})} \tag{32}$$

Following a similar procedure for z, one infers that

$$\|z\|_{L_T^\infty(B^{s_1',s_2'})} + \frac{1}{\varepsilon}\|z\|_{L_T^1(B^{s_1',s_2'})} \lesssim \|z_0\|_{B^{s_1',s_2'}} + \varepsilon\|\Omega\|_{B^{s_1-2,s_2+2}} + \|g\|_{L_T^1(B^{s_1',s_2'})}. \tag{33}$$

Notice that the linear term $\|z\|_{B^{s_1-1,s_2+1}}$ in (32) can be absorbed by the left-hand side of (33) if $s_1' \geqq s_1$ and $s_2' \leqq s_2$ via Lemma B.5 with $s = 1$. The linear term $\|\Omega\|_{B^{s_1-2,s_2+2}}$ will be absorbed in a similar fashion once the estimates for Ω are obtained.

Since $\Omega = z + \varepsilon\mathcal{R}_1 b$, this linear analysis together with Lemma 3.3 suggests us to choose that

$$s_1 = s_1' = \frac{3}{2} \quad \text{and} \quad s_2 = s_2' = \frac{1}{2}, \text{ to ensure that one controls} \|\nabla\mathcal{R}_1\Omega\|_{L_T^1(L^\infty)},$$
$$s_1 = s_1' = \frac{1}{2} \quad \text{and} \quad s_2 = s_2' = \frac{1}{2}, \text{ to ensure that one controls } \|\mathcal{R}_1\Omega\|_{L_T^1(L^\infty)}.$$

Remark 4.1. The term $\|\mathcal{R}_1\Omega\|_{L_T^1(L^\infty)}$ is not needed to close the a priori estimates in Sobolev spaces (proof of Proposition 4.1), but it is actually crucial to deal with some nonlinear terms appearing in this anisotropic analysis. The control of $\|\mathcal{R}_1\Omega\|_{L_T^1(W^{1,\infty})}$ requires a priori estimates in two different regularity settings (as just remarked above).

With these regularity indexes, one obtains

$$\|b\|_{L^\infty_T(B^{\frac{3}{2},\frac{1}{2}})} + \varepsilon\|b\|_{L^1_T(B^{-\frac{1}{2},\frac{5}{2}})} + \sqrt{\varepsilon}\|b\|_{L^2_T(B^{\frac{1}{2},\frac{3}{2}})} \lesssim \|b_0\|_{B^{\frac{3}{2},\frac{1}{2}}} + \|h\|_{L^1_T(B^{\frac{3}{2},\frac{1}{2}})},$$

$$\|b\|_{L^\infty_T(B^{\frac{1}{2},\frac{1}{2}})} + \varepsilon\|b\|_{L^1_T(B^{-\frac{3}{2},\frac{5}{2}})} + \sqrt{\varepsilon}\|b\|_{L^2_T(B^{-\frac{1}{2},\frac{3}{2}})} \lesssim \|b_0\|_{B^{\frac{1}{2},\frac{1}{2}}} + \|h\|_{L^1_T(B^{\frac{1}{2},\frac{1}{2}})},$$

and

$$\|z\|_{L^\infty_T(B^{\frac{3}{2},\frac{1}{2}})} + \frac{1}{\varepsilon}\|z\|_{L^1_T(B^{\frac{3}{2},\frac{1}{2}})} + \frac{1}{\sqrt{\varepsilon}}\|z\|_{L^2_T(B^{\frac{3}{2},\frac{1}{2}})} \lesssim \|z_0\|_{B^{\frac{3}{2},\frac{1}{2}}} + \|g\|_{L^1_T(B^{\frac{3}{2},\frac{1}{2}})},$$

$$\|z\|_{L^\infty_T(B^{\frac{1}{2},\frac{1}{2}})} + \frac{1}{\varepsilon}\|z\|_{L^1_T(B^{\frac{1}{2},\frac{1}{2}})} + \frac{1}{\sqrt{\varepsilon}}\|z\|_{L^2_T(B^{\frac{1}{2},\frac{1}{2}})} \lesssim \|z_0\|_{B^{\frac{1}{2},\frac{1}{2}}} + \|g\|_{L^1_T(B^{\frac{1}{2},\frac{1}{2}})},$$

where we used interpolation inequalities to recover the L^2_T terms. Since $\Omega = z - \varepsilon\mathcal{R}_1 b$, one derives the following bounds

$$\begin{aligned}
&\|\Omega\|_{L^\infty_T(B^{\frac{3}{2},\frac{1}{2}})} + \|\Omega\|_{L^1_T(B^{\frac{1}{2},\frac{3}{2}})} + \|\Omega\|_{L^2_T(B^{\frac{3}{2},\frac{1}{2}})} \\
&\quad \lesssim \|(\Omega_0, b_0)\|_{B^{\frac{3}{2},\frac{1}{2}}} + \|(g,h)\|_{L^1_T(B^{\frac{3}{2},\frac{1}{2}})}, \\
&\|\Omega\|_{L^\infty_T(B^{\frac{1}{2},\frac{1}{2}})} + \|\Omega\|_{L^1_T(B^{-\frac{1}{2},\frac{3}{2}})} + \|\Omega\|_{L^2_T(B^{\frac{1}{2},\frac{1}{2}})} \\
&\quad \lesssim \|(\Omega_0, b_0)\|_{B^{\frac{1}{2},\frac{1}{2}}} + \|(g,h)\|_{L^1_T(B^{\frac{1}{2},\frac{1}{2}})}.
\end{aligned}$$

As expected, via Lemma 3.3, one has that

$$\|\nabla\mathcal{R}_1\Omega\|_{L^1_T(L^\infty)} \lesssim \|\Omega\|_{L^1_T(B^{\frac{1}{2},\frac{3}{2}})} \quad \text{and} \quad \|\mathcal{R}_1\Omega\|_{L^1_T(L^\infty)} \lesssim \|\Omega\|_{L^1_T(B^{-\frac{1}{2},\frac{3}{2}})}. \tag{34}$$

Accordingly, we define the functional

$$\begin{aligned}
Y(t) = {} & \|(b,z,\Omega)\|_{L^\infty_T(B^{\frac{1}{2},\frac{1}{2}}\cap B^{\frac{3}{2},\frac{1}{2}})} + \varepsilon\|b\|_{L^1_T(B^{-\frac{3}{2},\frac{5}{2}}\cap B^{-\frac{1}{2},\frac{5}{2}})} \\
& + \sqrt{\varepsilon}\|b\|_{L^2_T(B^{-\frac{1}{2},\frac{3}{2}}\cap B^{\frac{1}{2},\frac{3}{2}})} \\
& + \frac{1}{\varepsilon}\|z\|_{L^1_T(B^{\frac{3}{2},\frac{1}{2}}\cap B^{\frac{1}{2},\frac{1}{2}})} + \frac{1}{\sqrt{\varepsilon}}\|z\|_{L^2_T(B^{\frac{3}{2},\frac{1}{2}}\cap B^{\frac{1}{2},\frac{1}{2}})} \\
& + \|\Omega\|_{L^1_T(B^{-\frac{1}{2},\frac{3}{2}}\cap B^{\frac{1}{2},\frac{3}{2}})} + \|\Omega\|_{L^2_T(B^{\frac{3}{2},\frac{1}{2}}\cap B^{\frac{1}{2},\frac{1}{2}})},
\end{aligned} \tag{35}$$

which will be useful to close the estimates.

5. Estimates of the Nonlinearities

The term $\|(h,g)\|_{L^1_T(B^{\frac{3}{2},\frac{1}{2}}\cap B^{\frac{1}{2},\frac{1}{2}})}$ is bounded by the following lemma:

Lemma 5.1. *Let (b,Ω) be a smooth solution of (2D-B), then one has the following estimate:*

$$\|(h,g)\|_{L^1_T(B^{\frac{3}{2},\frac{1}{2}}\cap B^{\frac{1}{2},\frac{1}{2}})} \lesssim Y(t)^2 + Y(t)X(t) + X(t)^2$$

The proof of Lemma 5.1 is postponed to Section A and it is based on product laws in anisotropic Besov spaces (again, postponed to Section A).

