RELAXATION LIMIT OF THE EULER-MAXWELL SYSTEM IN THE WHOLE SPACE: QUANTITATIVE ERROR ESTIMATES

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ABSTRACT. We investigate the compressible Euler-Maxwell system, a model for simulating the transport of electrons interacting with propagating electromagnetic waves in semiconductor devices. First, we show the global well-posedness of strong solutions being small perturbations of constant equilibrium states in a critical regularity setting, uniformly with respect to the relaxation parameter $\varepsilon > 0$. Then, for all time $t > 0$, we derive quantitative error at the rate $O(\varepsilon)$ between the diffusively rescaled Euler-Maxwell system and the limit drift-diffusion model. To the best of our knowledge, this provides the first global-in-time *strong* convergence result for this relaxation procedure in the whole space and in an ill-prepared setting.

1. INTRODUCTION

It is well-known that the Euler-Maxwell system for plasma physics is widely used to simulate phenomena such as photoconductive switches, electro-optics, semiconductor lasers, high-speed computers and so on. In these applications, the electrons transport in devices interacts with electromagnetic waves, which takes the form of Euler equations for the conservation laws of mass density, current density and energy density for electrons, coupled to Maxwell's equations for self-consistent electromagnetic field. See [4, 5, 37] for more explanation. In this paper, we consider the following isentropic Euler-Maxwell system in $\mathbb{R}^d$ $(d = 2, 3)$, which reads as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = -\rho(E + u \times B) - \dfrac{1}{\varepsilon}\rho u, \\ \partial_t E - \nabla \times B = \rho u, \\ \partial_t B + \nabla \times E = 0, \end{cases} \tag{1.1}$$

with constraints

$$\operatorname{div} E = \bar{\rho} - \rho, \quad \operatorname{div} B = 0 \tag{1.2}$$

for $(t, x) \in [0, +\infty) \times \mathbb{R}^d$. Here $\rho = \rho(t, x) > 0$, $u = u(t, x) \in \mathbb{R}^d$ are the density and the velocity of electrons, and $E = E(t, x) \in \mathbb{R}^d, B = B(t, x) \in \mathbb{R}^d$ denote the electric field and the magnetic field, respectively. In the momentum equation in (1.1), $\rho(E + u \times B)$ stands for the Lorentz force and $\frac{1}{\varepsilon}\rho u$ is a damping term with the relaxation parameter $\varepsilon > 0$, which is associated with friction forces. The pressure $P(\rho)$ is assumed to be smooth with respect to the density satisfying $P(\bar{\rho}) > 0$, where $\bar{\rho} > 0$ is a constant and stands for the density of charged background ions. We are concerned with (1.1)-(1.2) with initial data

$$(\rho, u, E, B)(0, x) = (\rho_0, u_0, E_0, B_0)(x), \quad x \in \mathbb{R}^3, \tag{1.3}$$

and we focus on solutions that are close to some constant state $(\bar{\rho}, 0, 0, \bar{B})$, at infinity, where $\bar{B} \in \mathbb{R}^d$ is a constant vector. Note that the constraint condition (1.2) remains true for every $t > 0$, if holds at time $t = 0$, namely,

$$\operatorname{div} E_0 = \bar{\rho} - \rho_0, \quad \operatorname{div} B_0 = 0, \quad x \in \mathbb{R}^3. \tag{1.4}$$

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The main interest of this paper is to justify the relaxation limit of solutions to (1.1) as $\varepsilon \to 0$. For that end, we change the time variable by considering an $\mathcal{O}(1/\varepsilon)$ time scale:

$$(\rho^\varepsilon, u^\varepsilon, E^\varepsilon, B^\varepsilon)(t,x) := (\rho, \frac{1}{\varepsilon}u, E, B)(\frac{t}{\varepsilon}, x). \tag{1.5}$$

Then the new variables satisfy

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \varepsilon^2 \partial_t(\rho^\varepsilon u^\varepsilon) + \varepsilon^2 \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \nabla P(\rho^\varepsilon) = -\rho^\varepsilon(E^\varepsilon + \varepsilon u^\varepsilon \times B^\varepsilon) - \rho^\varepsilon u^\varepsilon, \\ \varepsilon \partial_t E^\varepsilon - \nabla \times B^\varepsilon = \varepsilon \rho^\varepsilon u^\varepsilon, \\ \varepsilon \partial_t B^\varepsilon + \nabla \times E^\varepsilon = 0, \\ \operatorname{div} E^\varepsilon = \bar\rho - \rho^\varepsilon, \\ \operatorname{div} B^\varepsilon = 0, \end{cases} \tag{1.6}$$

with the initial data

$$(\rho^\varepsilon, u^\varepsilon, E^\varepsilon, B^\varepsilon)(0,x) = (\rho_0^\varepsilon, \frac{1}{\varepsilon}u_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)(x), \quad x \in \mathbb{R}^d. \tag{1.7}$$

Formally, as $\varepsilon \to 0$, $(\rho^\varepsilon, u^\varepsilon, E^\varepsilon, B^\varepsilon)$ converges to the corresponding limit (ρ^*, u^*, E^*, B^*), which solves that

$$\begin{cases} \partial_t \rho^* + \operatorname{div}(\rho^* u^*) = 0, \\ \rho^* u^* = -\nabla P(\rho^*) - \rho^* E^*, \\ \nabla \times B^* = 0, \\ \nabla \times E^* = 0, \\ \operatorname{div} E^* = \bar\rho - \rho^*, \\ \operatorname{div} B^* = 0. \end{cases} \tag{1.8}$$

Clearly, if $\nabla \times B^* = \operatorname{div} B^* = 0$, then $B^* = \bar B$. Due to $\nabla \times E^* = 0$, there exists a potential function ϕ^* such that $E^* = \nabla\phi^* = \nabla(-\Delta)^{-1}(\rho^* - \bar\rho)$. Thus, (1.8) can be reformulated as

$$\begin{cases} \partial_t \rho^* - \Delta P(\rho^*) - \operatorname{div}(\rho^* \nabla\phi^*) = 0, \\ \Delta\phi^* = \bar\rho - \rho^*. \end{cases} \tag{1.9}$$

The velocity field u^* satisfies the Darcy's law:

$$u^* = -\nabla(h(\rho^*) + \phi^*), \tag{1.10}$$

where the enthalpy $h(\rho)$ is defined by

$$h(\rho) := \int_{\bar\rho}^{\rho} \frac{P'(s)}{s} ds. \tag{1.11}$$

System (1.9) is referred as the classical drift-diffusion model for semiconductor.

1.1. **Exist literature.** So far there are a number of results on global existence, large-time behavior and asymptotic limit for the isentropic Euler-Maxwell system (1.1). In one dimension, by using the Godunov scheme with the fractional step and the compensated compactness theory, Chen, Jerome and Wang [5] constructed the existence of a global weak solution to the initial boundary value problem for arbitrarily large initial data. In multidimensional case, the question of global weak solutions is quite open and only smooth solutions are studied. Jerome [21] established the local unique smooth solutions to the Cauchy problem (1.1)-(1.3) in the framework of Sobolev spaces $H^s(\mathbb{R}^d)$ with $s > \frac{d}{2} + 1$ according to the standard theory for symmetrizable hyperbolic systems. The global smooth solutions near constant equilibrium states were obtained independently by Peng, Wang & Gu, Duan and Xu in [15, 36, 49]. Xu [49] introduced the inhomogeneous Besov space and established the global existence of classical solutions in B^{s_*} with the critical regularity index $s_* = \frac{d}{2}+1$. Furthermore, some singular limits, like the relaxation limit, the non-relativistic limit as well as combined nonrelativistic and relaxation limits were justified. Ueda, Wang and Kawashima [43] pointed out that System (1.1) was of regularity-loss type and the time-decay estimates of solutions were shown by [15, 46], respectively. Hajjej and Peng [17] employed the asymptotic expansion method and obtained the relaxation convergence rates of local-in-time solutions to (1.1) in the cases of both well-prepared data and ill-prepared data. Recently, Li, Peng and Zhao [28] studied the relaxation limit for global smooth solutions from (1.1) to (1.9). The error estimates between smooth periodic solutions between (1.1) and (1.9) are established by stream function techniques. Concerning the stability of steady-states, we refer to the works [33, 35, 31]. Let us mention also those results in [38, 56, 34] on global solutions for two-fluid Euler-Maxwell equations near constant states.

In order to investigate the large-time behavior of solutions to the system (1.1), as observed by Duan [15] and Ueda, Kawashima and Wang [46, 43], there is a non-symmetric dissipation due to the coupled electric field and the magnetic field, which leads to the one regularity loss when recovering time-decay estimates. More precisely, assume that U_L is the solution to the linearized system of (1.1) around $(\bar{\rho}, 0, 0, \bar{B})$ with $\varepsilon = 1$, then the Fourier image of $\widehat{U_L}$ verifies the following pointwise estimate:

$$|\widehat{U_L}(t,\xi)|^2 \lesssim e^{-\frac{c|\xi|^2}{(1+|\xi|^2)^2}t}|\widehat{U_L}(0,\xi)|^2, \tag{1.12}$$

for all $t > 0$, $\xi \in \mathbb{R}^d$ and $c > 0$, which leads to the time decay property of L^2-L^1-L^2 type, see [46], where the solution decays like heat kernel at low frequencies, and for the high-frequency part, it decays in times in the price that the additional regularity is assumed on the initial data. Later, Ueda, Duan and Kawashima [44] formulated a new structural condition to analyze the weak dissipation mechanism for generally hyperbolic systems with non-symmetric relaxation (including the Euler-Maxwell system (1.1)). Xu, Mori and Kawashima [54] developed a more general time-decay inequality of L^p-L^q-L^r type, where the minimal decay regularity is available. In the absence of damping term, we refer to those works by Germain and Masmoudi [16] and Deng, Ionescu, Pausader [13], where the global existence of smooth solutions was constructed by using dispersive estimates.

Without the effect of electric and magnetic fields, i.e. $B = E = 0$, the system (1.1) reduces to the isentropic damped compressible Euler equations:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) + \dfrac{1}{\varepsilon}\rho u = 0, \quad x \in \mathbb{R}^d. \end{cases} \tag{1.13}$$

Many advances have been made in the analysis of global solutions to system (1.13). For small initial data being small perturbations in Sobolev spaces $H^s(\mathbb{R}^d)$ $(s > \frac{d}{2} + 1)$, the global well-posedness and asymptotic behaviours of classical solutions for (1.13) have been studied in numerous works. In Sobolev spaces, Sideris, Thomases and Wang [41] and Wang and Tang [48] studied the optimal time-decay rates of solutions. In [6, 29], Coulombel, Goudon and Lin justified the converge of the system, in a diffusive scaling, toward the porous media equation. Then, the fourth author, Wang and Kawashima [50, 55] extended theses results to the framework of inhomogeneous Besov spaces. Recently, the first author and Danchin [7, 8] established the global well-posedness and optimal time-decay rates for (1.13) in the critical homogeneous Besov space $\dot{B}^{\frac{d}{2},\frac{d}{2}+1} = \dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}$.

Both (1.1) and (1.13) can be symmetrized and reformulated as partially dissipative hyperbolic systems of the form

$$\frac{\partial V}{\partial t} + \sum_{j=1}^{d} A^j(V)\frac{\partial V}{\partial x_j} = LV, \tag{1.14}$$

where $V = V(t,x) \in \mathbb{R}^n$ $(n \geq 2)$ is the unknown depending on the time and space variables $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $A^j(V)$ $(j = 1,..,d)$ are symmetric matrices, and L is a nonnegative definite matrix with a nontrivial kernel. From the result of Kato [22], Majda [32] and Serre [39], it is well-known that for smooth initial data, there exist local-in-time solutions to (1.14), and without the relaxation term LV, the solutions may develop singularities (shock waves) in finite time, see the result of Dafermos [11] and Lax [25].

In general, the degenerate relaxation matrix L is not enough to stabilize the whole solution and control the nonlinearities due to its partially dissipative nature. In the case L is symmetric, to overcome the lack of coercivity of the system, Shizuta and Kawashima [40] introduced the celebrated Shizuta-Kawashima condition, which describes the interaction between the hyperbolic and dissipative parts of the system, to stabilize the whole system. Based on this approach, for a global existence result in Sobolev spaces and a proper way of symmetrizing partially dissipative systems, we refer to the works of Kawashima and

Yong [23, 57]. In a similar framework, Bianchini, Hanouzet and Natalini [3] analyzed the Green function and studied the large-time behaviour of solutions. We also refer to the works of the fourth author and Kawashima [51, 52, 53] concerning the well-posedness in the inhomogeneous Besov space $B^{\frac{d}{2}+1}$. More recently, Beauchard and Zuazua [2] framed this phenomenon in the spirit of Villani's hypocoercivity [47] and showed the equivalency of the Shizuta-Kawashima condition and the Kalman rank condition in control theory. We also refer to [7, 8, 9, 12, 52, 53] and references therein for recent developments using Littlewood-Paley decomposition. Among them, under the Shizuta-Kawashima condition, Danchin [12] observed that partially dissipative hyperbolic systems (1.14) can be characterized by a parabolic system and a damped system in the frequency regime $|\xi|\lesssim\varepsilon^{-1}$ and justify the strong relaxation limit of (1.14) with an explicit convergence rate. This method is extended to study the relaxation limit of some Euler-Possion type models [10, 26].

However, the classical theorems by Shizuta and Kawashima [40] and Beauchard and Zuazua [2] may not be applied to the Euler-Maxwell system since it does not fulfill the Shizuta–Kawashima condition or the Kalman rank condition. As observed by many works [15, 46, 43], the 0th-order skew-symmetric part of the relaxation matrix plays a key role in capturing the dissipation of the electromagnetic part and leads to a loss of regularity when the frequencies are away from 0. To the best of our knowledge, there are no results characterizing the dissipation structures of the Euler-Maxwell system with respect to ε.

1.2. **Overview of the paper's findings.** In this paper, we enhance the comprehension of the partially dissipative nature of the Euler-Maxwell system. While this structure has been investigated in existing literature, there is room for improvement in the analysis. We frame the analysis of (1.6) in a hypocoercive way with elaborate dependence with respect to the relaxation parameter ε. This provides us good intuitions on how to deal with systems of the form (1.14) when L is non-symmetric.

First, we extend the current well-posedness theory for (1.6) to a larger class of initial perturbations in the critical homogeneous Besov space $\dot{B}^{\frac{d}{2}-1,\frac{d}{2}+1}=\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}+1}$. The high-frequency (critical) $\dot{B}^{\frac{d}{2}+1}$ assumption on the initial data is necessary to ensure a Lipschitz bound on the solution, which can be obtained via the end-point embedding $\dot{B}^{\frac{d}{2}+1}\hookrightarrow\dot{W}^{1,\infty}$. Without such assumption, we mention the works of Li, Yu and Zhu [27] and Linares, Pilod and Saut [30] concerning the ill-posedness of Burgers-type equations for initial data in H^s with $s<\frac{d}{2}+1$. On the other hand, the choice of $\dot{B}^{\frac{d}{2}-1}$ is due to the heat-like behavior of the system in low frequencies and leads to a $L^1_t(\dot{B}^{\frac{d}{2}+1})$-bound on the solution.

Next, we derive uniform-in-ε a priori estimates for the solution. For $U_{L,\varepsilon}=(\rho-\bar{\rho},\varepsilon u,E,B-\bar{B})$ with (ρ,u,E,B) being the solution to the linearization (3.4) of (1.6), we observe the following new pointwise behaviour (see Proposition 3.1):

$$|U_{L,\varepsilon}(t,\xi)|^2\lesssim e^{\lambda_\varepsilon(|\xi|)t}|U_{L,\varepsilon}(0,\xi)|^2,\qquad \lambda_\varepsilon(|\xi|):=-\frac{c_0|\xi|^2}{(1+\varepsilon^2|\xi|^2)(1+|\xi|^2)},\tag{1.15}$$

where c_0 is a uniform constant. Compared to (1.12), the estimate (1.15) allows us to keep track of the parameter ε. In accordance with (1.15), the behaviour of the solution can be analyzed as follows:

- In the low-frequency region $|\xi|\lesssim 1$, we have $\lambda_\varepsilon(|\xi|)\sim -c_0|\xi|^2$.
- In the medium-frequency region $1\lesssim|\xi|\lesssim 1/\varepsilon$, we have $\lambda_\varepsilon(|\xi|)\sim -c_0$.
- In the high-frequency region $|\xi|\gtrsim 1/\varepsilon$, we have $\lambda_\varepsilon(|\xi|)\sim -\frac{c_0}{|\xi|^2}$.

This reveals different behaviours: the solutions behave like the heat flow in low frequencies; the medium frequencies are exponentially damped; in high frequencies, a loss of derivative occurs. For a more precise analysis of the behavior of each component see (3.5) or Table 1. From this spectral behaviour, it is natural

	$\lvert\xi\rvert\leq 1$	$1\leq\lvert\xi\rvert\leq\frac{C}{\varepsilon}$	$\lvert\xi\rvert\geq\frac{C}{\varepsilon}$
$\rho^\varepsilon-\bar{\rho}$	Damped	Heat	Damped
u^ε	Damped	Damped	Damped
E^ε	Damped	Damped	Regularity-loss
$B^\varepsilon-\bar{B}$	Heat	Damped	Regularity-loss

TABLE 1. Behaviors of each component of the Euler-Maxwell system (3.2)

to split the analysis of the solution in low, medium and high frequencies. In each regime, we develop different hypocoercive methods combined with the Littlewood-Paley theory to capture sharp dissipation rates (refer to Figure 1).

Therefore, we deduce that one needs to decompose the analysis in three frequency regimes to recover the best stability properties for the Euler-Maxwell system. This contrasts with the classical Shizuta-Kawashima theory where only two frequency-regime are usually employed. Here, to derive uniform a priori estimates, following Proposition 3.1, we construct Lyapunov functionals localized in frequencies (using the Littlewood-Paley theory) which encodes enough information to recover the spectrally expected dissipative properties of the solutions, e.g. stated in Table 1. Then, to deal with the nonlinear terms, we perform energy estimates at different regularity levels in low, medium and high frequencies (see Figure 1). Moreover, in Subsection 3.2, we use product, composition and commutator estimates, adapted to the frequency decomposition, to control the nonlinearities. In particular, our functional framework allows us to track the regularity evolution and obtain uniform estimates with respect to the relaxation parameter ε.

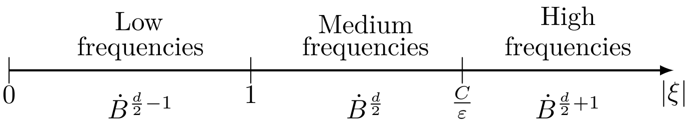

FIGURE 1. Frequency splitting and regularity for the Euler-Maxwell system (3.2).

With these uniform estimates in hand, we establish quantitative error estimates between the solution of the Euler-Maxwell system (1.6) and the drift-diffusion system (1.9). Note that, as the relaxation parameter $\varepsilon \to 0$, the frequency regime $|\xi| \lesssim \varepsilon^{-1}$ will cover the whole frequencies and the high-frequency regime will disappear. This is coherent as the density ρ^ε in the low and medium frequencies behaves similarly to the solution ρ^* of the limit drift-diffusion system (cf. Figure 2). To establish the error estimates, the key ingredient in our proof is the introduction of the effective velocity (damped mode)

$$z^\varepsilon := u^\varepsilon + \nabla h(\rho^\varepsilon) + E^\varepsilon + \varepsilon u^\varepsilon \times \bar{B}, \tag{1.16}$$

which is associated with Darcy's law (1.10). Based on the $\mathcal{O}(\varepsilon)$-bounds on this effective velocity z^ε, we are able to obtain an explicit convergence rate in ill-prepared scenarios.

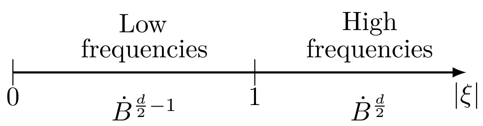

FIGURE 2. Frequency splitting for the drift-diffusion model (1.9).

1.3. **Outline of the paper.** The rest of the paper unfolds as follows. In Section 3.2, we derive uniform a-priori estimates for (3.2), and then based on these a-priori estimates, we prove the global well-posedness of the Cauchy problem (1.6)-(1.7) (Theorem 2.1) in Section 3. Section 4 is dedicated to the justification of the strong relaxation limit from (1.6) to (1.9) (Theorems 2.3 and 2.4). Appendix A collects some technical lemmas that are used throughout the text.

2. PRELIMINARIES AND MAIN RESULTS

2.1. **Notations and functional spaces.** Before stating our main results, we explain the notations and definitions used throughout this paper. $C > 0$ denotes a constant independent of ε and time, $f \lesssim g$ (resp. $f \gtrsim g$) means $f \leq Cg$ (resp. $f \geq Cg$), and $f \sim g$ stands for $f \lesssim g$ and $f \gtrsim g$. For any Banach space X and the functions $f, g \in X$, let $\|(f,g)\|_X := \|f\|_X + \|g\|_X$. For any $T > 0$ and $1 \leq \varrho \leq \infty$, we denote by $L^\varrho(0,T;X)$ the set of measurable functions $g : [0,T] \to X$ such that $t \mapsto \|g(t)\|_X$ is in $L^\varrho(0,T)$ and write $\|\cdot\|_{L^\varrho(0,T;X)} := \|\cdot\|_{L^\varrho_T(X)}$. $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for the Fourier transform and its inverse. In addition, let Λ^σ be defined by $\Lambda^\sigma f := \mathcal{F}^{-}(|\xi|^\sigma \mathcal{F} f)$. Then we have $\Lambda^2 = -\Delta$ and $\Lambda^{-2} = (-\Delta)^{-1}$.

We present succinctly the notations of the Littlewood-Paley decomposition and Besov spaces that we employ in this manuscript. The reader can refer to [1, Chapter 2] for a complete overview. Choose a smooth radial non-increasing function $\chi(\xi)$ with compact supported in $B(0,4/3)$ and $\chi(\xi)=1$ in $B(0,3/4)$ such that

$$\varphi(\xi):=\chi(\frac{\xi}{2})-\chi(\xi),\quad \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\cdot)=1,\quad \mathrm{Supp}\ \varphi\subset\{\xi\in\mathbb{R}^d\ :\ \frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}.$$

For any $j\in\mathbb{Z}$, the usual homogeneous dyadic blocks $\dot{\Delta}_j$ and the low-frequency cut-off operator $\dot{S}_j$ are defined by

$$\dot{\Delta}_j u:=\mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}u),\qquad \dot{S}_j u:=\mathcal{F}^{-1}(\chi(2^{-j}\cdot)\mathcal{F}u).$$

The homogeneous Besov space $\dot{B}^s_{p,r}$ for any $p,r\in[1,\infty]$ and $s\in\mathbb{R}$ is defined by

$$\dot{B}^s_{p,r}:=\{u\in\mathcal{S}'_h\ :\ \|u\|_{\dot{B}^s_{p,r}}:=\|\{2^{js}\|u_j\|_{L^p}\}_{j\in\mathbb{Z}}\|_{l^r}<\infty\}.$$

From now on, we use the shorthand notations

$$\dot{\Delta}_j u=u_j,\qquad \dot{B}^s=\dot{B}^s_{2,1}.$$

We also introduce the hybrid Besov spaces

$$\dot{B}^{s_1,s_2}:=\{u\in\mathcal{S}'_h\ :\ \|u\|_{\dot{B}^{s_1,s_2}}:=\sum_{j\leq 0}2^{js_1}\|u_j\|_{L^2}+\sum_{j\geq -1}2^{js_2}\|u_j\|_{L^2}<\infty\}.$$

We have

$$\begin{aligned}\dot{B}^{s_1,s_2}&=\dot{B}^{s_1} &&\text{if}\quad s_1=s_2,\\ \dot{B}^{s_1,s_2}&=\dot{B}^{s_1}\cap\dot{B}^{s_2} &&\text{if}\quad s_1<s_2,\\ \dot{B}^{s_1,s_2}&=\dot{B}^{s_1}+\dot{B}^{s_2} &&\text{if}\quad s_1>s_2.\end{aligned}$$

One of our key ideas involves partitioning the frequency space into three distinct regions. In each of these regimes, the solution demonstrates significantly different behaviours from the others, prompting the development of distinct methods for each. In this regard, we set the threshold J_ε between medium and high frequencies as

$$J_\varepsilon=-[\log_2\varepsilon]-k_0, \tag{2.1}$$

such that $2^{J_\varepsilon}\sim 1/\varepsilon$ and k_0 is a sufficiently large integer independent of ε chosen in Proposition 3.6. Then, we define the frequency-restricted Besov spaces

$$\|u\|^\ell_{\dot{B}^s}:=\sum_{j\leq 0}2^{js}\|u_j\|_{L^2},\quad \|u\|^m_{\dot{B}^s}:=\sum_{-1\leq j\leq J_\varepsilon}2^{js}\|u_j\|_{L^2},\quad \|u\|^h_{\dot{B}^s}:=\sum_{j\geq J_\varepsilon-1}2^{js}\|u_j\|_{L^2}.$$

Analogously, we decompose $u=u^\ell+u^m+u^h$ as

$$u^\ell:=\sum_{j\leq -1}u_j,\qquad u^m:=\sum_{0\leq j\leq J_\varepsilon-1}u_j,\qquad u^h:=\sum_{j\geq J_\varepsilon}u_j.$$

Note that using Young's and Bernstein's inequalities, it is easy to see that

$$\|u^\ell\|_{\dot{B}^s}\lesssim\|u\|^\ell_{\dot{B}^s},\quad \|u^m\|_{\dot{B}^s}\lesssim\|u\|^m_{\dot{B}^s},\quad \|u^h\|_{\dot{B}^s}\lesssim\|u^h\|_{\dot{B}^s},$$

and for any $s'>0$

$$\begin{cases}\|u\|^\ell_{\dot{B}^s}\lesssim\|u\|^\ell_{\dot{B}^{s-s'}}, & \|u\|^m_{\dot{B}^s}\lesssim\|u\|^m_{\dot{B}^{s+s'}},\\ \|u\|^m_{\dot{B}^s}\lesssim 2^{J_\varepsilon s'}\|u\|^m_{\dot{B}^{s-s'}}, & \|u\|^h_{\dot{B}^s}\lesssim 2^{-J_\varepsilon s'}\|u\|^h_{\dot{B}^{s+s'}}.\end{cases} \tag{2.2}$$

Furthermore, we denote the Chemin-Lerner type space $\widetilde{L}^\varrho(0,T;\dot{B}^s_{p,r})$ by the function set in $L^\varrho(0,T;\mathcal{S}'_h)$ endowed with the norm

$$\|u\|_{\widetilde{L}^\varrho_T(\dot{B}^s)}:=\begin{cases}\sum_{j\in\mathbb{Z}}2^{js}\|u_j\|_{L^\varrho_T(L^p)}<\infty, & \text{if } 1\leq\varrho<\infty,\\ \sum_{j\in\mathbb{Z}}2^{js}\sup_{t\in[0,T]}\|u_j\|_{L^p}<\infty, & \text{if } \varrho=\infty.\end{cases}$$

By the Minkowski inequality, it holds that

$$\|u\|_{\widetilde{L}^1_T(\dot{B}^s)}=\|u\|_{L^1_T(\dot{B}^s)}\quad\text{and}\quad \|u\|_{\widetilde{L}^\varrho_T(\dot{B}^s)}\geq\|u\|_{L^\varrho_T(\dot{B}^s)}\ \text{for}\ \varrho>1,$$

where $\|\cdot\|_{L^{\varrho}_T(\dot B^s)}$ is the usual Lebesgue-Besov norm. We omit the similar definitions of time-space hybrid spaces and the ones restricted in low, medium and high frequencies.

2.2. **Main results.** Before stating our main results, we introduce the energy functional

$$\mathcal{E}(a,u,E,H):=\|(a,\varepsilon u,E,H)\|_{\widetilde{L}^{\infty}_t(\dot B^{\frac d2-1,\frac d2})}+\varepsilon\|(a,\varepsilon u,E,H)\|^h_{\widetilde{L}^{\infty}_t(\dot B^{\frac d2+1})},\tag{2.3}$$

and the dissipation functional

$$\begin{aligned}\mathcal{D}(a,u,E,H)=&\|a\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1,\frac d2+1})}+\|a\|^{\ell}_{L^1_t(\dot B^{\frac d2})}+\|a\|^m_{L^1_t(\dot B^{\frac d2+1})}\\&+\|u\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1,\frac d2})}+\varepsilon\|u\|^h_{\widetilde{L}^2_t(\dot B^{\frac d2+1})}+\|u\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})}+\|u\|^m_{L^1_t(\dot B^{\frac d2-1})}\\&+\|E\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1,\frac d2})}+\|E\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})}+\|E\|^m_{L^1_t(\dot B^{\frac d2})}\\&+\|H\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}+\|H\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})}+\|H\|^m_{L^1_t(\dot B^{\frac d2})}.\end{aligned}\tag{2.4}$$

We also denote the initial energy

$$\mathcal{E}^{\varepsilon}_0:=\|(\rho^{\varepsilon}_0-\bar\rho,u^{\varepsilon}_0,E^{\varepsilon}_0,B^{\varepsilon}_0-\bar B)\|_{\dot B^{\frac d2-1,\frac d2}}+\varepsilon\|(\rho^{\varepsilon}_0-\bar\rho,u^{\varepsilon}_0,E^{\varepsilon}_0,B^{\varepsilon}_0-\bar B)\|^h_{\dot B^{\frac d2+1}}.\tag{2.5}$$

In our first theorem, we prove the global existence and uniqueness of classical solutions and derive uniform regularity estimates with respect to the relaxation parameter ε.

Theorem 2.1. *Let $0<\varepsilon\le\varepsilon_0$ with a suitably small constant ε_0. There exists a constant α_0 independent of ε such that if the initial datum $(\rho^{\varepsilon}_0,u^{\varepsilon}_0,E^{\varepsilon}_0,B^{\varepsilon}_0)$ satisfies*

$$\mathcal{E}^{\varepsilon}_0\le\alpha_0,\tag{2.6}$$

then the Cauchy problem (1.6)-(1.7) admits a unique global classical solution $(\rho^{\varepsilon},u^{\varepsilon},E^{\varepsilon},B^{\varepsilon})$ satisfying $(\rho^{\varepsilon}-\bar\rho,u^{\varepsilon},E^{\varepsilon},B^{\varepsilon}-\bar B)\in\mathcal{C}(\mathbb{R}^+;\dot B^{\frac d2-1,\frac d2+1})$ and

$$\mathcal{E}(\rho^{\varepsilon}-\bar\rho,u^{\varepsilon},E^{\varepsilon},B^{\varepsilon}-\bar B)+\mathcal{D}(\rho^{\varepsilon}-\bar\rho,u^{\varepsilon},E^{\varepsilon},B^{\varepsilon}-\bar B)\le C\mathcal{E}^{\varepsilon}_0,\quad t\in\mathbb{R}_+.\tag{2.7}$$

In addition, it holds that

$$\|z^{\varepsilon}\|^{\ell}_{L^1_t(\dot B^{\frac d2})}+\|z^{\varepsilon}\|^m_{L^1_t(\dot B^{\frac d2-1})}+\|z^{\varepsilon}\|^h_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\le C\mathcal{E}^{\varepsilon}_0\varepsilon,\quad t\in\mathbb{R}_+,\tag{2.8}$$

and

$$\|z^{\varepsilon}\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\lesssim\big(\mathcal{E}^{\varepsilon}_0+\|z^{\varepsilon}_0\|_{\dot B^{\frac d2-1}}\big)\varepsilon,\quad t\in\mathbb{R}_+,\tag{2.9}$$

where $C>0$ is a constant independent of ε and time, and z^{ε} is the effective velocity given by (1.16) with its initial value $z^{\varepsilon}_0:=\frac1\varepsilon u^{\varepsilon}_0+\nabla h(\rho^{\varepsilon}_0)+u^{\varepsilon}_0\times\bar B$.

Remark 2.2. Some remarks are in order.

- Theorem 2.1 extends the previous results about the global well-posedness of the Euler-Maxwell system to a broader functional framework. This can be seen with the following chain of embeddings
$$H^s(s>\frac d2+1)\hookrightarrow B^{\frac d2+1}\hookrightarrow\dot B^{\frac d2-1,\frac d2+1}(d=3)\hookrightarrow\mathcal{C}^1\cap W^{1,\infty}.$$
- In the limit $\varepsilon\to0$, the regularity properties obtained in Theorem 2.1 match the properties of the solution for the limit drift-diffusion model given in Theorem 4.1. This is consistent with the relaxation limit process described in Theorem 2.3.
- In low and medium frequencies, we obtain (end-point) L^1-in-time estimates for the solutions in (2.7). However, in high frequencies, due to the regularity-loss property of B^{ε} when estimating the nonlinear term $u^{\varepsilon}\times B^{\varepsilon}$ associated to the Lorentz force, we only obtain L^2-in-time estimates.
- It should be noted that the effective velocity z^{ε}, related to Darcy's law (1.10), verifies stronger regularity properties than the whole solution. This fact is key in proving our relaxation result: Theorem 2.3.

Next, we justify rigorously the relaxation limit from (1.6) to (1.8) and exhibit quantitative convergence estimates which are uniform with respect to time. Below, We state our relaxation limit result for ill-prepared initial data.