Proof of Proposition 4.2. It is a direct consequence of Lemma 5.1. □

5.1. Conclusion of the Proof of Theorem 2.1

Gathering the estimates from Proposition 4.1 and 4.2, we obtain

$$X(t) \lesssim X(0) + X(t)^3 + X(t)^2 + Y(t)X(t) + Y(t)^2. \tag{36}$$

Then, by Lemma 3.2, one has

$$Y(t) \lesssim X(t),$$

which yields

$$X(t) \lesssim X(0) + X(t)^2 + X(t)^3. \tag{37}$$

From there, a standard bootstrap argument leads to the existence of global-in-time solutions of (2D-B). Then the uniqueness follows from stability estimate below and Theorem 2.1 is proven.

5.2. Uniqueness: Stability Estimate

Let (ρ_1, ρ_2) be two solutions of (IPM-diss) associated to the same initial data. We Defining that $w = \rho_1 - \rho_2$, thus satisfies that

$$\partial_t w - \mathcal{R}_1^2 w = (\mathcal{R}_2\mathcal{R}_1\rho_1)\partial_x\rho_1 - (\mathcal{R}_1^2\rho_1)\partial_y\rho_1 - (\mathcal{R}_2\mathcal{R}_1\rho_2)\partial_x\rho_2 + (\mathcal{R}_1^2\rho_2)\partial_y\rho_2. \tag{38}$$

The right-hand side terms may be rewritten as follows:

$$\partial_t w - \mathcal{R}_1^2 w = (\mathcal{R}_2\mathcal{R}_1 w, -\mathcal{R}_1^2 w) \cdot \nabla\rho_1 + (\mathcal{R}_2\mathcal{R}_1\rho_2, -\mathcal{R}_1^2\rho_2) \cdot \nabla w. \tag{39}$$

Applying $\Lambda^s = (-\Delta)^{\frac{s}{2}}$ to (39) and using the notation $\dot{w} = \Lambda^s w$, one infers that

$$\partial_t \dot{w} - \mathcal{R}_1^2 \dot{w} = \Lambda^s((\mathcal{R}_2\mathcal{R}_1 w, -\mathcal{R}_1^2 w) \cdot \nabla\rho_1) + \Lambda^s((\mathcal{R}_2\mathcal{R}_1\rho_2, -\mathcal{R}_1^2\rho_2) \cdot \nabla w). \tag{40}$$

Combining the use of commutator estimates for the nonlinear terms (as in the previous section) with the bounds on (ρ_1, ρ_2) from Theorem 2.1 and the Gronwall inequality easily gives $w = 0$, from which the uniqueness of smooth solutions for (IPM-diss) follows. Analogous arguments lead to the uniqueness of smooth solutions to system (2D-B).

6. Proof of the Relaxation Limit Theorem 2.2

Recall the scaled variables

$$(\widetilde{b}^\varepsilon, \widetilde{\Omega}^\varepsilon)(\tau, x) \triangleq (b, \frac{\Omega}{\varepsilon})(t, x) \quad \text{with} \quad \tau = \varepsilon t.$$

The system reads as

$$\begin{cases} \partial_t \widetilde{b}^\varepsilon - \mathcal{R}_1 \widetilde{\Omega}^\varepsilon = (\mathcal{R}_2 \widetilde{\Omega}^\varepsilon)\partial_x \widetilde{b}^\varepsilon - (\mathcal{R}_1 \widetilde{\Omega}^\varepsilon)\partial_y \widetilde{b}^\varepsilon, \\ \varepsilon^2 \partial_t \widetilde{\Omega}^\varepsilon - \mathcal{R}_1 \widetilde{b}^\varepsilon + \widetilde{\Omega}^\varepsilon = \varepsilon^2 \Lambda^{-1}[(\mathcal{R}_2 \widetilde{\Omega}^\varepsilon, -\mathcal{R}_1 \widetilde{\Omega}^\varepsilon) \cdot (\nabla \Lambda \widetilde{\Omega}^\varepsilon)]. \end{cases} \tag{41}$$

Let $(\widetilde{b}^\varepsilon, \widetilde{\Omega}^\varepsilon)$ be the unique solution of (41) from Theorem 2.1. Under such scaling, that satisfies the following estimate:

$$\begin{aligned} &\|\widetilde{b}^\varepsilon\|_{L^\infty_T(\dot{H}^{1-\tau}\cap\dot{H}^s)} + \varepsilon\|\widetilde{\Omega}^\varepsilon\|_{L^\infty_T(\dot{H}^{1-\tau}\cap\dot{H}^s)} + \|\mathcal{R}_1\widetilde{b}^\varepsilon\|_{L^2_T(\dot{H}^{1-\tau}\cap\dot{H}^s)} \\ &\quad + \|\widetilde{\Omega}^\varepsilon\|_{L^2_T(\dot{H}^{1-\tau}\cap\dot{H}^s)} + \|\widetilde{\Omega}^\varepsilon - \mathcal{R}_1\widetilde{b}^\varepsilon\|_{L^2_T(\dot{H}^{1-\tau}\cap\dot{H}^s)} \leqq \widetilde{\mathcal{M}}_0. \end{aligned} \tag{42}$$

there $\widetilde{\mathcal{M}}_0 = \|\widetilde{b}_0^\varepsilon\|_{\dot{H}^{1-\tau}\cap\dot{H}^s} + \|\widetilde{\Omega}_0^\varepsilon\|_{\dot{H}^{1-\tau}\cap\dot{H}^s}$. Owing to (42), $\varepsilon\widetilde{\Omega}^\varepsilon$ and $\widetilde{\Omega}^\varepsilon$ (and therefore $\mathcal{R}_1\Omega$ and $\mathcal{R}_2\Omega$) are uniformly bounded in the spaces $L^\infty(\mathbb{R}^+; \dot{H}^{1-\tau}\cap\dot{H}^s)$ and $L^2(\mathbb{R}^+; \dot{H}^{1-\tau} \cap \dot{H}^s)$, respectively. This implies that

$$\varepsilon^2\Lambda^{-1}[(\mathcal{R}_2\widetilde{\Omega}^\varepsilon, -\mathcal{R}_1\widetilde{\Omega}^\varepsilon) \cdot (\nabla\Lambda\widetilde{\Omega}^\varepsilon)] \to 0 \quad \text{in} \quad L^2(\mathbb{R}^+; \dot{H}^{s-1}).$$

Therefore $\varepsilon^2\partial_t\widetilde{\Omega}^\varepsilon$ goes to 0 in the sense of distributions. Putting this information into the second equation of (41), one infers that

$$\widetilde{\Omega}^\varepsilon - \widetilde{\mathcal{R}}_1 b^\varepsilon \rightharpoonup 0 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d). \tag{43}$$

Concerning the other unknown, $\widetilde{b}^\varepsilon$ is uniformly bounded in $L^\infty(\mathbb{R}^+; \dot{H}^{1-\tau} \cap \dot{H}^s)$ and $\partial_t\widetilde{b}^\varepsilon$ is bounded in $L^2_T\dot{H}^{s-1}$. Therefore, there exists $\rho \in L^\infty(\mathbb{R}^+; \dot{H}^{1-\tau} \cap \dot{H}^s)$ such that, up to subsequence,

$$\widetilde{b}^\varepsilon \overset{*}{\rightharpoonup} \rho \ \text{ in } L^\infty(\mathbb{R}^+; \dot{H}^{1-\tau} \cap \dot{H}^s) \cap H^1(\mathbb{R}_+; \dot{H}^{s-1}). \tag{44}$$

Then, as the bounds from (42) easily ensure bounds for the time-derivative of the solution $(\partial_t\widetilde{b}^\varepsilon, \partial_t\widetilde{\Omega}^\varepsilon)$, a standard procedure involving compactness argument and Aubin-Lions lemma leads to $\widetilde{b}^\varepsilon \to \rho$ strongly[2] in $C([0, T], \dot{H}^{1-\tau'}_{loc} \cap \dot{H}^{s-s'}_{loc})$ for $0 < \tau < \tau' < 1$ and $0 < s' < s$. For more details, we refer to Coulombel and Lin in [28] or Xu and Wang in [36] for the relaxation limit of the compressible Euler system with damping in the inhomogeneous Sobolev and Besov settings respectively. Now, defining

$$\widetilde{Z}^\varepsilon := \widetilde{\Omega}^\varepsilon - \mathcal{R}_1\widetilde{b}^\varepsilon \tag{45}$$

[2] At first, the convergence takes place only for a subsequence, then it is deduced for the whole sequence because the limit system has a unique solution so all the sequences will converge to the same limit.

the first equation of (41) may be rewritten as

$$\partial_t \widetilde{b}^\varepsilon - \mathcal{R}_1^2 \widetilde{b}^\varepsilon = \widetilde{S}^\varepsilon \quad \text{with} \quad \widetilde{S}^\varepsilon = \mathcal{R}_1 \widetilde{Z}^\varepsilon + (\mathcal{R}_2 \widetilde{\Omega}^\varepsilon, -\mathcal{R}_1 := \widetilde{\Omega}^\varepsilon) \cdot \nabla \widetilde{b}^\varepsilon. \tag{46}$$

Hence, combining (42), (43), (44) and (45), one can deduce that ρ satisfies

$$\partial_t \rho - \mathcal{R}_1^2 \rho = -(\mathcal{R}_2 \mathcal{R}_1 \rho, -\mathcal{R}_1^2 \rho) \cdot \nabla \rho. \tag{47}$$

The convergence rate of $\widetilde{Z}^\varepsilon$ stated in (19) comes directly from the control of $Y(t)$ (35), which is ensured by Theorem 2.1.

7. Extensions

In this section, we discuss the adaptations of our proof to the case of the periodic domain $\mathbb{T}^2$ and slightly more general density profiles.

7.1. Asymptotic Stability on the Torus $\mathbb{T}^2$

from the whole space $\mathbb{R}^2$, Elgindi [16] provides asymptotic stability of (IP-M-diss) on the 2D torus $\mathbb{T}^2$. His proof is completely different from the analogous result in the whole space. In particular, in the periodic domain $\mathbb{T}^2$ the density ρ does not decay in $L^2(\mathbb{T}^2)$ and proving decay requires a loss of derivatives. It is natural to inquire whether our approach is also applicable to the torus $\mathbb{T}^2$. We observe that the anisotropic Littlewood-Paley decomposition (anisotropic Besov spaces) and anisotropic Bernstein inequalities indeed extend to the torus $\mathbb{T}^2$. Our approach of exploiting the anisotropic dissipation through a suitable (anisotropic) functional space works on $\mathbb{T}^2$ as well. The only additional difficulty lies in the control of the horizontal average of the density $\tilde{\rho}(y) = \fint \rho(x, y)\,\mathrm{d}x$. To overcome this problem, we can rely on Elgindi's idea (see also [34]) of decomposing $\rho(x, y) = \bar{\rho}(x, y) + \tilde{\rho}(y)$ and writing the two equations

$$\partial_t \bar{\rho} - \mathcal{R}_1^2 \bar{\rho}(1 - \partial_y \tilde{\rho}) = -\overline{\boldsymbol{u} \cdot \nabla \bar{\rho}}, \tag{48}$$

$$\partial_t \tilde{\rho} + \partial_y \widetilde{(u_2 \bar{\rho})} = 0 \quad \Leftrightarrow \quad \partial_t \tilde{\rho} + \partial_y (\bar{\rho} \mathcal{R}_1^2 \tilde{\rho}) = 0. \tag{49}$$

As showed by Elgindi in the proof of [16, Proposition 4.1], the horizontal average $\tilde{\rho}$ is preserved by the *linear* evolution, i.e. neglecting the nonlinear terms yields

$$\begin{aligned} \partial_t \tilde{\rho} &= \partial_t \int_{-\pi}^{\pi} \rho(t, x, y)\,\mathrm{d}x = \int_{-\pi}^{\pi} \mathcal{R}_1^2 \rho (1 - \partial_y \tilde{\rho})\,\mathrm{d}x \\ &= \int_{-\pi}^{\pi} \partial_x (-\Delta) s^{-1} \partial_x \rho (1 - \partial_y \tilde{\rho})\,\mathrm{d}x = 0, \end{aligned}$$

as well as any vertical derivative $\frac{\mathrm{d}}{\mathrm{d}t} \partial_y^i \tilde{\rho} = 0$, i.e. $\partial_y^i \tilde{\rho}(t) = \partial_y^i \tilde{\rho}_0$ for all $i \in \mathbb{N}$. On the other hand, the linear evolution of $\bar{\rho}$ displays anisotropic dissipation and decay. Towards a proof of our result on $\mathbb{T}^2$, taking horizontal average at the initial time $\|\tilde{\rho}_0\|_{\dot{H}^{1-\tau} \cap \dot{H}^s} \leqq \delta_0$ with δ_0 small enough (and $s > 3$) we should achieve the

control of our functional $X(t)$ in (16) (adjusted for $\bar{\rho}$) with very similar computations. More precisely, the Sobolev estimates for controlling $\mathcal{M}(t)$ would resemble the computations in Section 4.1, further capitalizing on the smallness of $\tilde{\rho}$. The primary challenge lies in the (linear part of the) Besov estimate in Section 4.2. More precisely, using the same ideas requires controlling the term $\|\dot{\Delta}_j \dot{\Delta}_q^h(\partial_y \tilde{\rho} \mathcal{R}_1^2 \bar{\rho})\|_{L^2}$. Being $\partial_y \tilde{\rho}$ a function of y only, it commutes with $\dot{\Delta}_q^h$ and therefore that term can be written as $\|\dot{\Delta}_j(\partial_y \tilde{\rho} \mathcal{R}_1^2 \dot{\Delta}_q^h \bar{\rho})\|_{L^2}$. In turn, the latter can be bounded using the usual (isotropic) product laws in Besov spaces, yielding

$$\begin{aligned}\|2^{\frac{3}{2}j} 2^{\frac{q}{2}} \|\dot{\Delta}_j(\partial_y \tilde{\rho} \mathcal{R}_1^2 \dot{\Delta}_q^h \bar{\rho})\|_{L^2(\mathbb{R}^2)}\|_{\ell^1} &\lesssim \|\partial_y \tilde{\rho}\|_{L^\infty} \|\mathcal{R}_1^2 \bar{\rho}\|_{\dot{B}^{\frac{3}{2},\frac{1}{2}}} \\ &\quad + \|\partial_y \tilde{\rho}\|_{\dot{B}^{\frac{3}{2},\frac{1}{2}}} \|\mathcal{R}_1^2 \bar{\rho}\|_{L^\infty} \\ &\lesssim \|\partial_y \tilde{\rho}\|_{\dot{B}^1} \|\bar{\rho}\|_{\dot{B}^{-\frac{1}{2},\frac{5}{2}}} + \|\partial_y \tilde{\rho}\|_{\dot{B}^{\frac{3}{2}}} \|\bar{\rho}\|_{\dot{B}^{-\frac{3}{2},\frac{5}{2}}}.\end{aligned}$$

Integrating in time,

$$\begin{aligned}&\int_0^t \|2^{\frac{3}{2}j} 2^{\frac{q}{2}} \|\dot{\Delta}_j(\partial_y \tilde{\rho} \mathcal{R}_1^2 \dot{\Delta}_q^h \bar{\rho})\|_{L^2(\mathbb{R}^2)}\|_{\ell^1} \, \mathrm{d}t \\ &\quad \lesssim \|\partial_y \tilde{\rho}\|_{L_T^\infty \dot{B}^1} \|\bar{\rho}\|_{L_T^1 \dot{B}^{-\frac{1}{2},\frac{5}{2}}} + \|\partial_y \tilde{\rho}\|_{L_T^\infty \dot{B}^{\frac{3}{2}}} \|\bar{\rho}\|_{L_T^1 \dot{B}^{-\frac{3}{2},\frac{5}{2}}},\end{aligned}$$

where $\|\bar{\rho}\|_{L_T^1 \dot{B}^{-\frac{1}{2},\frac{5}{2}}}$ and $\|\bar{\rho}\|_{L_T^1 \dot{B}^{-\frac{3}{2},\frac{5}{2}}}$ would be controlled by (the adaptation to $\bar{\rho}$ of) the functional $Y(t)$ in (35). Of course some extra work would be needed to manage (and ensure the smallness of) the additional term $\partial_y \tilde{\rho}$ in $\dot{B}^1 \cap \dot{B}^{\frac{3}{2}}$. In summary, we believe that our results for $\mathbb{R}^2$ can be extended to the torus $\mathbb{T}^2$ under the same regularity setting.