Theorem 2.3 (Ill-prepared case)**.** *Let $0<\varepsilon\leq\varepsilon_0$ with a suitably small constant ε_0. Let $(\rho^\varepsilon,u^\varepsilon,E^\varepsilon,B^\varepsilon)$ and ρ^* be the solutions to the Cauchy problems (1.6)-(1.7) and (1.9) obtained from Theorems 2.1 and 4.1 and associated to the initial data $(\rho_0^\varepsilon,u_0^\varepsilon,E_0^\varepsilon,B_0^\varepsilon)$ and ρ_0^* respectively. Let $E^*=\nabla(-\Delta)^{-1}(\rho^*-\bar\rho)$ with its initial datum $E_0^*=\nabla(-\Delta)^{-1}(\rho_0^*-\bar\rho)$, and $B^*=\bar B$.*

Suppose that

$$\|E_0^\varepsilon-E_0^*\|_{\dot B^{\frac d2,\frac d2-1}}+\|B_0^\varepsilon-B^*\|_{\dot B^{\frac d2,\frac d2-1}}\leq\varepsilon. \tag{2.10}$$

Then for all $t\in\mathbb{R}_+$, we have

$$\begin{aligned}&\|\rho^\varepsilon-\rho^*\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1,\frac d2-2})\cap\widetilde L^2_t(\dot B^{\frac d2-1})}+\|E^\varepsilon-E^*\|_{\widetilde L^\infty_t(\dot B^{\frac d2,\frac d2-1})\cap\widetilde L^2_t(\dot B^{\frac d2,\frac d2-1})}\\&\qquad+\|B^\varepsilon-B^*\|_{\widetilde L^\infty_t(\dot B^{\frac d2,\frac d2-1})\cap\widetilde L^2_t(\dot B^{\frac d2+1,\frac d2-1})}\leq C_1\varepsilon,\end{aligned} \tag{2.11}$$

where $C_1>0$ is a constant independent of ε and time.

We also prove stronger convergence estimates for well-prepared initial data.

Theorem 2.4 (Well-prepared case)**.** *Let $0<\varepsilon\leq\varepsilon_0$ with a suitably small constant ε_0. Let $(\rho^\varepsilon,u^\varepsilon,E^\varepsilon,B^\varepsilon)$ and ρ^* be the solutions to the Cauchy problems (1.6)-(1.7) and (1.9) obtained from Theorems 2.1 and 4.1 and associated to the initial data $(\rho_0^\varepsilon,u_0^\varepsilon,E_0^\varepsilon,B_0^\varepsilon)$ and ρ_0^* respectively. Let $E^*=\nabla(-\Delta)^{-1}(\rho^*-\bar\rho)$ with its initial datum $E_0^*=\nabla(-\Delta)^{-1}(\rho_0^*-\bar\rho)$, and $B^*=\bar B$. If we assume that*

$$\|E_0^\varepsilon-E_0^*\|_{\dot B^{\frac d2}}+\|B_0^\varepsilon-B^*\|_{\dot B^{\frac d2-1}}\leq\varepsilon, \tag{2.12}$$

and

$$\|z_0^\varepsilon\|_{\dot B^{\frac d2-1}}\leq 1, \tag{2.13}$$

where $z_0^\varepsilon:=\frac1\varepsilon u_0^\varepsilon+\nabla h(\rho_0^\varepsilon)+u_0^\varepsilon\times\bar B$, then, for all $t\in\mathbb{R}_+$, the convergence holds in stronger norms, namely

$$\begin{aligned}&\|\rho^\varepsilon-\rho^*\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1})\cap\widetilde L^2_t(\dot B^{\frac d2-1,\frac d2})}+\|u^\varepsilon-u^*\|_{\widetilde L^2_t(\dot B^{\frac d2,\frac d2-1})}\\&\qquad+\|E^\varepsilon-E^*\|_{\widetilde L^\infty_t(\dot B^{\frac d2})\cap\widetilde L^2_t(\dot B^{\frac d2})}+\|B^\varepsilon-B^*\|_{\widetilde L^\infty_t(\dot B^{\frac d2})\cap\widetilde L^2_t(\dot B^{\frac d2+1,\frac d2})}\leq C_2\varepsilon,\end{aligned} \tag{2.14}$$

where $C_2>0$ is a constant independent of ε and time.

Remark 2.5. Some comments on Theorem 2.3 are in order.

- Concerning the regularity assumption on the initial data in Theorem 2.3: we do not impose conditions on the initial error $\rho_0^\varepsilon-\rho_0^*$. In fact, the regularity of the density and the electric field are linked since (1.4) implies

$$\rho_0^\varepsilon-\rho_0^*=-\mathrm{div}\,(E_0^\varepsilon-E_0^*). \tag{2.15}$$

- The condition (2.13) on z_0^ε is equivalent to the fact that u_0^ε has a $\mathcal{O}(\varepsilon)$-bound in $\dot B^{\frac d2-1}$. It enables us to establish additional regularity estimates for the effective velocity z^ε and obtain a stronger convergence result.
- We are able to establish the strong convergence for high-order norms. For example, using a interpolation argument between (2.7) and (2.14), for all $\sigma\in[\frac d2-1,\frac d2)$, one has

$$\begin{aligned}&\|\rho^\varepsilon-\rho^*\|_{L^\infty(\mathbb{R}_+;\dot B^\sigma)\cap L^2(\mathbb{R}_+;\dot B^{\sigma+1})}+\|u^\varepsilon-u^*\|_{L^2(\mathbb{R}_+;\dot B^\sigma)}\\&\quad+\|E^\varepsilon-E^*\|_{L^\infty(\mathbb{R}_+;\dot B^\sigma)\cap L^2(\mathbb{R}_+;\dot B^\sigma)}+\|B^\varepsilon-B^*\|_{L^\infty(\mathbb{R}_+;\dot B^\sigma)\cap L^2(\mathbb{R}_+;\dot B^{\sigma+1,\frac d2})}\xrightarrow[\varepsilon\to0]{}0.\end{aligned}$$

We now explain the strategies for establishing the error estimates of the relaxation limit for (1.6). Different from the damped Euler equations [9] or Euler-Possion type models [10, 26], we need to overcome the regularity-loss phenomenon and carry out convergence estimates on the error unknowns in three frequency-regimes due to the elaborate dissipative structures of solutions. In the low and medium frequencies, our key step is to establish maximal decay estimates of the effective velocity z^ε defined in (1.16) associated to the L^1_t-regularity estimates of solutions in (2.7). To this end, we decouple the equations $(1.6)_1$-$(1.6)_3$ and treat the terms involving H^ε as source terms. In fact, the effective velocity z^ε allows us to reformulate the density equation $(1.6)_1$ as

$$\partial_t\rho^\varepsilon - P'(\bar\rho)\Delta\rho^\varepsilon + \bar\rho\rho^\varepsilon = -\bar\rho\,\mathrm{div}\, z^\varepsilon + \varepsilon\bar\rho u^\varepsilon \times \bar B + \text{nonlinear terms}, \tag{2.16}$$

which has a structure similar to the drift-diffusion model (1.9). Then, we have to derive decay-in-ε for the remainder term $\bar\rho\,\mathrm{div}\, z^\varepsilon$ to justify the relaxation limit. The effective velocity[1] z^ε satisfies the damped equation

$$\varepsilon\partial_t z^\varepsilon + \frac{1}{\varepsilon}z^\varepsilon = \varepsilon\nabla\partial_t n^\varepsilon + \varepsilon\partial_t E^\varepsilon - z^\varepsilon \times \bar B + \text{nonlinear terms}. \tag{2.17}$$

To handle the $\varepsilon\partial_t E^\varepsilon$ term on the right-hand side of (2.17), we observe that

$$\varepsilon\partial_t E^\varepsilon + \varepsilon\bar\rho E^\varepsilon = \varepsilon\bar\rho z^\varepsilon + \varepsilon\bar\rho\nabla n^\varepsilon - \varepsilon^2\bar\rho(z^\varepsilon - \nabla n^\varepsilon - E^\varepsilon)\times\bar B + \nabla H^\varepsilon + \text{nonlinear terms}. \tag{2.18}$$

Given the L^1_t-regularity estimate of ∇H^ε on the right-hand side of (2.18), we are able to treat (2.16)-(2.18) separately as purely dissipative equations, where the higher order linear terms on the right-hand sides can be absorbed if the threshold J_ε takes the form (2.1) with a k_0 chosen small enough. This enables us to obtain the $\mathcal{O}(\varepsilon)$-bound (2.8) for the L^1_t-regularity of z^ε.

Let $(\delta\rho, \delta u, \delta E, \delta B) := (\rho^\varepsilon - \rho^*, u - u^*, E - E^*, B - B^*)$ be the error unknowns. Regarding the low and medium frequencies, we observe that the error $\delta\rho$ satisfies

$$\partial_t\delta\rho - P'(\bar\rho)\Delta\delta\rho + \bar\rho\delta\rho = -\bar\rho\,\mathrm{div}\, z^\varepsilon + \varepsilon\bar\rho\,\mathrm{div}\,(u^\varepsilon\times\bar B) + \text{nonlinear terms}. \tag{2.19}$$

Thence, the error estimate of $\delta\rho$ essentially comes from maximal regularity estimates for (2.19) and the $\mathcal{O}(\varepsilon)$-bound (2.8) for the effective velocity z^ε. To handle $(\delta E, \delta B)$, we observe the following partially dissipative equations:

$$\begin{cases} \partial_t\delta E - \dfrac{1}{\varepsilon}\nabla\times\delta B + \bar\rho\delta E = \nabla\times B^{1,*} + \bar\rho z^\varepsilon - \varepsilon\bar\rho u^\varepsilon\times\bar B - P'(\bar\rho)\nabla\delta\rho + \text{nonlinear terms},\\ \partial_t\delta B + \dfrac{1}{\varepsilon}\nabla\times\delta E = 0,\\ \mathrm{div}\,\delta E = -\delta\rho, \qquad \mathrm{div}\,\delta B = 0, \end{cases} \tag{2.20}$$

where the term $B^{1,*} = -(-\Delta)^{-1}\nabla\times(\rho^* u^*)$ causes an additional difficulty in the analysis of (2.20). To overcome it, we introduce the modified error unknown

$$\delta\mathcal{B} := \delta B + \varepsilon B^{1,*}.$$

By employing a hypocoercive argument for $(\delta E, \delta\mathcal{B})$, we are able to recover the error estimates of $(\delta E, \delta B)$. Furthermore, the convergence estimates for high-frequency norms can be deduced from the uniform regularity estimate established in (2.7) and the cut-off properties in (2.2). With these observations, we prove the desired converge estimate (2.11), cf. Subsection 4.1.

Finally, under the stronger assumption (2.13), deriving L^2-type estimates from (2.17), one can establish the stronger decay estimate (2.9) of the effective velocity z^ε in the whole frequency space, which enables us to perform the L^2_t-type energy argument on (2.19) and (2.20) to obtain the stronger convergence estimate (2.14) in Theorem 2.3, cf. Subsection 4.2.

3. Global well-posedness for the Euler-Maxwell system

This section is devoted to the proof of Theorem 2.1. In this section, we omit the superscript ε in the solution (ρ, u, E, H) for (1.6) to lighten the notations. We introduce the perturbations of n and H as

$$n := h(\rho), \qquad H = B - \bar B. \tag{3.1}$$

[1]The effective velocity z^ε is reminiscent of the "good unknown" introduced by Hoff and Haspot in [18, 19] to treat the compressible Navier-Stokes system and by the first two authors in [7, 8, 9] to analyze the compressible Euler type equations with damping.

Due to (**??**) and the fact that ρ is close to the positive constant $\bar{\rho}$, the implicit function theorem implies that the density ρ is a C^5 function in a small neighborhood of 0 with respect to the enthalpy $n = h(\rho)$. The system (1.6) can be rewritten as

$$
\begin{cases}
\partial_t n + u\cdot\nabla n + (P'(\bar{\rho}) + G(n))\operatorname{div} u = 0,\\
\varepsilon^2(\partial_t u + u\cdot\nabla u) + \nabla n + E + u + \varepsilon u\times\bar{B} = -\varepsilon u\times H,\\
\varepsilon\partial_t E - \nabla\times H - \varepsilon\bar{\rho}u = \varepsilon F(n)u,\\
\varepsilon\partial_t H + \nabla\times E = 0,\\
\operatorname{div} E = -Kn - \Phi(n),\\
\operatorname{div} H = 0,\\
(n,u,E,H)(0,x) = (n_0^\varepsilon, \frac{1}{\varepsilon}u_0^\varepsilon, E_0^\varepsilon, H_0^\varepsilon)(x),
\end{cases}
\tag{3.2}
$$

with

$$
\begin{cases}
n_0^\varepsilon := h(\rho_0^\varepsilon), \quad H_0^\varepsilon := B_0^\varepsilon - \bar{B},\\
K := \rho'(0) = \dfrac{\bar{\rho}}{P'(\bar{\rho})},\\
G(n) := P'(\rho) - P'(\bar{\rho}), \qquad F(n) := \rho - \bar{\rho}, \qquad \Phi(n) := \rho - \bar{\rho} - Kn.
\end{cases}
\tag{3.3}
$$

Remark that $\Phi(n)$ is a quadratic nonlinear term with respect to n. Such formulation allows us to consider general pressure laws.

3.1. **Pointwise estimates for the linear Euler-Maxwell system.** To better understand the dissipative structures of the solutions to (3.2), we provide pointwise estimates of the linearized system

$$
\begin{cases}
\partial_t n + P'(\bar{\rho})\operatorname{div} u = 0,\\
\varepsilon^2\partial_t u + \nabla n + E + u + \varepsilon u\times\bar{B} = 0,\\
\varepsilon\partial_t E - \nabla\times H - \varepsilon\bar{\rho}u = 0,\\
\varepsilon\partial_t H + \nabla\times E = 0,\\
\operatorname{div} E = -Kn, \qquad \operatorname{div} H = 0.
\end{cases}
\tag{3.4}
$$

Since the seminal works by [40, 42], significant advancements have been made in employing the energy method within Fourier spaces for analyzing various types of hyperbolic systems. These include systems pertaining to viscoelasticity, radiating gas, compressible Euler-Maxwell equations, Timoshenko systems, among others (see [14, 20, 45, 46, 24] and references therein for comprehensive discussions). Additionnaly, [2] offers valuable insights into the perspective of hypocoercivity, while [44] delves into hyperbolic systems featuring non-symmetric dissipation.

In this study, we introduce a parameter-dependent energy method using Fourier decomposition tailored for (3.4) (see (3.5)-(3.6) below). These newly elucidated pointwise frequency behaviors provide insights into the evolution of the dissipation rate concerning the relaxation parameter ε.

Proposition 3.1. *For all $0<\varepsilon\le 1$, let (n,u,E,H) be a solution to system (3.4). Then, there exists a functional $\mathcal{L}_\xi(t)\sim|(\widehat{n},\varepsilon\widehat{u},\widehat{E},\widehat{H})(t,\xi)|^2$ and a constant $c_0 = c_0(\bar{\rho},\bar{B},P'(\bar{\rho}))>0$ such that*

$$
\frac{d}{dt}\mathcal{L}_\xi(t) + c_0|\widehat{u}|^2 + \frac{c_0(1+|\xi|^2)}{1+\varepsilon^2|\xi|^2}|\widehat{n}|^2 + \frac{c_0}{1+\varepsilon^2|\xi|^2}|\widehat{E}|^2 + \frac{c_0|\xi|^2}{(1+\varepsilon^2|\xi|^2)(1+|\xi|^2)}|\widehat{H}|^2 \le 0.
\tag{3.5}
$$

Furthermore, we have

$$
|(\widehat{n},\varepsilon\widehat{u},\widehat{E},\widehat{H})(t,\xi)|^2 \lesssim e^{\lambda_\varepsilon(|\xi|)t}(\widehat{n},\varepsilon\widehat{u},\widehat{E},\widehat{H})(0,\xi)|^2, \qquad t>0, \quad \xi\in\mathbb{R}^d,
\tag{3.6}
$$

where $\lambda_\varepsilon(|\xi|)$ is given by

$$
\lambda_\varepsilon(|\xi|) = -\frac{c_0|\xi|^2}{(1+\varepsilon^2|\xi|^2)(1+|\xi|^2)}.
$$

3.2. **Uniform regularity estimates for the problem** (3.2)**.** In the next proposition, we derive uniform a priori estimates for the Cauchy problem (3.2).

Proposition 3.2. *For given time $T>0$, suppose that, for $t\in(0,T)$, (n,u,E,H) is a classical solution to the Cauchy problem* (3.2)*. Let*

$$\mathcal{X}(t):=\mathcal{E}(n,u,E,H)+\mathcal{D}(n,u,E,H), \tag{3.7}$$

with $\mathcal{E}$ and $\mathcal{D}$ defined in (2.3) *and* (2.4)*. Under the assumption*

$$\|n\|_{L^\infty_t(L^\infty)}\ll 1,\qquad 0<t<T, \tag{3.8}$$

it holds that

$$\mathcal{X}(t)\le C_0(\mathcal{X}_0+\mathcal{X}(t)^2+\mathcal{X}(t)^3),\qquad 0<t<T, \tag{3.9}$$

where $C_0>0$ is a constant independent of T and ε.

Proposition 3.2 is a direct consequence of Lemmas 3.3, 3.4 and 3.5 dedicated to the analysis of the low, medium and high-frequency regimes respectively.

3.2.1. *Low-frequency analysis.* First, we derive low-frequency a priori estimates.

Lemma 3.3. *Let (n,u,E,H), for $t\in(0,T)$, be a classical solution to the Cauchy problem* (3.2) *satisfying* (3.8)*. It holds that*

$$\begin{aligned}&\|(n,\varepsilon u,E,H)\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac d2-1})}+\|(n,u,E,H)\|^\ell_{L^1_t(\dot{B}^{\frac d2+1})}\\&\quad+\|(n,u,E)\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac d2-1,\frac d2+1})}+\|H\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac d2}\cap\dot{B}^{\frac d2+1})}\lesssim\mathcal{E}^\varepsilon_0+\mathcal{X}(t)^2.\end{aligned} \tag{3.10}$$

Proof. First, we construct a localized Lyapunov inequality in the low-frequency regime. We aim to show that for all $j\le 0$, there exists a functional $\mathcal{L}^\ell_j(t)\sim\|(n_j,\varepsilon u_j,E_j,H_j)\|^2_{L^2}$ such that

$$\frac{d}{dt}\mathcal{L}^\ell_j(t)+2^{2j}\mathcal{L}^\ell_j(t)+\|u_j\|^2_{L^2}+\|(n_j,E_j)\|^2_{L^2}+2^{2j}\|H_j\|^2_{L^2}\lesssim G^\ell_j(t)\sqrt{\mathcal{L}^\ell_j(t)} \tag{3.11}$$

with

$$G^\ell_j(t):=\|\dot\Delta_j(u\cdot\nabla n,G(n)\mathrm{div}\,u,\varepsilon u\cdot\nabla u,u\times H,\varepsilon F(n)u,\Phi(n))\|_{L^2}.$$

To achieve it, applying the frequency-localization operator $\dot\Delta_j$ to (3.2), we obtain

$$\begin{cases}\partial_t n_j+P'(\bar\rho)\mathrm{div}\,u_j=-\dot\Delta_j(u\cdot\nabla n)-\dot\Delta_j(G(n)\mathrm{div}\,u),\\ \varepsilon^2\partial_t u_j+\nabla n_j+E_j+u_j+\varepsilon u_j\times\bar B=-\dot\Delta_j(\varepsilon^2u\cdot\nabla u)-\dot\Delta_j(\varepsilon u\times H),\\ \varepsilon\partial_t E_j-\nabla\times H_j-\varepsilon\bar\rho u_j=\dot\Delta_j(\varepsilon F(n)u),\\ \varepsilon\partial_t H_j+\nabla\times E_j=0,\\ \mathrm{div}\,E_j=-Kn_j-\dot\Delta_j\Phi(n),\qquad \mathrm{div}\,H_j=0.\end{cases} \tag{3.12}$$

Taking the L^2-inner product of $(3.12)_1$ with n_j, we have

$$\frac12\frac{d}{dt}\int|n_j|^2\,dx+P'(\bar\rho)\int\mathrm{div}\,u_jn_j\,dx\le(\|\dot\Delta_j(u\cdot\nabla n)\|_{L^2}+\|\dot\Delta_j(G(n)\mathrm{div}\,u)\|_{L^2})\|n_j\|_{L^2}. \tag{3.13}$$

To cancel the second term on the left-hand side of (3.13), we take the L^2-inner product of $(3.12)_2$ with $P'(\bar\rho)u_j$ and use $(u_j\times\bar B)\cdot u_j=(u_j\times u_j)\cdot\bar B=0$ so that

$$\begin{aligned}&\frac{P'(\bar\rho)\varepsilon^2}{2}\frac{d}{dt}\int|u_j|^2\,dx+P'(\bar\rho)\int\nabla n_j\cdot u_j\,dx+P'(\bar\rho)\int|u_j|^2\,dx+P'(\bar\rho)\int E_j\cdot u_j\,dx\\&\le P'(\bar\rho)\|\dot\Delta_j(\varepsilon u\cdot\nabla u,u\times H)\|_{L^2}\varepsilon\|u_j\|_{L^2}.\end{aligned} \tag{3.14}$$

In addition, one deduces from $(3.12)_3$-$(3.12)_4$ that

$$\frac{P'(\bar\rho)}{2\bar\rho}\frac{d}{dt}\|(E_j,H_j)\|^2_{L^2}-P'(\bar\rho)\int u_j\cdot E_j\,dx\le\frac1K\|\dot\Delta_j(F(n)u)\|_{L^2}\|E_j\|_{L^2}, \tag{3.15}$$

where one has used

$$\int (\nabla\times f)\cdot g-(\nabla\times g)\cdot f\,dx=\int \mathrm{div}\,(f\times g)\,dx=0,\qquad \forall f,g\in\mathcal{S}'(\mathbb{R}^3).$$

Combining (3.13)-(3.15) together, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\big(|n_j|^2+P'(\bar\rho)\varepsilon^2|u_j|^2+\frac{1}{K}|E_j|^2+\frac{1}{K}|H_j|^2\big)\,dx+P'(\bar\rho)\int|u_j|^2\,dx\\ &\le(\|\dot\Delta_j(u\cdot\nabla n)\|_{L^2}+\|\dot\Delta_j(G(n)\mathrm{div}\,u)\|_{L^2})\|n_j\|_{L^2}\\ &\quad+P'(\bar\rho)\|\dot\Delta_j(\varepsilon u\cdot\nabla u,u\times H)\|_{L^2}\varepsilon\|u_j\|_{L^2}+\frac{1}{K}\|\dot\Delta_j(F(n)u)\|_{L^2}\|E_j\|_{L^2}.\end{aligned}\tag{3.16}$$

In order to derive the dissipative effect for n_j, we multiply $(3.12)_2$ by ∇n_j, make use of $(3.12)_1$ and integrate by parts. In fact, by $\mathrm{div}\,E_j=-Kn_j-\dot\Delta_j\Phi(n)$, we see that n_j satisfies

$$\int E_j\cdot\nabla n_j\,dx=-\int\mathrm{div}\,E_jn_j\,dx=K\|n_j\|_{L^2}^2+\int\dot\Delta_j\Phi(n)n_j\,dx.$$

Thus, we have

$$\begin{aligned}&\varepsilon^2\frac{d}{dt}\int u_j\cdot\nabla n_j\,dx+\int\big(|\nabla n_j|^2+K|n_j|^2-P'(\bar\rho)\varepsilon^2|\mathrm{div}\,u_j|^2+u_j\cdot\nabla n_j\big)\,dx\\ &\le\varepsilon\|\dot\Delta_j(\varepsilon u\cdot\nabla u,u\times H)\|_{L^2}\|\nabla n_j\|_{L^2}+\varepsilon\|\nabla\dot\Delta_j(u\cdot\nabla n,G(n)\mathrm{div}\,u)\|_{L^2}\varepsilon\|u_j\|_{L^2}\\ &\quad+\|\dot\Delta_j\Phi(n)\|_{L^2}\|n_j\|_{L^2}.\end{aligned}\tag{3.17}$$

Concerning dissipation for E_j, it comes from the interaction between the symmetric and skew-symmetric part of the order 0 dissipation matrix. Indeed, taking the inner product of $(3.12)_2$ with E_j, using $(3.12)_3$ and $(3.12)_5$ and noticing that $n_j=-\frac{1}{K}\mathrm{div}\,E_j-\frac{1}{K}\dot\Delta_j\Phi(n)$, we get

$$\begin{aligned}&\varepsilon^2\frac{d}{dt}\int u_j\cdot E_j\,dx+\int(|E_j|^2+\frac{1}{K}|\mathrm{div}\,E_j|^2)\,dx\\ &+\int\big(u_j\cdot E_j+\varepsilon(u_j\times\bar B)\cdot E_j-\varepsilon u_j\cdot(\nabla\times H_j)-\varepsilon^2\bar\rho|u_j|^2\big)\,dx\\ &\le\varepsilon\|\dot\Delta_j(\varepsilon u\cdot\nabla u,u\times H)\|_{L^2}\|E_j\|_{L^2}+\varepsilon\|\dot\Delta_j(F(n)u)\|_{L^2}\varepsilon\|u_j\|_{L^2}+\frac{1}{K}\|\dot\Delta_j\Phi(n)\|_{L^2}\|\mathrm{div}\,E_j\|_{L^2}.\end{aligned}\tag{3.18}$$

Finally, taking the inner product of $(3.12)_3$ with $-\nabla\times H_j$ and using $(3.12)_4$, we get dissipation for H_j:

$$\begin{aligned}&-\varepsilon\frac{d}{dt}\int E_j\cdot\nabla\times H_j\,dx+\int(|\nabla\times H_j|^2+\varepsilon\bar\rho u_j\cdot\nabla\times H_j-|\nabla\times E_j|^2)\,dx\\ &\qquad\le\varepsilon\|\dot\Delta_j(F(n)u)\|_{L^2}\|\nabla\times H_j\|_{L^2}.\end{aligned}\tag{3.19}$$

Let $\eta_1\in(0,1)$ be a constant to be chosen later, we define the low-frequency functional

$$\begin{aligned}\mathcal{L}_{\ell,j}(t):=&\frac{1}{2}\int\big(|n_j|^2+P'(\bar\rho)\varepsilon^2|u_j|^2+\frac{1}{K}|E_j|^2+\frac{1}{K}|H_j|^2\big)\,dx\\ &+\varepsilon^2\eta_1\int u_j\cdot\nabla n_j\,dx+\varepsilon^2\eta_1\int u_j\cdot E_j\,dx-\eta_1^{\frac54}\varepsilon\int E_j\cdot\nabla\times H_j\,dx,\end{aligned}$$

and

$$\begin{aligned}D_{\ell,j}(t):=&P'(\bar\rho)\int|u_j|^2+\eta_1\int\big(|\nabla n_j|^2+K|n_j|^2-P'(\bar\rho)|\mathrm{div}\,u_j|^2+u_j\cdot\nabla n_j\big)\,dx\\ &+\eta_1\int\big(|E_j|^2+\frac{1}{K}|\mathrm{div}\,E_j|^2+u_j\cdot E_j+\varepsilon(u_j\times\bar B)\cdot E_j-\varepsilon u_j\cdot(\nabla\times H_j)-\varepsilon^2\bar\rho|u_j|^2\big)\,dx\\ &+\eta_1^{\frac54}\int(|\nabla\times H_j|^2-\varepsilon\bar\rho u_j\cdot\nabla\times H_j-|\nabla\times E_j|^2)\,dx.\end{aligned}$$

Combining (3.8), (3.16)-(3.19), Bernstein's inequality and $2^j\le1$ with $j\le J_0$ leads to

$$\begin{aligned}&\frac{d}{dt}\mathcal{L}_{\ell,j}(t)+D_{\ell,j}(t)\\ &\lesssim\|\dot\Delta_j(u\cdot\nabla n,G(n)\mathrm{div}\,u,\varepsilon u\cdot\nabla u,\varepsilon u\times H,\varepsilon F(n)u,\Phi(n))\|_{L^2}\|(n_j,\varepsilon u_j,E_j,H_j)\|_{L^2}.\end{aligned}\tag{3.20}$$

Hence, we claim that for $\varepsilon\in(0,1)$, there exists a suitable small constant $\eta_1>0$ independent of ε such that

$$\begin{cases}\mathcal{L}_{\ell,j}(t)\sim\|(n_j,\varepsilon u_j,E_j,H_j)\|_{L^2}^2,\\ D_{\ell,j}(t)\gtrsim\|(n_j,u_j,E_j)\|_{L^2}^2+2^{2j}\|H_j\|_{L^2}^2\gtrsim 2^{2j}\mathcal{L}_{\ell,j}(t).\end{cases}\tag{3.21}$$

Indeed, it follows from supp $(\widehat{\dot\Delta_j\cdot})\subset\{\frac{3}{4}2^j\leq|\xi|\leq\frac{8}{3}2^j\}$ and $2^j\leq 1$ that

$$\mathcal{L}_{\ell,j}(t)\leq\frac{1}{2}\int\big((1+\frac{8}{3}\eta_1)|n_j|^2+(P'(\bar\rho)+\frac{11}{3}\eta_1)\varepsilon^2|u_j|^2+(\frac{1}{K}+\frac{11}{3}\eta_1)|E_j|^2+(\frac{1}{K}+\frac{8}{3}\eta_1^{\frac{3}{2}})|H_j|^2\big)\,dx,$$
$$\mathcal{L}_{\ell,j}(t)\geq\frac{1}{2}\int\big((1-\frac{8}{3}\eta_1)|n_j|^2+(P'(\bar\rho)-\frac{11}{3}\eta_1)\varepsilon^2|u_j|^2+(\frac{1}{K}-\frac{11}{3}\eta_1)|E_j|^2+(\frac{1}{K}-\frac{8}{3}\eta_1^{\frac{3}{2}})|H_j|^2\big)\,dx.$$

Similar computations yield

$$\begin{aligned}D_{\ell,j}(t)\geq{}&P'(\bar\rho)\int|u_j|^2\,dx+\eta_1\int\big(\frac{1}{2}|\nabla n_j|^2\,dx+K|n_j|^2-P'(\bar\rho)|\mathrm{div}\,u_j|^2-\frac{1}{2}|u_j|^2\big)\,dx\\&+\eta_1\int\big(\frac{1}{2}|E_j|^2-(1+\bar B^2+\frac{1}{2\eta_1^{\frac{1}{4}}}+\bar\rho)|u_j|^2-\frac{1}{2}\eta_1^{\frac{1}{4}}|\nabla\times H_j|^2\big)\,dx\\&+\eta_1^{\frac{5}{4}}\int\big(\frac{1}{2}|\nabla\times H_j|^2-\frac{\bar\rho^2}{2}|u_j|^2-|\nabla\times E_j|^2\big)\,dx\\\geq{}&\int\big((P'(\bar\rho)-\frac{64P'(\bar\rho)}{9}\eta_1-\bar\rho\eta_1-\frac{\bar\rho^2}{2}\eta_1^{\frac{3}{4}})|u_j|^2+\eta_1K|n_j|^2)\big)dx\\&+\int\big(\frac{1}{2}\eta_1\varepsilon(1-\frac{32}{9}\eta_1^{\frac{1}{4}})|E_j|^2+\frac{9}{32}\eta_1^{\frac{5}{4}}2^{2j}|H_j|^2\big)\,dx,\end{aligned}$$

where we used that, since $\mathrm{div}\,H_j=0$, the div-curl lemma implies

$$\|\nabla\times H_j\|_{L^2}^2=\|\nabla H_j\|_{L^2}^2\geq\frac{9}{16}2^{2j}\|H_j\|_{L^2}^2.\tag{3.22}$$

Taking η_1 is sufficiently small, we have (3.21). Therefore, from (3.20) and (3.21), (3.11) follows.