7.2. Nearly Linear Density Profiles

Besides the linear density profile $\bar{\rho}_{\text{eq}} = \rho_0 - y$, we believe that our proof could be extended to the slightly more general case of nearly linear density profile $\bar{\rho}_{\text{eq}} = \rho_0 + G(y)$, with $G'(y) < 0$ and $\|1 + G'(y)\|_{H^r} \leqq \delta_0$ with δ_0 sufficiently small and r big enough. Linearizing the IPM equation around it gives

$$\partial_t \rho + G'(y) \mathcal{R}_1^2 \rho = \text{NL} \quad \Leftrightarrow \quad \partial_t \rho - \mathcal{R}_1^2 \rho + (1 + G'(y)) \mathcal{R}_1^2 \rho = \text{NL},$$

where NL stands for the nonlinear term. While the Sobolev estimates in Section 4.1 should work similarly exploiting the smallness of $\|1 + G'(y)\|_{H^r}$, the delicate point would be the (linear part of the) Besov estimate in Section 4.2. Similarly to the previous case, this point requires controlling $\|2^{\frac{3}{2}j} 2^{\frac{q}{2}} \|\dot{\Delta}_j((1 + G'(y)) \mathcal{R}_1^2 \dot{\Delta}_q^h \rho)\|_{L^2(\mathbb{R}^2)}\|_{\ell^1}$, where, once again $\dot{\Delta}_q^h$ and $1 + G'(y)$ commute.

This leads to

$$\begin{aligned}\int_0^t \|2^{\frac{3}{2}j} 2^{\frac{q}{2}} \|\dot{\Delta}_j((1 + G') \mathcal{R}_1^2 \dot{\Delta}_q^h \rho)\|_{L^2(\mathbb{R}^2)}\|_{\ell^1} \, \mathrm{d}t &\lesssim \|1 + G'\|_{\dot{B}^1} \|\rho\|_{L_T^1 \dot{B}^{-\frac{1}{2},\frac{5}{2}}} \\ &\quad + \|1 + G'\|_{\dot{B}^{\frac{3}{2}}} \|\rho\|_{L_T^1 \dot{B}^{-\frac{3}{2},\frac{5}{2}}},\end{aligned}$$

where $\|\rho\|_{L^1_T \dot{B}^{-\frac{1}{2},\frac{5}{2}}}$ and $\|\rho\|_{L^1_T \dot{B}^{-\frac{3}{2},\frac{5}{2}}}$ are controlled by the functional $Y(t)$ in (35) and $1+G'$ is small enough in a sufficiently regular Sobolev space.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Appendix A

Lemma A.1. (Anisotropic product laws) *Let* $-\frac{1}{2} \leqq s_1 \leqq \frac{5}{2}$ *and* $-\frac{1}{2} \leqq s_2 \leqq \frac{1}{2}$. *Let* $\delta_1, \delta_2, \delta_3, \delta_4 \geqq 0$ *and* f, g *be two smooth functions, one has*

$$\begin{aligned}\|fg\|_{B^{s_1,s_2}} \lesssim{}& \|f\|_{L^\infty}\|g\|_{B^{s_1,s_2}} + \|f\|_{B^{s_1,s_2}}\|g\|_{L^\infty} \\ &+ \|f\|_{B^{s_1+\frac{1}{2}+\delta_1,-\delta_2}}\|g\|_{B^{-\delta_1,s_2+\frac{1}{2}+\delta_2}} + \|f\|_{B^{-\delta_3,s_2+\frac{1}{2}+\delta_4}}\|g\|_{B^{s_1+\frac{1}{2}+\delta_3,-\delta_4}}.\end{aligned}$$

Moreover,

$$\begin{aligned}\|fg\|_{B^{-\frac{1}{2},\frac{1}{2}}} \lesssim{}& \|f\|_{L^\infty}\|g\|_{B^{-\frac{1}{2},\frac{1}{2}}} + \|f\|_{B^{\frac{1}{2},\frac{1}{2}}}\|\Lambda^{-1}g\|_{L^\infty} \\ &+ \|f\|_{B^{\delta_1,-\delta_2}}\|g\|_{B^{-\delta_1,1+\delta_2}} + \|f\|_{B^{-\delta_3,1+\delta_4}}\|g\|_{B^{\delta_3,-\delta_4}}.\end{aligned}$$

A.1 Proof of Lemma A.1

Proof of Lemma A.1. Recalling the definitions of the Littlewood-Paley blocks $\dot{\Delta}_j$ and $\dot{\Delta}^h_q$ in Section 3.1, we introduce the isotropic and anisotropic paradifferential decomposition due to Bony in [5]. Let $f, g \in \mathcal{S}'(\mathbb{R}^d)$,

$$fg = T(f,g) + R(f,g) \quad \text{and} \quad fg = T^h(f,g) + R^h(f,g)$$

where

$$T(f,g)=\sum_{j\in\mathbb{Z}} S_{j-1}f\dot{\Delta}_j g \quad\text{and}\quad R(f,g)=\sum_{j\in\mathbb{Z}}\dot{\Delta}_j f S_{j+2}g,$$

and in the horizontal direction

$$T^h(f,g)=\sum_{q\in\mathbb{Z}} S_{q-1}f\dot{\Delta}_q^h g \quad\text{and}\quad R^h(f,g)=\sum_{q\in\mathbb{Z}}\dot{\Delta}_q^h f S_{q+2}g.$$

Applying the double paraproduct decomposition to fg, one has

$$fg=TT^h(f,g)+TR^h(f,g)+RT^h(f,g)+RR^h(f,g). \tag{50}$$

Let us estimate each of these terms separately. For the first one, we have

$$\begin{aligned}\|\dot{\Delta}_j\dot{\Delta}_q^h TT^1(f,g)\|_{L^2} &\lesssim \sum_{|j'-j|\leqq 4,|q'-q|\leqq 4}\|S_{j'-1}S_{q'-1}f\dot{\Delta}_{j'}\dot{\Delta}_{q'}\partial_y g\|_{L^2}\\ &\lesssim \sum_{|j'-j|\leqq 4,|q'-q|\leqq 4}\|S_{j'-1}S_{q'-1}f\|_{L^\infty}\|\dot{\Delta}_{j'}\dot{\Delta}_{q'}\partial_y g\|_{L^2}\\ &\lesssim c_{j,q}2^{-js_1}2^{-qs_2}\|f\|_{L^\infty}\|g\|_{B^{s_1,s_2}}.\end{aligned}$$

Concerning the fourth term, one has

$$\begin{aligned}\|\dot{\Delta}_j\dot{\Delta}_q^h RR^1(f,g)\|_{L^2} &\lesssim \sum_{|j'-j|\leqq 4,|q'-q|\leqq 4}\|\dot{\Delta}_{j'}\dot{\Delta}_{q'}fS_{j'+2}S_{q'+2}g\|_{L^2}\\ &\lesssim \sum_{|j'-j|\leqq 4,|q'-q|\leqq 4}\|\dot{\Delta}_{j'}\dot{\Delta}_{q'}f\|_{L^2}\|S_{j'+2}S_{q'+2}g\|_{L^\infty}\\ &\lesssim c_{j,q}2^{-js_1}2^{-qs_2}\|f\|_{B^{s_1,s_2}}\|g\|_{L^\infty}.\end{aligned}$$

The following computations in the special case $s_1=-\frac{1}{2}$ and $s_2=\frac{1}{2}$ will be needed in the proof of Lemma A.1:

$$\begin{aligned}\|\dot{\Delta}_j\dot{\Delta}_q^h RR^1(f,g)\|_{L^2} &\lesssim \sum_{|j'-j|\leqq 4,|q'-q|\leqq 4}\|\dot{\Delta}_{j'}\dot{\Delta}_{q'}fS_{j'+2}S_{q'+2}g\|_{L^2}\\ &\lesssim \sum_{|j'-j|\leqq 4,|q'-q|\leqq 4}\|\dot{\Delta}_{j'}\dot{\Delta}_{q'}f\|_{2}\|S_{j'+2}S_{q'+2}g\|_{L^\infty}\\ &\lesssim \sum_{|j'-j|\leqq 4,|q'-q|\leqq 4}2^j\|\dot{\Delta}_{j'}\dot{\Delta}_{q'}f\|_{L^2}2^{-j}\|S_{j'+2}S_{q'+2}g\|_{L^2}\\ &\lesssim c_{j,q}2^{\frac{j}{2}}2^{-\frac{q}{2}}\|f\|_{B^{\frac{1}{2},\frac{1}{2}}}\|\Lambda^{-1}g\|_{L^\infty}.\end{aligned}$$