Then, with the aid of (3.11), we are ready to prove the estimates (3.10). Noticing that $1\geq 2^{2j}$ for $j\leq 0$, applying Lemma A.7-(1) to (3.11), we have

$$\begin{aligned}&\|(n_j,\varepsilon u_j,E_j,H_j)\|_{L_t^\infty(L^2)}+2^{2j}\|(n_j,\varepsilon u_j,E_j,H_j)\|_{L_t^1(L^2)}\\&\quad+\|(n_j,u_j,E_j)\|_{L_t^2(L^2)}+2^j\|H_j\|_{L_t^2(L^2)}\\&\lesssim\|(n_j,\varepsilon u_j,E_j,H_j)(0)\|_{L^2}+\|\dot\Delta_j(u\cdot\nabla n,G(n)\mathrm{div}\,u,\varepsilon u\cdot\nabla u,\varepsilon u\times H,\varepsilon F(n)u,\Phi(n))\|_{L^2}.\end{aligned}\tag{3.23}$$

Multiplying (3.23) by $2^{(\frac{d}{2}-1)j}$ and summing it over $j\leq 0$, we get

$$\begin{aligned}&\|(n,\varepsilon u,E,H)\|^{\ell}_{\widetilde{L}_t^\infty(\dot B^{\frac{d}{2}-1})}+\|(n,\varepsilon u,E,H)\|^{\ell}_{L_t^1(\dot B^{\frac{d}{2}+1})}\\&\quad+\|(n,u,E)\|^{\ell}_{\widetilde{L}^2(\dot B^{\frac{d}{2}-1})}+\|H\|^{\ell}_{\widetilde{L}_t^2(\dot B^{\frac{d}{2}})}\\&\lesssim\|(n_0,u_0,E_0,H_0)\|^{\ell}_{\dot B^{\frac{d}{2}-1}}+\|(u\cdot\nabla n,G(n)\mathrm{div}\,u,\varepsilon u\cdot\nabla u,\varepsilon u\times H,\varepsilon F(n)u,\Phi(n))\|^{\ell}_{L_t^1(\dot B^{\frac{d}{2}-1})}.\end{aligned}\tag{3.24}$$

Let us now turn to the analysis of the nonlinear terms. Using Bernstien inequality and the product law $\dot B^{\frac{d}{2}-1}\hookrightarrow\dot B^{\frac{d}{2}}\times\dot B^{\frac{d}{2}-1}$ in (A.2), we have

$$\|u\cdot\nabla n\|^{\ell}_{L_t^1(\dot B^{\frac{d}{2}-1})}\lesssim\|u\cdot\nabla n\|^{\ell}_{L_t^1(\dot B^{\frac{d}{2}-2})}\lesssim\|u\|_{\widetilde{L}_t^2(\dot B^{\frac{d}{2}-1})}\|n\|_{\widetilde{L}_t^2(\dot B^{\frac{d}{2}})}.\tag{3.25}$$

Similarly, as $\varepsilon\leq 1$, we get

$$\varepsilon\|u\cdot\nabla u\|_{L_t^1(\dot B^{\frac{d}{2}-1})}+\|u\times H\|_{L_t^1(\dot B^{\frac{d}{2}-1})}\lesssim\|u\|_{\widetilde{L}_t^2(\dot B^{\frac{d}{2}-1})}\|(u,H)\|_{\widetilde{L}_t^2(\dot B^{\frac{d}{2}})}.\tag{3.26}$$

By (A.2) and the composition estimate (A.4), it also holds that

$$\varepsilon\|F(n)u\|_{L_t^1(\dot B^{\frac{d}{2}-1})}\lesssim\|n\|_{\widetilde{L}_t^2(\dot B^{\frac{d}{2}})}\|u\|_{\widetilde{L}_t^2(\dot B^{\frac{d}{2}-1})}.\tag{3.27}$$

From the quadratic term, it follows from (??), (3.8), Lemma A.6 and the embedding $\dot{B}^{\frac{d}{2}} \hookrightarrow L^\infty$ that

$$\|\Phi(n)\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1})} \lesssim \|n\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}(\|n\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})} + \|n\|^m_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})} + \|n\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}). \tag{3.28}$$

Combining the above nonlinear estimates (3.25)-(3.28) with (3.24) leads to

$$\begin{aligned}\|(n,\varepsilon u,E,H)\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})} &+ \|(n,\varepsilon u,E,H)\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \|(n,u,E)\|^\ell_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}-1})} + \|H\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}} \\ &\lesssim \|(n_0,u_0,E_0,H_0)\|^\ell_{\dot{B}^{\frac{d}{2}-1}} + \mathcal{X}(t)^2.\end{aligned} \tag{3.29}$$

Furthermore, taking advantage of (2.2) and (3.24), one also has

$$\begin{aligned}\|(n,u,E)\|^\ell_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}+1})} + \|H\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})} &\lesssim \|(n,u,E)\|^\ell_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}-1})} + \|H\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})} \\ &\lesssim \|(n_0,u_0,E_0,H_0)\|^\ell_{\dot{B}^{\frac{d}{2}-1}} + \mathcal{X}(t)^2.\end{aligned} \tag{3.30}$$

Since u satisfies the damped equation

$$\varepsilon^2\partial_t u + u = -\nabla n - E - \varepsilon u\times\bar{B} - (u\cdot\nabla u) - \varepsilon(u\times H), \tag{3.31}$$

we conclude from Lemma A.8 for (3.31) and the estimate (3.29) that

$$\begin{aligned}\|u\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1})} &\lesssim \varepsilon^2\|u_0\|^\ell_{\dot{B}^{\frac{d}{2}+1}} + \|(\nabla n,\varepsilon u,E)\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \varepsilon\|(u\cdot\nabla u,u\times H)\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1})} \\ &\lesssim \varepsilon\|u_0\|^\ell_{\dot{B}^{\frac{d}{2}+1}} + \|(n,\varepsilon u,E)\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \varepsilon\|(u\cdot\nabla u,u\times H)\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1})}.\end{aligned} \tag{3.32}$$

By (3.29), (3.30) and (3.32), we derive (3.10) and complete the proof of Lemma 3.3. □

3.2.2. *Medium-frequency analysis.* Next, we establish the desired medium-frequency estimates.

Lemma 3.4. *Let (n,u,E,H) for $t\in(0,T)$ be a classical solution to the Cauchy problem (3.2) satisfying (3.8). Then it holds that*

$$\begin{aligned}\|(n,\varepsilon u,E,H)\|^m_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})} &+ \|(n,E,H)\|^m_{L^1_t(\dot{B}^{\frac{d}{2}})} + \|u\|^m_{L^1_t(\dot{B}^{\frac{d}{2}-1})} \\ &+ \|n\|^m_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})} + \|(u,E,H)\|^m_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})} \lesssim \mathcal{E}_0^\varepsilon + \mathcal{X}(t)^2.\end{aligned} \tag{3.33}$$

Proof. Before proving (3.33), we need the following localized Lyapunov inequality for $-1\le j\le J_\varepsilon$:

$$\frac{d}{dt}\mathcal{L}^m_j(t) + \mathcal{L}^m_j(t) + 2^{2j}\|n_j\|^2_{L^2} + \|(n_j,u_j,E_j,H_j)\|^2_{L^2} \lesssim G^m_j(t)\sqrt{\mathcal{L}^m_j(t)}, \tag{3.34}$$

with $\mathcal{L}^m_j(t)\sim\|(n_j,\varepsilon u_j,E_j,H_j)\|^2_{L^2}$,

$$G^m_j(t) := \|\dot{\Delta}_j(u\cdot\nabla n,\varepsilon u\cdot\nabla u,\varepsilon F(n)u,\varepsilon u\times H,\Psi(n))\|_{L^2} + \|\partial_t n\|_{L^\infty}\varepsilon\|u_j\|_{L^2} + \|\mathcal{R}_{1,j}\|_{L^2}$$

and the commutator term $\mathcal{R}_{1,j} := [G(n),\dot{\Delta}_j]\mathrm{div}\,u$. We now provide the proof of (3.34). To avoid using the product law for the high-order term $G(n)\mathrm{div}\,u$ at $\dot{B}^{\frac{d}{2}}$-level, we rewrite $(3.12)_1$ as

$$\partial_t n_j + (P'(\bar{\rho}) + G(n))\mathrm{div}\,u_j = \mathcal{R}_{1,j} - \dot{\Delta}_j(u\cdot\nabla n). \tag{3.35}$$

Taking the inner product of (3.35) with n_j, we obtain

$$\frac{1}{2}\frac{d}{dt}\|n_j\|^2_{L^2} + \int(P'(\bar{\rho}) + G(n))\mathrm{div}\,u_j n_j\,dx \le (\|\mathcal{R}_{1,j}\|_{L^2} + \|\dot{\Delta}_j(u\cdot\nabla n)\|_{L^2})\|n_j\|_{L^2}. \tag{3.36}$$

In order to cancel the second term on the left-hand side of (3.36), we multiply $(3.12)_2$ by $(P'(\bar{\rho})+G(n))u_j$ and integrate over $\mathbb{R}^d$ such that

$$\begin{aligned}&\frac{\varepsilon^2}{2}\frac{d}{dt}\int(P'(\bar{\rho}) + G(n))|u_j|^2\,dx - \int(\bar{\rho} + G(n))\mathrm{div}\,u_j n_j\,dx \\ &\quad + \int\big((P'(\bar{\rho}) + G(n))|u_j|^2 + (P'(\bar{\rho}) + G(n))E_j\cdot u_j\big)\,dx \\ &\le \frac{\varepsilon^2}{2}\|\partial_t G(n)\|_{L^\infty}\|u_j\|^2_{L^2} + \|\nabla G(n)\|_{L^\infty}\|u_j\|_{L^2}\|n_j\|_{L^2} \\ &\quad + (\bar{\rho} + \|G(n)\|_{L^\infty})\|\dot{\Delta}_j(\varepsilon u\cdot\nabla u,u\times H)\|_{L^2}\varepsilon\|u_j\|_{L^2}.\end{aligned} \tag{3.37}$$

By (3.15), (3.36) and (3.37), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \left(|n_j|^2+(P'(\bar\rho)+G(n))\varepsilon^2|u_j|^2+\frac{1}{K}|E_j|^2+\frac{1}{K}|H_j|^2\right)\,dx\\
&\quad+\int \left((P'(\bar\rho)+G(n))|u_j|^2+G(n)E_j\cdot u_j\right)\,dx\\
&\le (\|\mathcal{R}_{1,j}\|_{L^2}+\|\dot\Delta_j(u\cdot\nabla n)\|_{L^2})\|n_j\|_{L^2}\\
&\quad+\frac{\varepsilon^2}{2}\|\partial_t G(n)\|_{L^\infty}\|u_j\|_{L^2}^2+\|\nabla G(n)\|_{L^\infty}\|u_j\|_{L^2}\|n_j\|_{L^2}\\
&\quad+(\bar\rho+\|G(n)\|_{L^\infty})\|\dot\Delta_j(\varepsilon u\cdot\nabla u,u\times h)\|_{L^2}\varepsilon\|u_j\|_{L^2}\\
&\quad+\frac{1}{K}\varepsilon\|\dot\Delta_j(F(n)u)\|_{L^2}\|E_j\|_{L^2}.
\end{aligned}
\tag{3.38}
$$

Again, dissipation for n_j can be obtained from $(3.12)_2$ and (3.35) as follows

$$
\begin{aligned}
&\varepsilon^2\frac{d}{dt}\int u_j\cdot\nabla n_j\,dx+\int \left(|\nabla n_j|^2+K|n_j|^2-(P'(\bar\rho+G(n))\varepsilon^2|\operatorname{div} u_j|^2+u_j\cdot\nabla n_j\right)\,dx\\
&\le \|\dot\Delta_j(\varepsilon u\cdot\nabla u,u\times H)\|_{L^2}\varepsilon\|\nabla n_j\|_{L^2}+\varepsilon\|\nabla\dot\Delta_j(u\cdot\nabla n)\|_{L^2}\varepsilon\|u_j\|_{L^2}\\
&\quad+\varepsilon\|\nabla\mathcal{R}_{1,j}\|_{L^2}\varepsilon\|u_j\|_{L^2}+\|\dot\Delta_j\Phi(n)\|_{L^2}\|n_j\|_{L^2},
\end{aligned}
\tag{3.39}
$$

for some constant $\eta_2\in(0,1)$ to be determined later. In view of (3.18)-(3.19) and (3.38)-(3.39), we introduce the medium-frequency functionals

$$
\begin{aligned}
\mathcal{L}_{m,j}(t):=&\frac{1}{2}\int \left(|n_j|^2+(P'(\bar\rho)+G(n))\varepsilon^2|u_j|^2+\frac{1}{K}|E_j|^2+\frac{1}{K}|H_j|^2\right)\,dx\\
&+\eta_2\varepsilon^2\int u_j\cdot\nabla n_j\,dx+\eta_2\varepsilon^2\int u_j\cdot E_j\,dx-\eta_2^{\frac54}\varepsilon 2^{-2j}\int E_j\cdot\nabla\times H_j\,dx,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{D}_{m,j}(t):=&\int \left((P'(\bar\rho)+G(n))|u_j|^2+G(n)E_j\cdot u_j\right)\,dx\\
&+\eta_2\int \left(|\nabla n_j|^2+K|n_j|^2-(P'(\bar\rho)+G(n))\varepsilon^2|\operatorname{div} u_j|^2+u_j\cdot\nabla n_j\right)\,dx\\
&+\eta_2\int \left(|E_j|^2+\frac{1}{K}|\operatorname{div} E_j|^2+u_j\cdot E_j+\varepsilon(u_j\times\bar B)\cdot E_j-\varepsilon u_j\cdot(\nabla\times H_j)-\varepsilon^2\bar\rho|u_j|^2\right)\,dx\\
&+\eta_2^{\frac54}2^{-2j}\int (|\nabla\times H_j|^2-\varepsilon\bar\rho u_j\cdot\nabla\times H_j-|\nabla\times E_j|^2)\,dx.
\end{aligned}
$$

From (3.8) and composition estimates, one has

$$
\frac{\bar\rho}{2}\le\bar\rho+G(n)\le\frac{3\bar\rho}{2}.
\tag{3.40}
$$

Since $2^{-1}\le 2^j\lesssim 1/\varepsilon$, as in the low-frequency setting, applying Bernstein's inequality and choosing the constant η_2 small enough, we have

$$
\begin{cases}
\mathcal{L}_{m,j}(t)\sim\|(n_j,\varepsilon u_j,E_j,H_j)\|_{L^2}^2,\\
\mathcal{D}_{m,j}(t)\gtrsim 2^{2j}\|n_j\|_{L^2}^2+\|(n_j,u_j,E_j,H_j)\|_{L^2}^2\gtrsim\mathcal{L}_{m,j}(t), \qquad -1\le j\le J_\varepsilon.
\end{cases}
\tag{3.41}
$$

For brevity, we omit the details. By virtue of (3.8), (3.18)-(3.19), (3.38)-(3.39) and (3.41), we get $\mathcal{L}_j^m(t)\sim\|(n_j,\varepsilon u_j,E_j,H_j)\|_{L^2}^2$ and (3.34).

Then, for $-1\le j\le J_\varepsilon$, we obtain from Lemma A.7 applied to (3.34) that

$$
\begin{aligned}
\|(n_j,\varepsilon u_j,E_j,H_j)\|_{L^\infty_t(L^2)}+\|(n_j,\varepsilon u_j,E_j,H_j)\|_{L^1_t(L^2)}+2^j\|n_j\|_{L^2_t(L^2)}+\|(u_j,E_j,H_j)\|_{L^2_t(L^2)}\\
\lesssim\|(n_j,\varepsilon u_j,E_j,H_j)(0)\|_{L^2}+\|G_j^m\|_{L^1_t(L^2)},
\end{aligned}
$$

which implies

$$\begin{aligned}\|(n,\varepsilon u,E,H)\|^m_{\widetilde{L}^\infty_t(\dot B^{\frac d2})}&+\|(n,\varepsilon u,E,H)\|^m_{L^1_t(\dot B^{\frac d2})}+\|n\|^m_{\widetilde{L}^2_t(\dot B^{\frac d2+1})}+\|(u,E,H)\|^m_{\widetilde{L}^2_t(\dot B^{\frac d2})}\\&\lesssim \mathcal{E}_0^\varepsilon+\|(u\cdot\nabla n,\varepsilon u\cdot\nabla u,u\times H,\varepsilon F(n)u,\Phi(n))\|^m_{L^1_t(\dot B^{\frac d2})}\\&\quad+\varepsilon\|\partial_t n\|_{L^2_t(L^\infty)}\|u\|^m_{\widetilde{L}^2_t(\dot B^{\frac d2})}+\sum_{j\mathbb{Z}}2^{\frac d2 j}\|\mathcal{R}_{1,j}\|_{L^1_t(L^2)}.\end{aligned}\tag{3.42}$$

Let us now turn to the analysis of the nonlinear terms. It follows from the product law (A.2) that

$$\|(u\cdot\nabla n,\varepsilon u\cdot\nabla u)\|_{L^1_t(\dot B^{\frac d2})}\lesssim\|u\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}(\|n\|_{\widetilde{L}^2_t(\dot B^{\frac d2+1})}+\varepsilon\|u\|_{\widetilde{L}^2_t(\dot B^{\frac d2+1})}).\tag{3.43}$$

Similarly, one has

$$\|u\times H\|_{L^1_t(\dot B^{\frac d2})}\lesssim\|u\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}\|H\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}.\tag{3.44}$$

Using (A.2) and (A.4), we also get

$$\|F(n)u\|_{L^1_t(\dot B^{\frac d2})}\lesssim\|u\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}\|n\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}.\tag{3.45}$$

Employing (**??**), (3.8) and the composition law in Lemma A.6, we deduce that

$$\|\Phi(n)\|^m_{L^1_t(\dot B^{\frac d2})}\lesssim\|n\|^2_{\widetilde{L}^2_t(\dot B^{\frac d2})}.\tag{3.46}$$

According to (2.2), (3.2) and the embedding $\dot B^{\frac d2}\hookrightarrow L^\infty$, there holds that

$$\begin{aligned}\varepsilon\|\partial_t n\|_{L^2_t(L^\infty)}&\lesssim\varepsilon\|u\|_{L^\infty_t(L^\infty)}\|\nabla n\|_{\widetilde{L}^2_t(L^\infty)}+(\bar\rho+\|G(n)\|_{L^\infty_t(L^\infty)})\varepsilon\|\operatorname{div}u\|_{L^2_t(L^\infty)}\\&\lesssim\|n\|_{\widetilde{L}^2_t(\dot B^{\frac d2+1})}+\varepsilon\|u\|_{\widetilde{L}^2_t(\dot B^{\frac d2+1})}.\end{aligned}\tag{3.47}$$

In addition, due to (**??**), we know that $G(n)$ is a $C^{[\frac d2]+3}$ function and satisfies $G(0)=0$. It thus follow from the bound (3.8), the commutator estimate (A.3) and the composition estimate (A.4) imply that

$$\sum_{j\in\mathbb{Z}}2^{\frac d2 j}\|\mathcal{R}_{1,j}\|_{L^1_t(L^2)}\lesssim\|G(n)\|_{\widetilde{L}^2_t(\dot B^{\frac d2+1})}\|\operatorname{div}u\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\lesssim\|n\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}\|u\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}.\tag{3.48}$$

Substituting the above estimates (3.43)-(3.48) into (3.42), we have

$$\begin{aligned}\|(n,\varepsilon u,E,H)\|^m_{\widetilde{L}^\infty_t(\dot B^{\frac d2})}+\|(n,\varepsilon u,E,H)\|^m_{L^1_t(\dot B^{\frac d2})}&+\|n\|^m_{\widetilde{L}^2_t(\dot B^{\frac d2+1})}\\&+\|(u,E,H)\|^m_{\widetilde{L}^2_t(\dot B^{\frac d2})}\lesssim\mathcal{E}_0^\varepsilon+\mathcal{X}(t)^2.\end{aligned}\tag{3.49}$$

Moreover, we capture some additional dissipation estimates as follows. By (2.2) and (3.49) one gets

$$\|n\|^m_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}+\|(u,E,H)\|^m_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}+\varepsilon\|(u,E,H)\|^m_{\widetilde{L}^2_t(\dot B^{\frac d2+1})}\mathcal{E}_0^\varepsilon+\mathcal{X}(t)^2.\tag{3.50}$$

Using (2.2), (3.49) and that u verifies the damped equation (3.31), we get

$$\begin{aligned}\|u\|^m_{L^1_t(\dot B^{\frac d2-1})}&\lesssim\varepsilon^2\|u_0\|^m_{\dot B^{\frac d2-1}}+\|(\nabla n,\varepsilon u,E)\|^m_{L^1_t(\dot B^{\frac d2-1})}+\|(\varepsilon^2u\cdot\nabla u,\varepsilon u\times H)\|^m_{L^1_t(\dot B^{\frac d2-1})}\\&\lesssim\varepsilon\|u_0\|^m_{\dot B^{\frac d2}}+\|(n,\varepsilon u,E)\|^m_{L^1_t(\dot B^{\frac d2})}+\|(\varepsilon u\cdot\nabla u,u\times H)\|^m_{L^1_t(\dot B^{\frac d2})}\\&\lesssim\mathcal{E}_0^\varepsilon+\mathcal{X}(t)^2.\end{aligned}\tag{3.51}$$

Combining (3.49), (3.50) and (3.51) together, we have (3.33). This completes the proof of Lemma **??**. □

3.2.3. *High-frequency analysis.* We now derive the desired high-frequency a priori estimates.