For the third term, using the anisotropic Bernstein inequality, one has

$$
\begin{aligned}
&\|\dot{\Delta}_j \dot{\Delta}_q^h TR^1(f,g)\|_{L^2} \sum_{|j'-j|\leqq 4, |q'-q|\leqq 4} \|\dot{\Delta}_{j'} S_{q'+2} f S_{j'+2} \dot{\Delta}_{q'} g\|_{L^2} \\
&\lesssim \sum_{|j'-j|\leqq 4, |q'-q|\leqq 4} \|\dot{\Delta}_{j'} S_{q'+2} f\|_{L_x^2 L_y^\infty} \|S_{j'+2} \dot{\Delta}_{q'} g\|_{L_x^\infty L_y^2} \\
&\lesssim \sum_{|j'-j|\leqq 4, |q'-q|\leqq 4} 2^{\frac{j'}{2}} \|\dot{\Delta}_{j'} S_{q'+2} f\|_{L^2} 2^{\frac{q'}{2}} \|S_{j'+2} \dot{\Delta}_{q'} g\|_{L^2} \\
&\lesssim 2^{-js_1} 2^{-qs_2} \sum_{|j'-j|\leqq 4, |q'-q|\leqq 4} 2^{j'(s_1+\frac{1}{2})} \|\dot{\Delta}_{j'} S_{q'+2} f\|_{L^2} 2^{q'(s_2+\frac{1}{2})} \|S_{j'+2} \dot{\Delta}_{q'} g\|_{L^2} \\
&\lesssim 2^{-js_1} 2^{-qs_2} \sum_{|j'-j|\leqq 4, |q'-q|\leqq 4} 2^{j'(s_1+\frac{1}{2}+\delta_1)} 2^{-q\delta_2} \|\dot{\Delta}_{j'} S_{q'+2} f\|_{L^2} 2^{q'(s_2+\frac{1}{2}+\delta_2)} 2^{-j'\delta_1} \|S_{j'+2} \dot{\Delta}_{q'} g\|_{L^2} \\
&\lesssim 2^{-js_1} 2^{-qs_2} c_{j,q} \begin{cases} \|f\|_{B^{s_1+\frac{1}{2}+\delta_1, -\delta_2}} \|g\|_{B^{-\delta_1, s_2+\frac{1}{2}+\delta_2}} & \text{if } \delta_1, \delta_2 > 0 \\ \|f\|_{B^{s_1+\frac{1}{2}}} \|g\|_{B_h^{s_2+\frac{1}{2}}} & \text{if } \delta_1, \delta_2 = 0, \end{cases}
\end{aligned}
$$

where B_h^s refers to the Besov spaces with horizontal localisation $\dot{\Delta}_q$. Since

$$
\|f\|_{B^{s_1+\frac{1}{2}}} \|g\|_{B_h^{s_2+\frac{1}{2}}} \lesssim \|f\|_{B^{s_1+\frac{1}{2},0}} \|g\|_{B^{0,s_2+\frac{1}{2}}},
$$

we obtain the desired estimate. Similar computations lead to

$$
\|\dot{\Delta}_j \dot{\Delta}_q^h RT^1(f,g)\|_{L^2} \lesssim 2^{-js_1} 2^{-qs_2} c_{j,q} \|f\|_{B^{-\delta_3, s_2+\frac{1}{2}+\delta_4}} \|g\|_{B^{s_1+\frac{1}{2}+\delta_3, -\delta_4}}.
$$

Multiplying the above estimates by $2^{js_1} 2^{qs_2}$ and summing on $j, q \in \mathbb{Z}$, the desired result follows. □

A.2 Proof of Lemma 5.1

Proof of Lemma 5.1. I) Estimates for $h = (\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega) \cdot \nabla b$. **i)** Applying Lemma A.1 with $f = \mathcal{R}_1\Omega$, $g = \partial_y b$, $s_1 = \frac{3}{2}$, $s_2 = \frac{1}{2}$, $\delta_1 = \delta_2 = \frac{1}{2}$ and $\delta_3 = \delta_4 = 0$, one has

$$
\begin{aligned}
\|\mathcal{R}_1\Omega \partial_y b\|_{L_T^1(B^{\frac{3}{2},\frac{1}{2}})} &\lesssim \|\mathcal{R}_1\Omega\|_{L_T^1(L^\infty)} \|\partial_y b\|_{L_T^\infty(B^{\frac{3}{2},\frac{1}{2}})} \\
&\quad + \|\mathcal{R}_1\Omega\|_{L_T^1(B^{\frac{3}{2},\frac{1}{2}})} \|\partial_y b\|_{L_T^\infty(L^\infty)} \\
&\quad + \|\mathcal{R}_1\Omega\|_{L_T^2(B^{\frac{5}{2},-\frac{1}{2}})} \|\partial_y b\|_{L_T^2(B^{-\frac{1}{2},\frac{3}{2}})} \\
&\quad + \|\mathcal{R}_1\Omega\|_{L_T^1(B^{0,1})} \|\partial_y b\|_{L_T^\infty(B^{2,0})}.
\end{aligned} \tag{51}
$$

Now, using Lemma 3.2 and B.5 we have

- $\|\mathcal{R}_1\Omega\|_{L_T^1(L^\infty)} \|\partial_y b\|_{L_T^\infty(B^{\frac{3}{2},\frac{1}{2}})} \lesssim X(t) \|b\|_{L^\infty(\dot{H}^s \cap H^{3-\varepsilon})} \lesssim X(t)^2$,
- $\|\mathcal{R}_1\Omega\|_{L_T^1(B^{\frac{3}{2},\frac{1}{2}})} \|\partial_y b\|_{L_T^\infty(L^\infty)} \lesssim Y(t) \|b\|_{L^\infty(H^{2+\varepsilon})} \lesssim Y(t) X(t)$,
- $\|\mathcal{R}_1\Omega\|_{L_T^2(B^{\frac{5}{2},-\frac{1}{2}})} \|\partial_y b\|_{L_T^2(B^{-\frac{1}{2},\frac{3}{2}})} \lesssim \|\Omega\|_{L_T^2(B^{\frac{3}{2},\frac{1}{2}})} \|b\|_{L_T^2(B^{\frac{1}{2},\frac{3}{2}})} \lesssim Y(t)^2$,

- $\|\mathcal{R}_1\Omega\|_{L^1_T(B^{0,1})}\|\partial_y b\|_{L^\infty_T(B^{2,0})} \lesssim \|\Omega\|_{L^1_T(B^{-1,2})}\|b\|_{L^\infty_T(B^{3,0})} \lesssim \|\Omega\|_{L^1_T(B^{-\frac{1}{2},\frac{3}{2}})}$ $\|b\|_{L^\infty_T(B^{3,0})} \lesssim X(t)^2$.

Gathering the above estimates, one obtains

$$\|\dot{\Delta}_j\dot{\Delta}^h_q\mathcal{R}_1\Omega\partial_y b\|_{L^1_T(B^{\frac{3}{2},\frac{1}{2}})} \lesssim X(t)^2.$$

ii) Applying Lemma A.1 with $f = \mathcal{R}_1\Omega$, $g = \partial_y b$, $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2}$, $\delta_1 = \delta_2 = \frac{1}{2}$ and $\delta_3 = \delta_4 = 0$, we get

$$\begin{aligned}\|\mathcal{R}_1\Omega\partial_y b\|_{L^1_T(B^{\frac{1}{2},\frac{1}{2}})} \lesssim{}& \|\mathcal{R}_1\Omega\|_{L^1_T(L^\infty)}\|\partial_y b\|_{L^\infty_T(B^{\frac{1}{2},\frac{1}{2}})} + \|\mathcal{R}_1\Omega\|_{L^1_T(B^{\frac{1}{2},\frac{1}{2}})}\|\partial_y b\|_{L^\infty_T(L^\infty)}\\ &+ \|\mathcal{R}_1\Omega\|_{L^2_T(B^{\frac{3}{2},-\frac{1}{2}})}\|\partial_y b\|_{L^2_T(B^{-\frac{1}{2},\frac{3}{2}})}\\ &+ \|\mathcal{R}_1\Omega\|_{L^1_T(B^{0,1})}\|\partial_y b\|_{L^\infty_T(B^{1,0})}.\end{aligned}$$

Using Lemma 3.2 and B.5, we have

- $\|\mathcal{R}_1\Omega\|_{L^1_T(L^\infty)}\|\partial_y b\|_{L^\infty_T(B^{\frac{1}{2},\frac{1}{2}})} \lesssim Y(t)\|b\|_{L^\infty_T(\dot{H}^s\cap H^{2-\tau})} \lesssim X(t)^2$,
- $\|\mathcal{R}_1\Omega\|_{L^1_T(B^{\frac{1}{2},\frac{1}{2}})}\|\partial_y b\|_{L^\infty_T(L^\infty)} \lesssim Y(t)\|b\|_{L^\infty(H^{2+\tau})} \lesssim Y(t)X(t)$,
- $\|\mathcal{R}_1\Omega\|_{L^2_T(B^{\frac{3}{2},-\frac{1}{2}})}\|\partial_y b\|_{L^2_T(B^{-\frac{1}{2},\frac{3}{2}})} \lesssim \|\Omega\|_{L^2_T(B^{\frac{1}{2},\frac{1}{2}})}\|b\|_{L^2_T(B^{\frac{1}{2},\frac{3}{2}})} \lesssim Y(t)^2$,
- $\|\mathcal{R}_1\Omega\|_{L^1_T(B^{0,1})}\|\partial_y b\|_{L^\infty_T(B^{1,0})} \lesssim \|\Omega\|_{L^1_T(B^{-1,2})}\|b\|_{L^\infty_T(B^{2,0})} \lesssim \|\Omega\|_{L^1_T(B^{-\frac{1}{2},\frac{3}{2}})}$ $\|b\|_{L^\infty_T(B^{2,0})} \lesssim X(t)^2$.

iii) For the second addend of h, applying Lemma A.1 with $f = \mathcal{R}_2\Omega$, $g = \partial_x b$, $s_1 = \frac{3}{2}$, $s_2 = \frac{1}{2}$, $\delta_1 = \delta_2 = \frac{1}{2}$ and $\delta_3 = \delta_4 = 0$, one obtains

$$\begin{aligned}\|\mathcal{R}_2\Omega\partial_y b\|_{L^1_T(B^{\frac{3}{2},\frac{1}{2}})} \lesssim{}& \|\mathcal{R}_2\Omega\|_{L^2_T(L^\infty)}\|\partial_x b\|_{L^2_T(B^{\frac{3}{2},\frac{1}{2}})}\\ &+ \|\mathcal{R}_2\Omega\|_{L^2_T(B^{\frac{3}{2},\frac{1}{2}})}\|\partial_x b\|_{L^2_T(L^\infty)}\\ &+ \|\mathcal{R}_2\Omega\|_{L^\infty_T(B^{\frac{5}{2},-\frac{1}{2}})}\|\partial_x b\|_{L^1_T(B^{-\frac{1}{2},\frac{3}{2}})}\\ &+ \|\mathcal{R}_2\Omega\|_{L^2_T(B^{0,1})}\|\partial_x b\|_{L^2_T(B^{2,0})}.\end{aligned}$$

Let us deal with each r.h.s. term:

- $\|\mathcal{R}_2\Omega\|_{L^2_T(L^\infty)}\|\partial_x b\|_{L^2_T(B^{\frac{3}{2},\frac{1}{2}})} \lesssim \|\mathcal{R}_2\Omega\|_{L^2_T(B^{\frac{1}{2},\frac{1}{2}})}\|\mathcal{R}_1 b\|_{L^2_T(\dot{H}^s\cap H^{3-\tau})} \lesssim Y(t)$ $X(t)$,
- $\|\mathcal{R}_2\Omega\|_{L^2_T(B^{\frac{3}{2},\frac{1}{2}})}\|\partial_x b\|_{L^2_T(L^\infty)} \lesssim Y(t)X(t)$,
- $\|\mathcal{R}_2\Omega\|_{L^\infty_T(B^{\frac{5}{2},-\frac{1}{2}})}\|\partial_x b\|_{L^1_T(B^{-\frac{1}{2},\frac{3}{2}})} \lesssim \|\mathcal{R}_2\Omega\|_{L^\infty_T(B^{\frac{5}{2},-\frac{1}{2}})}\|b\|_{L^1_T(B^{-\frac{1}{2},\frac{5}{2}})} \lesssim X(t)$ $Y(t)$,
- $\|\mathcal{R}_2\Omega\|_{L^2_T(B^{0,1})}\|\partial_x b\|_{L^2_T(B^{2,0})} \lesssim \|\mathcal{R}_2\Omega\|_{L^2_T(B^{\frac{1}{2},\frac{1}{2}})}\|\partial_x b\|_{L^2_T(B^{2,0})} \lesssim Y(t)X(t)$.

iv) Applying Lemma A.1 with $f = \mathcal{R}_2\Omega$, $g = \partial_x b$, $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2}$, $\delta_1 = \delta_2 = 0$ and $\delta_3 = \delta_4 = 0$,

$$\begin{aligned}\|\mathcal{R}_2\Omega\partial_x b\|_{L^1_T(B^{\frac{1}{2},\frac{1}{2}})} \lesssim\ & \|\mathcal{R}_2\Omega\|_{L^2_T(L^\infty)}\|\partial_x b\|_{L^2_T(B^{\frac{1}{2},\frac{1}{2}})} \\ &+ \|\mathcal{R}_2\Omega\|_{L^2_T(B^{\frac{1}{2},\frac{1}{2}})}\|\partial_x b\|_{L^2_T(L^\infty)} \\ &+ \|\mathcal{R}_2\Omega\|_{L^2_T(B^{1,0})}\|\partial_x b\|_{L^2_T(B^{0,1})} \\ &+ \|\mathcal{R}_2\Omega\|_{L^2_T(B^{0,1})}\|\partial_x b\|_{L^2_T(B^{1,0})}.\end{aligned}$$

Again, we deal with each term separately:

- $\|\mathcal{R}_2\Omega\|_{L^2_T(L^\infty)}\|\partial_x b\|_{L^2_T(B^{\frac{1}{2},\frac{1}{2}})} \lesssim \|\Omega\|_{L^2_T(B^{\frac{1}{2},\frac{1}{2}})}\|b\|_{L^2_T(B^{\frac{1}{2},\frac{3}{2}})} \leqq Y(t)^2$,
- $\|\mathcal{R}_2\Omega\|_{L^2_T(B^{\frac{1}{2},\frac{1}{2}})}\|\partial_x b\|_{L^2_T(L^\infty)} \lesssim \|\Omega\|_{L^2_T(B^{\frac{1}{2},\frac{1}{2}})}\|b\|_{L^2_T(B^{\frac{1}{2},\frac{3}{2}})} \lesssim Y(t)^2$,
- $\|\mathcal{R}_2\Omega\|_{L^2_T(B^{1,0})}\|\partial_x b\|_{L^2_T(B^{0,1})} \lesssim \|\Omega\|_{L^2_T(B^{1,0})}\|\mathcal{R}_1 b\|_{L^2_T(B^{2,0})} \lesssim X(t)^2$,
- $\|\mathcal{R}_2\Omega\|_{L^2_T(B^{0,1})}\|\partial_x b\|_{L^2_T(B^{1,0})} \lesssim \|\Omega\|_{L^2_T(B^{1,0})}\|\mathcal{R}_1 b\|_{L^2_T(B^{2,0})} \lesssim X(t)^2$.

Gathering the above estimates yields

$$\|\dot{\Delta}_j\dot{\Delta}^h_q\mathcal{R}_1\Omega\partial_y b\|_{L^1_T(B^{\frac{1}{2},\frac{1}{2}})} \lesssim X(t)^2.$$

Adding the estimates from $i) - iv)$ and using that $Y(t) \lesssim X(t)$ thanks to Lemma 3.2, one obtains

$$\|h\|_{L^1_T(B^{\frac{3}{2},\frac{1}{2}}\cap B^{\frac{1}{2},\frac{1}{2}})} \lesssim X(t)^2. \tag{52}$$

II) Estimates for $g = \varepsilon\mathcal{R}_1(\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot\nabla b + \Lambda^{-1}((\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot(\nabla\Lambda\Omega))$. The estimates of the first term $\varepsilon\mathcal{R}_1(\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot\nabla b$ can be obtained in the same way as the previous terms. Indeed, notice that by Lemma B.5 and thanks to the boundedness of the Riesz transform $\mathcal{R}_1 : B^{1,0} \to B^{1,0}$, one has

$$\|\mathcal{R}_1(\mathcal{R}_2\Omega\partial_x b)\|_{L^1_T(B^{\frac{1}{2},\frac{1}{2}})} \lesssim \|\mathcal{R}_1(\mathcal{R}_2\Omega\partial_x b)\|_{L^1_T(B^{1,0})} \lesssim \|\mathcal{R}_2\Omega\partial_x b\|_{L^1_T(B^{1,0})},$$

which is exactly the same term as the one we treated in $\boldsymbol{I}$). A similar argument can be applied for the bound in $B^{\frac{3}{2},\frac{1}{2}}$. We then turn to the second addend of g.
i) First, observing that

$$\|\Lambda^{-1}((\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot(\nabla\Lambda\Omega))\|_{B^{\frac{3}{2},\frac{1}{2}}} \leqq \|(\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot(\nabla\Lambda\Omega)\|_{B^{\frac{1}{2},\frac{1}{2}}}$$

and since Ω has better decay properties than b and we control Ω in $L^2_T(H^s)$ for $s > 3$, we can directly deduce from our previous computations that

$$\|\Lambda^{-1}((\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot(\nabla\Lambda\Omega))\|_{B^{\frac{3}{2},\frac{1}{2}}} \leqq X(t)^2.$$

ii) For the second regularity setting it is a bit trickier as one has

$$\|\Lambda^{-1}((\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot(\nabla\Lambda\Omega))\|_{B^{\frac{1}{2},\frac{1}{2}}} \lesssim \|(\mathcal{R}_2\Omega, -\mathcal{R}_1\Omega)\cdot(\nabla\Lambda\Omega)\|_{B^{-\frac{1}{2},\frac{1}{2}}}.$$