Lemma 3.5. *Let (n,u,E,H) for $t\in(0,T)$ be a classical solution to the Cauchy problem (3.2) satisfying (3.8). It holds that*

$$\begin{aligned}&\varepsilon\|(n,\varepsilon u,E,H)\|^h_{\widetilde{L}^\infty_t(\dot B^{\frac d2+1})}+\|(n,\varepsilon u)\|^h_{\widetilde{L}^2_t(\dot B^{\frac d2+1})}+\|(u,E,H)\|^h_{\widetilde{L}^2_t(\dot B^{\frac d2})}\\&\lesssim\varepsilon\|(n_0,u_0,E_0,H_0)\|^h_{\dot B^{\frac d2}}+\mathcal{X}(t)^2+\mathcal{X}(t)^3.\end{aligned}\tag{3.52}$$

Proof. We claim that for all $j\leq J_\varepsilon-1$, there exists a functional $\mathcal{L}_j^h(t)\sim\|(n_j,u_j,E_j,H_j)\|_{L^2}^2$ such that

$$\begin{aligned}&\frac{d}{dt}\mathcal{L}_j^h(t)+\frac{1}{\varepsilon^2}\mathcal{L}_j^h(t)+\frac{1}{\varepsilon^2}\|n_j\|_{L^2}^2+\|u_j\|_{L^2}^2+\frac{1}{\varepsilon^2}2^{-2j}\|(E_j,H_j)\|_{L^2}^2\\&\lesssim G_{1,j}^h(t)\sqrt{\mathcal{L}_j^h(t)}+G_{1,j}^h(t)\|u_j\|_{L^2},\end{aligned}\tag{3.53}$$

where

$$\begin{aligned}G_{1,j}^h(t):&=\|\dot\Delta_j(\varepsilon F(n)u,\Psi(n))\|_{L^2}+(\|\mathrm{div}\,u\|_{L^\infty}+\|\partial_t n\|_{L^\infty})\|(n_j,\varepsilon u_j)\|_{L^2}\\&\quad+(1+\varepsilon\|\nabla n\|_{L^\infty})\|u\|_{L^\infty}\|u_j\|_{L^2}+\|(\mathcal{R}_{1,j},\mathcal{R}_{2,j},\varepsilon\mathcal{R}_{3,j})\|_{L^2},\\G_{2,j}^h(t):&=\dot\Delta_j(u\times B)\|_{L^2}\end{aligned}$$

with the commutator terms $\mathcal{R}_{1,j}:=[G(n),\dot\Delta_j]\mathrm{div}\,u$, $\mathcal{R}_{2,j}:=[u,\dot\Delta_j]\nabla a$ and $\mathcal{R}_{3,j}:=[u,\dot\Delta_j]\nabla n$.

To show (3.53) in the high-frequency region, we shall use commutator estimates and rewrite (3.54) as

$$\begin{cases}\partial_t n_j+u\cdot\nabla n_j+(P'(\bar\rho)+G(n))\mathrm{div}\,u_j=\mathcal{R}_{1,j}+\mathcal{R}_{2,j},\\\varepsilon^2\partial_t u_j+\varepsilon^2u\cdot\nabla u_j+\nabla n_j+E_j+u_j+\varepsilon u_j\times\bar B=-\varepsilon\dot\Delta_j(u\times H)-\varepsilon^2\mathcal{R}_{3,j},\\\varepsilon\partial_t E_j-\nabla\times H_j-\bar\rho\varepsilon u_j=\varepsilon\dot\Delta_j(F(n)u),\\\varepsilon\partial_t H_j+\nabla\times E_j=0,\\\mathrm{div}\,E_j=--Kn_j-\dot\Delta_j\Phi(n),\qquad\mathrm{div}\,H_j=0.\end{cases}\tag{3.54}$$

Similarly to (3.37)-(3.38), by a direct computation on (3.54) we are able to get

$$\begin{aligned}&\frac12\frac{d}{dt}\int\big(|n_j|^2+(P'(\bar\rho)+G(n))\varepsilon^2|u_j|^2+\frac1K|E_j|^2+\frac1K|H_j|^2\big)\,dx\\&\int\big((P'(\bar\rho)+G(n))|u_j|^2+G(n)E_j\cdot u_j+\varepsilon G(n)(u_j\times\bar B)\cdot u_j\big)\,dx\\&\leq(\bar\rho+\|G(n)\|_{L^\infty})\varepsilon\|\dot\Delta_j(u\times H)\|_{L^2}\|u_j\|_{L^2}+\frac1K\varepsilon\|\dot\Delta_j(F(n)u)\|_{L^2}\|E_j\|_{L^2}\\&\quad+\frac12\|\mathrm{div}\,u\|_{L^\infty}\|n_j\|_{L^2}^2+\frac12\big((\bar\rho+\|G(n)\|_{L^\infty})\|\mathrm{div}\,u\|_{L^\infty}\varepsilon^2\|u_j\|_{L^2}^2+\|\nabla G(n)\|_{L^\infty}\|u\|_{L^\infty}\varepsilon^2\|u_j\|_{L^2}^2\\&\quad+\frac{\varepsilon^2}{2}\|\partial_tG(n)\|_{L^\infty}\|u_j\|_{L^2}^2+\big((\bar\rho+\|G(n)\|_{L^\infty})\|(\mathcal{R}_{1,j},\mathcal{R}_{2,j},\varepsilon\mathcal{R}_{3,j})\|_{L^2}\|(n_j,\varepsilon u_j)\|_{L^2}.\end{aligned}\tag{3.55}$$

From $(3.54)_1$-$(3.54)_2$ we modify the dissipation for n_j and perform the following cross estimate

$$\begin{aligned}&\varepsilon^2\frac{d}{dt}\int u_j\cdot\nabla n_j\,dx+\int\big(|\nabla n_j|^2+K|n_j|^2-(P'(\bar\rho+G(n))\varepsilon^2|\mathrm{div}\,u_j|^2+u_j\cdot\nabla n_j\big)\,dx\\&\leq2\varepsilon^2\|u\|_{L^\infty}\|\nabla u_j\|_{L^2}\|\nabla n_j\|_{L^2}+\varepsilon\|\dot\Delta_j(u\times H)\|_{L^2}\|\nabla n_j\|_{L^2}+\|(\mathcal{R}_{1,j},\mathcal{R}_{2,j},\varepsilon\mathcal{R}_{3,j})\|_{L^2}\|\nabla(\varepsilon u_j,n_j)\|_{L^2}.\end{aligned}$$

According to (3.18), (3.19), (3.55) and the previous inequality, we define

$$\begin{aligned}\mathcal{L}_{h,j}(t):&=\frac12\int\big(|n_j|^2+(P'(\bar\rho)+G(n))|u_j|^2+\frac1K|E_j|^2+\frac1K|H_j|^2\big)\,dx\\&\quad+\frac{1}{\varepsilon^2}\eta_3 2^{-2j}\int u_j\cdot\nabla n_j\,dx+\frac{1}{\varepsilon^2}\eta_3 2^{-2j}\int u_j\cdot E_j\,dx-\eta_3^{\frac54}\frac{1}{\varepsilon^2}2^{-4j}\int E_j\cdot\nabla\times H_j\,dx,\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_{h,j}(t):&=\int\big((P'(\bar\rho)+G(n))|u_j|^2+G(n)E_j\cdot u_j+\varepsilon G(n)(u_j\times\bar B)\cdot u_j\big)\,dx\\&\quad+\eta_3\frac{1}{\varepsilon^2}2^{-2j}\int\big(|\nabla n_j|^2+K|n_j|^2-(P'(\bar\rho+G(n))\varepsilon^2|\mathrm{div}\,u_j|^2+u_j\cdot\nabla n_j\big)\,dx\\&\quad+\eta_3\frac{1}{\varepsilon^2}2^{-2j}\int\big(|E_j|^2+\frac1K|\mathrm{div}\,E_j|^2+u_j\cdot E_j+\varepsilon(u_j\times\bar B)\cdot E_j-\varepsilon u_j\cdot(\nabla\times H_j)-\varepsilon^2\bar\rho|u_j|^2\big)\,dx\\&\quad+\eta_3^{\frac54}\frac{1}{\varepsilon^2}2^{-4j}\int(|\nabla\times H_j|^2-\varepsilon\bar\rho u_j\cdot\nabla\times H_j-|\nabla\times E_j|^2)\,dx.\end{aligned}$$

For all $j \geq J_\varepsilon - 1$, with the help of (3.40) and that $2^{-j} \lesssim \varepsilon$, one can verify that

$$\begin{cases} \mathcal{L}_{h,j}(t) \sim \|(n_j, \varepsilon u_j, E_j, H_j)\|_{L^2}^2, \\ \mathcal{D}_{h,j}(t) \gtrsim \dfrac{1}{\varepsilon^2}\|n_j\|_{L^2}^2 + \|u_j\|_{L^2}^2 + \dfrac{1}{\varepsilon^2} 2^{-2j}\|(E_j, H_j)\|_{L^2}^2 \gtrsim \dfrac{1}{\varepsilon^2} 2^{-2j}\mathcal{L}_{h,j}(t). \end{cases}$$

The details are omitted. These inequalities together with (3.18), (3.19), (3.40) and (3.55), yields (3.53).

Based on (3.52), we are able to get the high-frequency estimates (3.52). Indeed, it follows from Lemma A.7-(2) applied to (3.53) that

$$\begin{aligned} &\|(n_j, \varepsilon u_j, E_j, H_j)\|_{L^\infty_t(L^2)} + \frac{1}{\varepsilon}\|n_j\|_{L^2_t(L^2)} + \|u_j\|_{L^2_t(L^2)} + \frac{1}{\varepsilon}2^{-j}\|(E_j, H_j)\|_{L^2_t(L^2)} \\ &\lesssim \|(n_j, u_j, E_j, H_j)(0)\|_{L^2} + \|G^h_{1,j}\|_{L^1_t(L^2)} + \|G^h_{2,j}\|_{L^2_t(L^2)}, \end{aligned} \tag{3.56}$$

for $j \geq J_\varepsilon - 1$. It should be noted that in order to handle $G^h_{2,j}$, we have to perform L^2-type dissipation estimate rather than L^1-type due to the loss of regularity in high frequencies. Multiplying (3.56) by $2^{j(\frac{d}{2}+1)}$ and summing over $j \geq J_\varepsilon - 1$, we get

$$\begin{aligned} &\varepsilon\|(n, \varepsilon u, E, H)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})} + \|n\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})} + \varepsilon\|u\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})} + \|(E,H)\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})} \\ &\quad\lesssim \varepsilon\|(n_0, u_0, E_0, H_0)\|^h_{\dot{B}^{\frac{d}{2}+1}} + \varepsilon\|(F(n)u, \Psi(n))\|^h_{L^1_t(\dot{B}^{\frac{d}{2}+1})} \\ &\qquad + \varepsilon(\|\mathrm{div}\, u\|_{L^2_t(L^\infty)} + \|\partial_t n\|_{L^2_t(L^\infty)})\|(n, \varepsilon u)\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})} \\ &\qquad + (1 + \varepsilon\|\nabla n\|_{L^\infty_t(L^\infty)})\|u\|_{L^2_t(L^\infty)}\varepsilon\|u\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})} \\ &\qquad + \varepsilon\sum_{j\geq J_\varepsilon - 1} 2^{(\frac{d}{2}+1)j}\|(\mathcal{R}_{1,j}, \mathcal{R}_{2,j}, \mathcal{R}_{3,j})\|_{L^1_t(L^2)} + \varepsilon\|u \times H\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}. \end{aligned} \tag{3.57}$$

For the nonlinear terms, it follows from the product law (A.1) and the composition estimate (A.4) that

$$\varepsilon\|F(n)u\|^h_{L^1_t(\dot{B}^{\frac{d}{2}+1})} \lesssim \|n\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}\varepsilon\|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})} + \|n\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}\|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}$$

As $\Phi(n) \in C^{[\frac{d}{2}]+4}$ when $\|n\|_{L_t(L^\infty)} \ll 1$, employing (A.7) with $(s, \sigma) = (\frac{d}{2}+1, \frac{d}{2})$ yields

$$\varepsilon\|\Phi(n)\|^m_{L^1_t(\dot{B}^{\frac{d}{2}+1})} \lesssim \|n\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}(\|n\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})} + \|n\|^m_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})} + \varepsilon\|n\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}).$$

Moreover, it is easy to verify that

$$\varepsilon\|\nabla n\|_{L^\infty_t(L^\infty)} \lesssim \|n\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})} + \|n\|^m_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})} + \varepsilon\|n\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})}, \qquad \|u\|_{L^2_t(L^\infty)} \lesssim \|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}.$$

Furthermore, in view of the commutator estimate (A.3) and the composition estimate (A.4) for the $C^{[\frac{d}{2}]+3}$ function $G(n)$, it follows that

$$\varepsilon\sum_{j\in\mathbb{Z}} 2^{(\frac{d}{2}+1)j}\|(\mathcal{R}_{1,j}, \mathcal{R}_{2,j}, \mathcal{R}_{3,j})\|_{L^1_t(L^2)} \lesssim \|(n, \varepsilon u)\|^2_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}.$$

For the term $u \times H$ with L^2-in-time estimates, the product law (A.2) gives

$$\varepsilon\|u \times H\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})} \lesssim \|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}))}\varepsilon\|H\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})} + \varepsilon\|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}\|H\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})}.$$

The combination of the above estimates and $\|u\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})} \lesssim \varepsilon\|u\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}$ gives rise to (3.52). □

3.2.4. *Additional regularity estimates: the effective velocity.* As mentioned before, the effective velocity

$$z := u + \nabla n + E + \varepsilon u \times \bar{B}$$

plays a key role in justifying the strong relaxation limit. Inserting

$$u = z - \nabla n - E - \varepsilon u \times \bar{B}$$

into $(3.2)_1$ and using the facts that

$$\mathrm{div}\, E = -Kn - \Psi(n) \quad \text{and} \quad \partial_t u = -u \cdot \nabla u - \frac{1}{\varepsilon^2}z - \frac{1}{\varepsilon}u \times H,$$

we obtain the following (partially) diagonalized system

$$\begin{cases}\partial_t n - P'(\bar\rho)\Delta n + \bar\rho n = -P'(\bar\rho)\mathrm{div}\, z - \varepsilon P'(\bar\rho)\mathrm{div}\,(u\times\bar B) + F_1,\\ \varepsilon\partial_t z + \dfrac{1}{\varepsilon} z = \varepsilon\nabla\partial_t n + \varepsilon\partial_t E - z\times\bar B + F_2,\\ \varepsilon\partial_t E + \varepsilon\bar\rho E = \varepsilon\bar\rho z + \varepsilon\bar\rho\nabla n - \varepsilon^2\bar\rho(z-\nabla n - E)\times\bar B + \nabla\times H + F_3,\\ (n,z)|_{t=0} = (n_0, z_0),\end{cases} \tag{3.58}$$

with $z_0^\varepsilon := \frac{1}{\varepsilon}u_0 + \nabla n_0 + E_0 + u_0\times\bar B$ and

$$\begin{aligned}&F_1 := -u\cdot\nabla n - G(n)\mathrm{div}\, u - P'(\bar\rho)\mathrm{div}\,\Psi(n),\\ &F_2 := -\varepsilon u\cdot\nabla u - \varepsilon^2(u\cdot\nabla u)\times\bar B - u\times H - \varepsilon(u\times B)\times\bar B,\\ &F_3 := \varepsilon F(n)u.\end{aligned}$$

Such reformulation (3.58) of $(3.2)_1$-$(3.2)_3$ reveals that damping effect exists for both n, z and E if the linear higher order terms of (n,z,E) on the right-hand side can be absorbed, and the linear term associated with H will be treated as a given source satisfying the uniform bounds in (2.7). This can be done in low and medium frequencies and we establish additional estimates in the next proposition.