Applying the second inequality of Lemma A.1 with $f = \mathcal{R}_2\Omega$, $g = \partial_x\Lambda\Omega$, $s_1 = -\frac{1}{2}$, $s_2 = \frac{1}{2}$, $\delta_1 = 1$, $\delta_2 = 0$ and $\delta_3 = \delta_4 = 0$, we obtain

$$\begin{aligned}\|\mathcal{R}_2\Omega\partial_x\Lambda\Omega\|_{L^1_T(B^{-\frac{1}{2},\frac{1}{2}})} \lesssim\ & \|\mathcal{R}_2\Omega\|_{L^\infty_T(L^\infty)}\|\partial_x\Lambda\Omega\|_{L^1_T(B^{-\frac{1}{2},\frac{1}{2}})} \\ &+ \|\mathcal{R}_2\Omega\|_{L^\infty_T(B^{\frac{1}{2},\frac{1}{2}})}\|\Lambda^{-1}\partial_x\Lambda\Omega\|_{L^1_T(L^\infty)} \\ &+ \|\mathcal{R}_2\Omega\|_{L^2_T(B^{1,0})}\|\partial_x\Lambda\Omega\|_{L^2_T(B^{-1,1})} \\ &+ \|\mathcal{R}_2\Omega\|_{L^2_T(B^{0,1})}\|\partial_x\Lambda\Omega\|_{L^2_T(B^{0,0})}.\end{aligned}$$

□

Remark A.1. Alternatively, one could think to exploit the first inequality of Lemma A.1, which would require a control of $\|\mathcal{R}_2\Omega\|_{L^2_T(B^{-\frac{1}{2},\frac{1}{2}})}\|\partial_x\Lambda\Omega\|_{L^2_T(L^\infty)}$. However, such bounds hold under additional low-regularity assumptions on the initial data, for instance $B^{0,0}$, that we want to avoid here.

We estimate the above terms as follows.

- $\|\mathcal{R}_2\Omega\|_{L^\infty_T(L^\infty)}\|\partial_x\Lambda\Omega\|_{L^1_T(B^{-\frac{1}{2},\frac{1}{2}})} \lesssim \|\Omega\|_{L^\infty_T(B^{\frac{1}{2},\frac{1}{2}})}\|\Omega\|_{L^1_T(B^{\frac{1}{2},\frac{3}{2}})} \lesssim Y(t)^2$,
- $\|\mathcal{R}_2\Omega\|_{L^\infty_T(B^{\frac{1}{2},\frac{1}{2}})}\|\Lambda^{-1}\partial_x\Lambda\Omega\|_{L^1_T(L^\infty)} \lesssim \|\Omega\|_{L^\infty_T(B^{\frac{1}{2},\frac{1}{2}})}\|\Omega\|_{L^1_T(B^{\frac{1}{2},\frac{3}{2}})} \lesssim X(t)Y(t)$,
- $\|\mathcal{R}_2\Omega\|_{L^2_T(B^{1,0})}\|\partial_x\Lambda\Omega\|_{L^2_T(B^{-1,1})} \lesssim \|\Omega\|_{L^2_T(B^{1,0})}\|\Omega\|_{L^2_T(B^{0,2})} \lesssim \|\Omega\|_{L^2_T(B^{1,0})}\|\Omega\|_{L^2_T(B^{2,0})} \lesssim X(t)^2$,
- $\|\mathcal{R}_2\Omega\|_{L^2_T(B^{0,1})}\|\partial_x\Lambda\Omega\|_{L^2_T(B^{0,0})} \lesssim \|\Omega\|_{L^2_T(B^{1,0})}\|\Omega\|_{L^2_T(B^{1,1})} \lesssim \|\Omega\|_{L^2_T(B^{1,0})}\|\Omega\|_{L^2_T(B^{\frac{3}{2},\frac{1}{2}})} \lesssim X(t)Y(t)$.

The estimates for the last term $\mathcal{R}_1\Omega\partial_y\Lambda\Omega$ follow the exact same lines, then we omit them. We have therefore

$$\|g\|_{L^1_T(B^{\frac{3}{2},\frac{1}{2}}\cap B^{\frac{1}{2},\frac{1}{2}})} \lesssim X(t)^2. \tag{53}$$

Adding (52) and (53) concludes the proof of Lemma 5.1. □

Appendix B. Toolbox

We collect some technical lemmas that are used in the course of the proof the results of this article. In some case, we provide (short) proofs, while in other cases, appealing to the existing literature to which we refer explicitly, we omit the proofs. When no explicit reference is provided being the results classical, the reader can look for instance at [20].

Lemma B.1. *Let $X : [0,T] \to \mathbb{R}_+$ be a continuous function such that X^2 is differentiable. Assume that there exists a constant $B \geqq 0$ and a measurable function $A : [0,T] \to \mathbb{R}_+$ such that*

$$\frac{1}{2}\frac{d}{dt}X^2 + BX^2 \leqq AX \quad \textit{a.e. on } [0,T].$$

Then, for all $t \in [0, T]$, *we have*

$$X(t) + B\int_0^t X \leqq X_0 + \int_0^t A.$$

Lemma B.2. (Product estimates, [35, Lemma 2.1]) *Let* $s > 0$, $1 \leqq p, r \leqq \infty$, *then*

$$\|\Lambda^s(fg)\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^2)}\|\Lambda^s g\|_{L^{p_2}(\mathbb{R}^2)} + \|g\|_{L^{r_1}(\mathbb{R}^2)}\|\Lambda^s f\|_{L^{r_2}(\mathbb{R}^2)},$$

where $1 \leqq p_1, r_1 \leqq \infty$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$.

Lemma B.3. (Commutator estimates, [35, Lemma 2.1]) *Let* $s > 0$, $1 \leqq p_1, r_1 \leqq \infty$ *and* $1 < p, p_1, r_1 < 1$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$. *Then*

$$\|[\Lambda^s, f]g\|_{L^p} \lesssim (\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{r_1}}\|g\|_{L^{r_2}}).$$

Lemma B.4. (Interpolation) *Let* $s_0 \leqq s \leqq s_1$. *Then, for* $\theta \in (0, 1)$, *such that* $s = \theta s_0 + (1-\theta)s_1$, *it holds*

$$\|f\|_{\dot{H}^s} \lesssim \|f\|^{\theta}_{\dot{H}^{s_0}}\|f\|^{1-\theta}_{\dot{H}^{s_1}}.$$

Lemma B.5. (Embedding in Besov spaces) *For* $s > 0$, $s_1, s_2 \in \mathbb{R}$, *one has*

$$B^{s_1+s, s_2-s} \subset B^{s_1, s_2} \tag{54}$$

Proof. Let $f \in B^{s_1+s, s_2-s} \cap B^{s_1, s_2}$. By definition of the localisation $\dot{\Delta}_j$ and $\dot{\Delta}^h_q$, when applying $\dot{\Delta}_j\dot{\Delta}^h_q$ to a function, one can use that there exists N_0 such that $j \geqq q - N_0$. This implies that

$$\begin{aligned}\sum_{j,q\in\mathbb{Z}} 2^{js_1}2^{qs_2}\|\dot{\Delta}_j\dot{\Delta}^h_q f\|_{L^2} &\lesssim \sum_{j,q\in\mathbb{Z}} 2^{js_1}2^{js}2^{-js}2^{qs_2}\|\dot{\Delta}_j\dot{\Delta}^h_q f\|_{L^2}\\ &\lesssim \sum_{j,q\in\mathbb{Z}, j\geqq q-N_0} 2^{js_1}2^{js}2^{-qs}2^{qs_2}\|\dot{\Delta}_j\dot{\Delta}^h_q f\|_{L^2}\\ &\lesssim \sum_{j,q\in\mathbb{Z}} 2^{j(s_1+s)}2^{q(s_2-s)}\|\dot{\Delta}_j\dot{\Delta}^h_q f\|_{L^2}.\end{aligned}$$