Proposition 3.6. *Under the assumption of Theorem 2.1, there holds that*

$$\|z\|^\ell_{L^1_t(\dot B^{\frac d2})} + \|z\|^m_{L^1_t(\dot B^{\frac d2-1})} + \|z\|^h_{\widetilde L^2_t(\dot B^{\frac d2-1})} \lesssim \varepsilon\big(\mathcal{E}_0^\varepsilon + \mathcal{X}(t)^2\big), \tag{3.59}$$

and

$$\|n\|^\ell_{L^1_t(\dot B^{\frac d2})} + \|n\|^m_{L^1_t(\dot B^{\frac d2+1})} \le \mathcal{E}_0^\varepsilon + \mathcal{X}(t)^2, \tag{3.60}$$

where $\mathcal{E}_0^\varepsilon$ and $\mathcal{X}(t)$ are defined by (2.5) and (3.7), respectively.

Proof. To prove (3.59)-(3.60), we consider the estimates in the three regimes separately.

- Low-frequencies.

Employing Lemma A.8 concerning maximal regularity estimates for $(3.58)_1$, we have

$$\|n\|^\ell_{L^1_t(\dot B^{\frac d2})} + \|\partial_t n\|^\ell_{L^1_t(\dot B^{\frac d2})} \lesssim \|n_0^\varepsilon\|^\ell_{\dot B^{\frac d2}} + \|\mathrm{div}\, z\|^\ell_{L^1_t(\dot B^{\frac d2})} + \varepsilon\|u\|^\ell_{L^1_t(\dot B^{\frac d2+1})} + \|F_1\|^\ell_{L^1_t(\dot B^{\frac d2})}. \tag{3.61}$$

The low-frequency cut-off property in (2.2) guarantees that

$$\|\mathrm{div}\, z\|^\ell_{L^1_t(\dot B^{\frac d2})} \lesssim \varepsilon\big(\frac{1}{\varepsilon}\|z\|^\ell_{L^1_t(\dot B^{\frac d2})}\big). \tag{3.62}$$

Similarly, from $(3.58)_2$-$(3.58)_3$ one gets

$$\frac{1}{\varepsilon}\|z\|^\ell_{L^1_t(\dot B^{\frac d2})} \lesssim \varepsilon\|z_0^\varepsilon\|^\ell_{\dot B^{\frac d2}} + \varepsilon\big(\frac{1}{\varepsilon}\|z\|^\ell_{L^1_t(\dot B^{\frac d2})}\big) + \varepsilon\|\partial_t n\|^\ell_{L^1_t(\dot B^{\frac d2})} + \varepsilon\|\partial_t E\|^\ell_{L^1_t(\dot B^{\frac d2})} + \|F_2\|^\ell_{L^1_t(\dot B^{\frac d2})}. \tag{3.63}$$

and

$$\begin{aligned}&\varepsilon\|E\|^\ell_{L^1_t(\dot B^{\frac d2})} + \varepsilon\|\partial_t E\|^\ell_{L^1_t(\dot B^{\frac d2})}\\ &\quad\lesssim \varepsilon\|E_0^\varepsilon\|^\ell_{\dot B^{\frac d2}} + (\varepsilon^2+\varepsilon^3)\big(\frac{1}{\varepsilon}\|z\|^\ell_{L^1_t(\dot B^{\frac d2})}\big) + (\varepsilon+\varepsilon^2)\|n\|^\ell_{L^1_t(\dot B^{\frac d2})} + \varepsilon^2\|E\|^\ell_{L^1_t(\dot B^{\frac d2})}\\ &\qquad + \|\nabla\times H\|^\ell_{L^1_t(\dot B^{\frac d2})} + \|F_3\|^\ell_{L^1_t(\dot B^{\frac d2})}.\end{aligned} \tag{3.64}$$

Inserting (3.64) into (3.63), combining the resulting equation with (3.61)-(3.63) and letting $\varepsilon\le\varepsilon_0$ with ε_0 small enough, we arrive at

$$\begin{aligned}&\|n\|^\ell_{L^1_t(\dot B^{\frac d2})} + \frac{1}{\varepsilon}\|z\|^\ell_{L^1_t(\dot B^{\frac d2})}\\ &\quad\lesssim \|(n_0^\varepsilon, E_0^\varepsilon)\|^\ell_{\dot B^{\frac d2}} + \varepsilon\|z_0^\varepsilon\|^\ell_{\dot B^{\frac d2}} + \|(u,H)\|^\ell_{L^1_t(\dot B^{\frac d2+1})} + \|(F_1,F_2,F_3)\|^\ell_{L^1_t(\dot B^{\frac d2})}.\end{aligned} \tag{3.65}$$

Now we are in a position to handle the right-hand side of (3.65). Using (2.2), we obtain

$$\varepsilon\|z_0^\varepsilon\|^\ell_{\dot B^{\frac d2}} \lesssim \|(n_0^\varepsilon, u_0^\varepsilon, E_0^\varepsilon)\|^\ell_{\dot B^{\frac d2}} \lesssim \mathcal{E}_0^\varepsilon.$$

Recalling Lemma 3.3, we know that

$$\|(u,H)\|^{\ell}_{L^1_t(\dot B^{\frac d2+1})}\lesssim \mathcal{E}_0^{\varepsilon}+\mathcal{X}(t)^2.$$

Concerning the nonlinear terms, one deduces from (3.25)-(3.28) that

$$\|(F_1,F_2,F_3)\|^{\ell}_{L^1_t(\dot B^{\frac d2})}\lesssim \|(F_1,F_2,F_3)\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})}\lesssim \mathcal{X}(t)^2.$$

Therefore, we obtain

$$\|n\|^{\ell}_{L^1_t(\dot B^{\frac d2})}+\frac{1}{\varepsilon}\|z\|^{\ell}_{L^1_t(\dot B^{\frac d2})}\lesssim \mathcal{E}_0^{\varepsilon}+\mathcal{X}(t)^2. \tag{3.66}$$

- Medium-frequencies.

Applying Lemma A.8 to (3.58), we have

$$\begin{aligned}&\|n\|^{m}_{L^1_t(\dot B^{\frac d2+1})}+\|\partial_t n\|^{m}_{L^1_t(\dot B^{\frac d2-1})}+\frac{1}{\varepsilon}\|z\|^{m}_{L^1_t(\dot B^{\frac d2-1})}\\ &\lesssim \|n_0^{\varepsilon}\|^{m}_{\dot B^{\frac d2+1}}+\varepsilon\|z_0^{\varepsilon}\|^{m}_{\dot B^{\frac d2-1}}+(\varepsilon 2^{J_\varepsilon}+\varepsilon)\frac{1}{\varepsilon}\|z\|^{m}_{L^1_t(\dot B^{\frac d2-1})}+\varepsilon 2^{J_\varepsilon}\|\partial_t n\|^{m}_{L^1_t(\dot B^{\frac d2-1})}+\varepsilon\|\partial_t E\|^{m}_{L^1_t(\dot B^{\frac d2-1})}\\ &\quad+\varepsilon 2^{J_\varepsilon}\|u\|^{\ell}_{L^1_t(\dot B^{\frac d2-1})}+\|(F_1,F_2)\|^{m}_{L^1_t(\dot B^{\frac d2})}.\end{aligned} \tag{3.67}$$

and

$$\begin{aligned}&\varepsilon\|E\|^{m}_{L^1_t(\dot B^{\frac d2-1})}+\varepsilon\|\partial_t E\|^{m}_{L^1_t(\dot B^{\frac d2-1})}\\ &\lesssim \varepsilon\|E_0^{\varepsilon}\|^{m}_{\dot B^{\frac d2-1}}+(\varepsilon^2+\varepsilon^3)\big(\frac{1}{\varepsilon}\|z\|^{m}_{L^1_t(\dot B^{\frac d2-1})}\big)+(1+\varepsilon)\varepsilon 2^{J_\varepsilon}\|n\|^{m}_{L^1_t(\dot B^{\frac d2-1})}+\varepsilon^2\|E\|^{m}_{L^1_t(\dot B^{\frac d2-1})}\\ &\quad+\|H\|^{m}_{L^1_t(\dot B^{\frac d2})}+\|F_3\|^{m}_{L^1_t(\dot B^{\frac d2-1})}.\end{aligned} \tag{3.68}$$

Let $\varepsilon\le\varepsilon_0$ and $\varepsilon 2^{J_\varepsilon}\le 2^{-k_0}$ with ε_0 and 2^{-k_0} sufficiently small. It follows from (3.67) and (3.68) that

$$\begin{aligned}&\|n\|^{m}_{L^1_t(\dot B^{\frac d2+1})}+\frac{1}{\varepsilon}\|z\|^{m}_{L^1_t(\dot B^{\frac d2-1})}\\ &\quad\lesssim \|n_0^{\varepsilon}\|^{m}_{\dot B^{\frac d2+1}}+\varepsilon\|z_0^{\varepsilon}\|^{m}_{\dot B^{\frac d2-1}}+\|u\|^{m}_{L^1_t(\dot B^{\frac d2-1})}+\|H\|^{m}_{L^1_t(\dot B^{\frac d2})}+\|(F_1,F_2,F_3)\|^{m}_{L^1_t(\dot B^{\frac d2})}.\end{aligned}$$

Using the medium-frequency cut-off property in (2.2), we have

$$\varepsilon\|z_0^{\varepsilon}\|^{m}_{\dot B^{\frac d2-1}}\lesssim \|(n_0,u_0,E_0)\|^{m}_{\dot B^{\frac d2}}\lesssim \mathcal{E}_0^{\varepsilon}.$$

Note that Lemma **??** ensures that

$$\|u\|^{m}_{L^1_t(\dot B^{\frac d2-1})}+\|H\|^{m}_{L^1_t(\dot B^{\frac d2})}\lesssim \mathcal{E}_0^{\varepsilon}+\mathcal{X}(t)^2.$$

As in (3.43)-(3.46), one also gets

$$\|(F_1,F_2,F_3)\|^{m}_{L^1_t(\dot B^{\frac d2-1})}\lesssim \mathcal{X}(t)^2.$$

Thus, it follows that

$$\|n\|^{m}_{L^1_t(\dot B^{\frac d2+1})}+\frac{1}{\varepsilon}\|z\|^{m}_{L^1_t(\dot B^{\frac d2-1})}\lesssim \mathcal{E}_0^{\varepsilon}+\mathcal{X}(t)^2. \tag{3.69}$$

- High-frequency case.

In this case, we are able to obtain the expected estimate directly from Lemma 3.5 and the faster decay property of the high-frequency norm. Indeed, one concludes from directly (2.2), (3.52) and $2^{-J_\varepsilon}\lesssim\varepsilon$ that

$$\begin{aligned}\|z\|^{h}_{\widetilde L^2_t(\dot B^{\frac d2-1})}&\lesssim \|(u,E)\|^{h}_{\widetilde L^2_t(\dot B^{\frac d2-1})}+\|n\|^{h}_{\widetilde L^2_t(\dot B^{\frac d2})}+\varepsilon\|u\|^{h}_{\widetilde L^2_t(\dot B^{\frac d2-1})}\\ &\lesssim \varepsilon\|n\|^{h}_{\widetilde L^2_t(\dot B^{\frac d2+1})}+(\varepsilon+\varepsilon^2)\|u\|^{h}_{\widetilde L^2_t(\dot B^{\frac d2-1})}+\varepsilon\|E\|^{h}_{\widetilde L^2_t(\dot B^{\frac d2})}\\ &\lesssim \varepsilon\big(\mathcal{E}_0^{\varepsilon}+\mathcal{X}(t)^2\big).\end{aligned} \tag{3.70}$$

Combining (3.66), (3.69) and (3.70) together, we get (3.59)-(3.60) which concludes the proof of Proposition 3.6. □

Furthermore, we establish a stronger estimate for the effective velocity in every frequency regime.

Proposition 3.7. *Under the assumptions of Theorem 2.1, it holds that*

$$\|z\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})} \lesssim \big(\mathcal{E}_0^\varepsilon + \|z_0^\varepsilon\|_{\dot{B}^{\frac{d}{2}-1}} + \mathcal{X}(t)^2\big)\varepsilon, \tag{3.71}$$

with z defined by (3.59) and $z_0^\varepsilon = \dfrac{1}{\varepsilon}u_0^\varepsilon + \nabla h(\rho_0^\varepsilon) + E_0^\varepsilon$.

Proof. Recall that z satisfies the damped equation $(3.58)_2$. Thus, according to Lemma A.9 concerning L^2-in-time estimates on the whole frequencies, we have

$$\begin{aligned}
&\|z\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})} + \frac{1}{\varepsilon}\|z\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}-1})}\\
&\quad\lesssim \|z_0^\varepsilon\|_{\dot{B}^{\frac{d}{2}-1}} + \varepsilon\|(\partial_t\nabla n, \partial_t E)\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}-1})} + \varepsilon\|z\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}-1})} + \|F_2\|_{L^2(\dot{B}^{\frac{d}{2}-1})}.
\end{aligned}$$

It follows from the equation $(3.2)_1$ that

$$\begin{aligned}
\varepsilon\|\partial_t\nabla n\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}-1})} &\lesssim \varepsilon\|u\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}+1})} + \varepsilon\|u\cdot\nabla n\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}})} + \varepsilon\|G(n)\mathrm{div}\, u\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}})}\\
&\lesssim (1+\|n\|_{\widetilde{L}^\infty(\dot{B}^{\frac{d}{2}})})\varepsilon\|u\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}+1})} + \varepsilon\|u\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}})}\|n\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}+1})}\\
&\lesssim \varepsilon\mathcal{X}(t)^2.
\end{aligned}$$

Here we used (2.7), (A.2), (A.4) and the estimates obtained in Lemmas 3.3, ?? and 3.5. Similarly, one has

$$\varepsilon\|\partial_t E\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}-1})} \lesssim \|H\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}})} + \varepsilon\|u\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}-1})}(1+\|H\|_{\widetilde{L}^\infty(\dot{B}^{\frac{d}{2}})}) \lesssim \mathcal{X}(t)^2.$$

For the nonlinear term, one deduces that

$$\begin{aligned}
\|F_2\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}-1})} &\lesssim \|u\cdot\nabla u\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}-1})} + \|u\times H\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}-1})}\\
&\lesssim \|u\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}-1})}(\|u\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}})} + \|H\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}-1})}) \lesssim \mathcal{X}(t)^2.
\end{aligned}$$

Gathering the above estimates, we end up with (3.71). □

3.2.5. *Global existence.* Here, we construct a local Friedrichs approximation (see, e.g., [1, Page 440]) and extend it to a global one by the a priori estimates established in Section 3.2. Then, we show the convergence of the approximate sequence to the expected global solution to the Cauchy problem for the system (1.6).

Define the Friedrichs projector

$$\dot{\mathbb{E}}_k f := \mathcal{F}^{-1}(\mathbf{1}_{\mathcal{C}_k}\mathcal{F}f), \quad \forall f\in L^2_k,$$

where L^2_k is the set of L^2 functions spectrally supported in the annulus $\mathcal{C}_k := \{\xi\in\mathbb{R}^d \;:\; 1/k\le|\xi|\le k\}$ endowed with the standard L^2 topology, and $\mathbf{1}_{\mathcal{C}_k}$ is the characteristic function on the annulus $\mathcal{C}_k$.