□

The next lemma provides a generalized version of the Kenig–Ponce–Vega inequality (the fractional version of the Leibniz rule) for all $s > 0$, see [27] and [14]. Recall the notation $\alpha, \beta \in \mathbb{N}^2$ (multi-index) and $\nabla^\alpha = (\partial_x^{\alpha_1}, \partial_y^{\alpha_2})$, while the operator $\Lambda^{s,\alpha}$ is defined via Fourier transform as

$$\widehat{\Lambda^{s,\alpha} f}(\xi) = \widehat{\Lambda^{s,\alpha}}(\xi)\widehat{f}(\xi), \qquad \widehat{\Lambda^{s,\alpha}}(\xi) = i^{-|\alpha|}\partial^\alpha_\xi(|\xi|^s).$$

Lemma B.6. (Generalized Kenig–Ponce–Vega inequality [27, Theorem 5.1]) *Let* $s > 0$ *and* $1 < p < \infty$. *Then, for any* $s_1, s_2 \geqq 0$ *such that* $s_1 + s_2 = s$, *and any* $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}&\left\|\Lambda^s(fg) - \sum_{|\alpha|\leqq s_1}\frac{1}{\alpha!}(\nabla^\alpha f)(\Lambda^{s,\alpha}g) - \sum_{|\beta|\leqq s_2-1}\frac{1}{\beta!}(\nabla^\beta g)(\Lambda^{s,\beta}f)\right\|_{L^p}\\ &\lesssim \|\Lambda^{s_1}f\|_{BMO}\|\Lambda^{s_2}g\|_{L^p}.\end{aligned}$$

References

1. Beauchard, K., Zuazua, E.: Large time asymptotics for partially dissipative hyperbolic systems. *Arch. Rational Mech. Anal* **199**, 177–227, 2011
2. Bianchini, R., Dalibard, A.-L., Saint-Raymond, L.: Near-critical reflection of internal waves. *Anal. PDE* **14**(1), 205–249, 2021
3. Bianchini R. Natalini, R. Asymptotic behavior of 2d stably stratified fluids with a damping term in the velocity equation. ESAIM: COCV, 27 (2021)
4. Bianchini, S., Hanouzet, B., Natalini, R.: Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy. *Comm. Pure and Appl. Math.* **60**, 1559–1622, 2007
5. Bony, J.-M.: Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup.* **4**(14), 209–246, 1981
6. Castro, Á., Córdoba, D., Lear, D.: Global existence of quasi-stratified solutions for the confined IPM equation. *Arch. Ration. Mech. Anal.* **232**(1), 437–471, 2019
7. Castro, Á., Córdoba, D., Lear, D.: On the asymptotic stability of stratified solutions for the 2D Boussinesq equations with a velocity damping term. *Math. Models Methods Appl. Sci.* **29**(7), 1227–1277, 2019
8. Chemin, J.-Y., Zhang, P.: On the global wellposedness of the 3-d incompressible anisotropic Navier-Stokes equations. *Commun. Math. Phys.* **272**, 529–566, 2007
9. Coulombel, J.-F., Goudon, T.: The strong relaxation limit of the multidimensional isothermal Euler equations. *Trans. Amer. Math. Soc.* **359**(2), 637–648, 2007
10. Crin-Barat, T.: Partially dissipative hyperbolic systems and applications to fluid mechanics. University Paris-Est, Thesis (2021)
11. Crin-Barat T. Danchin, R. Global existence for partially dissipative hyperbolic systems in the L^p framework, and relaxation limit. Mathematische Annalen, 2022.
12. Crin-Barat, T., Danchin, R.: Partially dissipative hyperbolic systems in the critical regularity setting?: the multi-dimensional case. *Journal de Mathématiques Pures et Appliquées* **165**, 1–41, 2022
13. Crin-Barat, T., Danchin, R.: Partially dissipative one-dimensional hyperbolic systems in the critical regularity setting, and applications. *Pure and Applied Analysis* **4**(1), 85–125, 2022
14. D'Ancona, P.: A short proof of commutator estimates. *J. Fourier Anal. Appl.* **25**(3), 1134–1146, 2019
15. Desjardins, B., Lannes, D., Saut, J.-C.: Normal mode decomposition and dispersive and nonlinear mixing in stratified fluids. *Water Waves* **3**(1), 153–192, 2021
16. Elgindi, T.M.: On the asymptotic stability of stationary solutions of the inviscid incompressible porous medium equation. *Arch. Rational Mech. Anal.* **225**, 573–599, 2017
17. Elgindi, T. M., Shikh Khalil, K. R.: Strong ill-posedness in L^∞ for the Riesz transform problem. arXiv preprint arXiv:2207.04556 (2022).
18. Elgindi, T.M., Widmayer, K.: Sharp decay estimates for an anisotropic linear semigroup and applications to the surface quasi-geostrophic and inviscid Boussinesq systems. *SIAM J. Math. Anal.* **47**(6), 4672–4684, 2015
19. Gallay, T.: Stability of vortices in ideal fluids: the legacy of Kelvin and Rayleigh. In Hy*perbolic problems: theory, numerics, applications, volume 10 of AIMS Ser. Appl. Math.*, 42–59. Am. Inst. Math. Sci. (AIMS), Springfield, MO (2020).
20. Grafakos, L.: Classical Fourier analysis, volume 249 of Graduate Texts in Mathematics, third edition. Springer, New York (2014).
21. Guo, Y., Pausader, B., Widmayer, K.: Global axisymmetric euler flows with rotation. arXiv preprint arXiv:2109.01029 (2021).
22. Haspot, B.: Existence of global strong solutions in critical spaces for barotropic viscous fluids. *Arch. Rational Mech. Anal* **202**, 427–460, 2011
23. Hoff, D.: Uniqueness of weak solutions of the Navier-Stokes equations of multimensionnal, compressible flow. *SIAM J. Math. Anal* **37**, 1742–1760, 2006

24. Hsiao, L.: Quasilinear Hyperbolic Systems and Dissipative Mechanisms, volume Singapore. World Scientific Publishing (1997).
25. Junca, S., Rascle, M.: Strong relaxation of the isothermal Euler system to the heat equation. *Z. angew. Math. Phys.* **53**, 239–264, 2002
26. Kiselev, A., Yao, Y.: Small scale formations in the incompressible porous media equation. arXiv:2102.05213 (2021).
27. Li, D.: On Kato-Ponce and fractional Leibniz. *Rev. Mat. Iberoam.* **35**(1), 23–100, 2019
28. Lin, C., Coulombel, J.-F.: The strong relaxation limit of the multidimensional Euler equations. *Nonlinear Differential Equations and Applications NoDEA* **20**, 447–461, 2013
29. Lin, F., Xu, L., Zhang, P.: Global small solutions to 2-D incompressible MHD system. *J. Differ. Equ.* **259**, 10, 2013
30. Lin, F., Zhang, P.: Global small solutions to an MHD-type system: The three-dimensional case. Communications on Pure and Applied Mathematics, LXVII:0531-0580 (2014).
31. Marcati, P., Milani, A.: The one-dimensional Darcy's law as the limit of a compressible Euler flow. *J. Differ. Equ.* **84**, 129–147, 1990
32. Marcati, P., Rubino, B.: Hyperbolic to parabolic relaxation theory for quasilinear first order systems. *J. Differ. Equ.* **162**, 359–399, 2000
33. Paicu, M.: Equation anisotrope de Navier-Stokes dans des espaces critiques. *Rev. Mat. Iberoamericana* **21**, 179–235, 2005
34. Paicu, M.: Équation périodique de navier-stokes sans viscosité dans une direction. *Commun. Partial Differ. Equ.* **30**, 1107–1140, 2005
35. Wan, R.: Global well-posedness for the 2D Boussinesq equations with a velocity damping term. Discrete Contin. Dyn. Syst. **39**, 2019
36. Xu, J., Wang, Z.: Relaxation limit in Besov spaces for compressible Euler equations. *Journal de Mathématiques Pures et Appliquées* **99**, 43–61, 2013

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