Let $(\rho_0^\varepsilon, u_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)$ satisfying (2.6) with $n_0^\varepsilon = h(\rho_0^\varepsilon)$ and $H_0^\varepsilon = B_0^\varepsilon - \bar{B}$. For every $k\ge1$, we solve the following approximate problem of (3.2):

$$\begin{cases}
\partial_t n^k + \dot{\mathbb{E}}_k\big(u^k\cdot\nabla n^k + (P'(\bar\rho)+G(n^k))\mathrm{div}\, u^k\big) = 0,\\
\varepsilon^2(\partial_t u^k + \dot{\mathbb{E}}_k\big(u^k\cdot\nabla u^k\big) + \nabla n^k + E^k + u^k + \varepsilon u^k\times\bar{B} + \varepsilon u^k\times H^k\big) = 0,\\
\varepsilon\partial_t E^k - \dot{\mathbb{E}}_k\big(\nabla\times H^k + \varepsilon\bar\rho u^k + \varepsilon F(n^k)u^k\big) = 0,\\
\varepsilon\partial_t H^k + \dot{\mathbb{E}}_k\nabla\times E^k = 0,\\
\mathrm{div}\,\dot{\mathbb{E}}_k E^k = -K\dot{\mathbb{E}}_k n^k - \dot{\mathbb{E}}_k\Phi(n^k), \qquad \mathrm{div}\,\dot{\mathbb{E}}_k H^k = 0,\\
(n^k, u^k, E^k, H^k)(0,x) = (n_0^k, u_0^k, E_0^k, H_0^k)(x) := \dot{\mathbb{E}}_k(n_0^\varepsilon, \dfrac{1}{\varepsilon}u_0^\varepsilon, E_0^\varepsilon, H_0^\varepsilon).
\end{cases} \tag{3.72}$$

It is classical to show that $(n_0^k, u_0^k, E_0^k, H_0^k)$ satisfies (2.6) uniformly with respect to $k\ge1$ and converges to $(n_0^\varepsilon, \frac{1}{\varepsilon}u_0^\varepsilon, E_0^\varepsilon, H_0^\varepsilon)$ strongly in the sense (2.6). Since the Sobolev norms of any quantities localized with the project $\dot{\mathbb{E}}_k$ are equivalent (thanks to Bernstein inequality), we have that (3.72) is a system of ordinary differential equations in L^2_k . By virtue of the Cauchy-Lipschitz theorem in [1, Page 124], there exists a maximal time $T_k^*>0$ such that the problem (3.72) admits a unique solution $(n^k,u^k,E^k,H^k)\in C([0,T_k^*);L_k^2)$ on $[0,T_k^*)$.

Now we define the maximal time

$$T_k := \sup\{t\geq 0 \ : \ \mathcal{X}^k(t)\leq C_0\mathcal{E}_0^\varepsilon\} \tag{3.73}$$

where $\mathcal{X}^k(t)$ denotes the same norm $\mathcal{X}(t)$ given by (3.7) but for (n^k,u^k,E^k,H^k). Thus, T_* is well-defined and fulfills $0<T_k\leq T_k^*$.

We first claim $T_k=T_k^*$. To prove it, we assume that $T_k<T_k^*$ and use a contradiction argument. Since $(n^k,u^k,E^k,H^k)=\dot{\mathbb{E}}_k(n^k,u^k,E^k,H^k)$, the orthogonal projector $\dot{\mathbb{E}}_k$ has no effect on the energy estimates established in Lemmas 3.3, 3.4 and 3.5. By virtue of (3.10), (3.33), (3.52) and (3.73), as long as $\mathcal{E}_0^\varepsilon$ satisfies (2.6) such that (3.8) holds, we have

$$\mathcal{X}^k(t)\leq C_0\big(\mathcal{E}_0^\varepsilon+\mathcal{X}^k(t)^2+\mathcal{X}^k(t)^3\big),\quad 0<t<T_k. \tag{3.74}$$

By (3.74) and a standard bootstrap argument, one can choose a generic constant α_0 in (2.6) such that

$$\mathcal{X}^k(t)\leq \frac{1}{2}C_0\mathcal{E}_0^\varepsilon,\quad 0<t<T_k, \tag{3.75}$$

so T_k is not the maximal time such that $\mathcal{X}^k(t)\leq C_0\mathcal{E}_0^\varepsilon$ holds. This contradicts the definition of T_k. Let us now show that $T^*k=+\infty$. If $T_k^*<\infty$, by (3.75) and $T_k=T_k^*$, we can take $(n^k,u^k,E^k,H^k)(t)$ for t sufficiently close to T_k^* as the new initial data and obtain the existence from t to some $t+\eta^*>T_k^*$ with a suitably small constant $\eta^*>0$ by the Cauchy-Lipschitz theorem, which contradicts the definition of T_k^*. Therefore, we have $T_k^*=\infty$ and (n^k,u^k,E^k,H^k) is a global solution to (3.2).

From the uniform estimate $\mathcal{X}^k(t)\leq C_0\mathcal{E}_0^\varepsilon$ and (3.2), one can estimate the time derivatives $(\partial_t n^k,\partial_t u^k,\partial_t E_k,\partial_t H_k)$ uniformly with respect to k. According to these uniform estimates, the Aubin-Lions lemma and the Cantor diagonal process, there exists a limit (n,u,E,H) such that, as $k\to\infty$, it holds, up to a subsequence, that (n^k,u^k,E^k,H^k) converges to (n,u,E,H) strongly in $L^2_{loc}(\mathbb{R}_+;H^{\frac{d}{2}}_{loc})$. Thus, it is easy to prove that the limit (n,u,E,H) solves the system (3.2) in the sense of distributions. Thanks to Fatou's property $\mathcal{X}(t)\lesssim \liminf\limits_{k\to\infty}\mathcal{X}^k(t)$, we know that $\mathcal{X}(t)\leq C_0\mathcal{E}_0^\varepsilon$ for all $t>0$. Denote ρ and B by

$$\rho:=\bar{\rho}+Kn+\Phi(n),\qquad B:=H+\bar{B},$$

with $\Phi(n)$ given by (3.3). Then, one can show that (ρ,u,E,B) is a classical solution to the original system (1.6)-(1.7) subject to the initial datum (ρ_0,u_0,E_0,B_0). By standard product laws, composition estimates and Propositions 3.6-3.7, (ρ,u,E,B) satisfies the properties (2.7)-(2.9). In addition, following a similar argument as in [3, Page 196], one has $(\rho-\bar{\rho},u,E,B-\bar{B})\in\mathcal{C}(\mathbb{R}_+;\dot{B}^{\frac{d}{2}-1,\frac{d}{2}+1})$. To finish the proof of Theorem 2.1, we show that the solution constructed in this subsection is unique.

3.2.6. *Uniqueness.* The proof of the uniqueness does not require the smallness of regularity for initial data. Let (ρ_1,u_1,E_1,H_1) and (ρ_2,u_2,E_2,H_2) be two solutions of system (1.6) with the same initial datum $(\rho_0^\varepsilon,\frac{1}{\varepsilon}u_0^\varepsilon,E_0^\varepsilon,H_0^\varepsilon)$. For given time $T>0$, let (ρ_i,u_i,E_i,H_i), $i=1,2$, satisfy $(\rho_i-\bar{\rho},u_i,E_i,B_i-\bar{B})\in L^\infty(0,T;\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})$ and $\rho_-\leq\rho_i\leq\rho_+$ for some constants $0<\rho_-\leq\rho_+<\infty$. Since the relaxation parameter ε does not play a role in the proof of uniqueness, we set $\varepsilon=1$. Let

$$(\delta\rho,\delta u,\delta E,\delta B)=(\rho_1-\rho_2,u_1-u_2,E_1-E_2,B_1-B_2).$$

The error unknown $(\delta\rho,\delta u,\delta E,\delta B)$ solves

$$\begin{cases}\partial_t\delta\rho+u_1\cdot\nabla\delta\rho+\rho_1\mathrm{div}\,\delta u=\delta F^1,\\ \partial_t\delta u+u_1\cdot\nabla\delta u+M(\rho_1)\nabla\delta\rho+\delta u+\delta E+\delta u\times\bar{B}=\delta F^2,\\ \partial_t\delta E-\nabla\times\delta B-\bar{\rho}\delta u=\delta F^3,\\ \partial_t\delta B+\nabla\times\delta E=0,\\ \mathrm{div}\,\delta E=-\delta\rho,\qquad \mathrm{div}\,\delta B=0,\end{cases} \tag{3.76}$$

with $M(s) = P'(s)/s$ and

$$\begin{aligned}
&\delta F^1 = -\delta u\cdot\nabla\rho_2 - \delta\rho\,\mathrm{div}\, u_2,\\
&\delta F^2 = -\delta u\cdot\nabla u_2 - (M(\rho_1)-M(\rho_2))\nabla\rho_2 - u_1\times\delta B_2 - \delta u\times(B_2-\bar{B}),\\
&\delta F^3 = \delta\rho u_1 + (\rho_2-\bar{\rho})\delta u.
\end{aligned}$$

Applying $\dot{\Delta}_j$ to (3.76) leads to

$$\begin{cases}
\partial_t\delta\rho_j + u_1\cdot\nabla\delta\rho_j + \rho_1\mathrm{div}\,\delta u_j = \delta F_j^1 + \delta R_{1,j} + \delta R_{2,j},\\
\partial_t\delta u_j + u_1\cdot\nabla\delta u_j + M(\rho_1)\nabla\delta\rho_j + \delta u_j + \delta E_j + \delta u_j\times\bar{B} = \delta F_j^2 + \delta R_{3,j} + \delta R_{4,j},\\
\partial_t\delta E_j - \nabla\times\delta B_j - \bar{\rho}\delta u_j = \delta F_j^3,\\
\partial_t\delta B_j + \nabla\times\delta E_j = 0,\\
\mathrm{div}\,\delta E_j = -\delta\rho_j, \qquad \mathrm{div}\,\delta B_j = 0,
\end{cases} \tag{3.77}$$

where commutator terms are defined as $\delta R_{1,j} := [u_1,\dot{\Delta}_j]\nabla\delta\rho$, $\delta R_{2,j} := [\rho_1,\dot{\Delta}_j]\nabla\delta u$, $\delta R_{3,j} := [u_1,\dot{\Delta}_j]\nabla\delta u$ and $\delta R_{4,j} := [M(\rho_1),\dot{\Delta}_j]\nabla\delta\rho$.

After a direct computation on (3.77) one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\big(\frac{1}{\rho_1}|\delta\rho_j|^2 + \frac{1}{M(\rho_1)}|\delta u_j|^2 + \frac{1}{P'(\bar{\rho})}|E_j|^2 + \frac{1}{P'(\bar{\rho})}|B_j|^2\big)\,dx + \int\frac{1}{M(\rho_1)}|u_j|^2dx\\
&\le \frac{1}{2}(\|\partial_t\frac{1}{\rho_1}\|_{L^\infty} + \|\nabla\frac{u_1}{\rho_1}\|_{L^\infty})\|\delta\rho_j\|_{L^2}^2 + \frac{1}{2}(\|\partial_t\frac{1}{M(\rho_1)}\|_{L^\infty} + \|\nabla\frac{u_1}{M(\rho_1)}\|_{L^\infty})\|\delta u_j\|_{L^2}^2\\
&\quad + \|\frac{1}{M(\rho_1)} - \frac{1}{M(\bar{\rho})}\|_{L^\infty}\|u_j\|_{L^2}\|E_j\|_{L^2} + \|\frac{1}{\rho_1}\|_{L^\infty}\|(\delta F_j^1,\delta R_{1,j},\delta R_{2,j})\|_{L^2}\|\delta\rho_j\|_{L^2}\\
&\quad + \|\frac{1}{M(\rho_1)}\|_{L^\infty}\|(\delta F_j^2,\delta R_{3,j},\delta R_{4,j})\|_{L^2}\|\delta u_j\|_{L^2} + \frac{1}{P'(\bar{\rho})}\|\delta F_j^3\|_{L^2}\|\delta E_j\|_{L^2}.
\end{aligned}$$

This yields

$$\begin{aligned}
\|(\delta\rho,\delta u,\delta E,\delta B)\|_{\dot{B}^{\frac{d}{2}}} &\lesssim \int_0^T(1+\|(\partial_t\rho_1,\nabla\rho_1,\nabla u_1)\|_{L^\infty})(\|(\delta\rho,\delta u)\|_{\dot{B}^{\frac{d}{2}}}dt\\
&\quad + \int_0^T\big(\|(\delta F_1,\delta F_2,\delta F_2)\|_{\dot{B}^{\frac{d}{2}}} + \sum_{j\in\mathbb{Z}}2^{\frac{d}{2}j}\|(\delta R_{1,j},\delta R_{2,j},\delta R_{3,j},\delta R_{4,j})\|_{L^2}\big)d\tau.
\end{aligned} \tag{3.78}$$

Using the product law (A.2) and the composition estimates (A.4) and (A.5), we have the following estimates of the nonlinear terms

$$\|(\delta F_1,\delta F_2,\delta F_2)\|_{\dot{B}^{\frac{d}{2}}} \lesssim (\|\nabla(\rho_2,u_2)\|_{\dot{B}^{\frac{d}{2}}} + \|(\rho_2-\bar{\rho},u_1,B_2-\bar{B})\|_{\dot{B}^{\frac{d}{2}}})\|(\delta\rho,\delta u)\|_{\dot{B}^{\frac{d}{2}}}. \tag{3.79}$$

In view of the composition estimate (A.3), one arrives at

$$\sum_{j\in\mathbb{Z}}2^{\frac{d}{2}j}\|(\delta R_{1,j},\delta R_{2,j},\delta R_{3,j},\delta R_{4,j})\|_{L^2} \lesssim \|\nabla(\rho_1,u_2)\|_{\dot{B}^{\frac{d}{2}}}\|(\delta\rho,\delta u)\|_{\dot{B}^{\frac{d}{2}}}. \tag{3.80}$$

Putting (3.79) and (3.80) into (3.78) and taking advantage of Grönwall's inequality, we have $(\rho_1,u_1,E_1,H_1) = (\rho_2,u_2,E_2,H_2)$ for all $(x,t)\in\mathbb{R}^d\times[0,T]$. This concludes the proof of the uniqueness and of Theorem 2.1.

4. Strong relaxation limit

4.1. **Proof of Theorem 2.3.** In this subsection, we establish the error estimates between the solutions of (1.6) and (1.8) for ill-prepared initial data. Before this, we first provide a global well-posedness result for the drift-diffusion system (1.9). Since the proof can be done similarly as in [26], we omitted the details for brevity.

Theorem 4.1. *There exists a generic constant α_1 such that if*

$$\|\rho_0^* - \bar{\rho}\|_{\dot{B}^{\frac{d}{2}-1,\frac{d}{2}}} \le \alpha_1, \tag{4.1}$$

then the Cauchy problem (1.9) *has a unique global solution* ρ^* *fulfilling* $\rho^*-\bar\rho\in\mathcal{C}(\mathbb{R}^+;\dot B^{\frac d2-1,\frac d2})$ *and*

$$\begin{aligned}\|\rho^*-\bar\rho\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1}\cap\dot B^{\frac d2})}&+\|\rho^*-\bar\rho\|_{L^1_t(\dot B^{\frac d2-1,\frac d2+2})}\\&+\|\rho^*-\bar\rho\|_{\widetilde L^2_t(\dot B^{\frac d2-1,\frac d2+1})}\lesssim\|\rho^*_0-\bar\rho\|_{\dot B^{\frac d2-1,\frac d2}}.\end{aligned}\tag{4.2}$$

Denote the error unknowns

$$(\delta\rho,\delta u,\delta E,\delta B):=(\rho^\varepsilon-\rho^*,u^\varepsilon-u^*,E^\varepsilon-E^*,B^\varepsilon-B^*).\tag{4.3}$$

Recall that the effective velocity z_ε is given by (1.16). Substituting $u^\varepsilon=z^\varepsilon-\nabla h(\rho^\varepsilon)-E^\varepsilon-\varepsilon u^\varepsilon\times\bar B$ into (1.6), we obtain

$$\begin{cases}\partial_t\rho^\varepsilon-P'(\bar\rho)\Delta\rho^\varepsilon+\bar\rho\rho^\varepsilon=\operatorname{div}\big(-\rho^\varepsilon z^\varepsilon+\varepsilon\rho^\varepsilon u^\varepsilon\times\bar B+(P'(\rho^\varepsilon)-P'(\bar\rho))\nabla\rho^\varepsilon+(\rho^\varepsilon-\bar\rho)E^\varepsilon\big),\\ u^\varepsilon=z^\varepsilon-\nabla h(\rho^\varepsilon)-E^\varepsilon-\varepsilon u^\varepsilon\times\bar B,\\ \varepsilon\partial_tE^\varepsilon-\nabla\times B^\varepsilon+\varepsilon\rho^\varepsilon E^\varepsilon=\varepsilon\big(\rho^\varepsilon z^\varepsilon-\varepsilon u^\varepsilon\times\bar B-\nabla P(\rho^\varepsilon)\big),\\ \varepsilon\partial_tB^\varepsilon+\nabla\times E^\varepsilon=0,\\ \operatorname{div}E^\varepsilon=\bar\rho-\rho^\varepsilon,\qquad \operatorname{div}B^\varepsilon=0.\end{cases}\tag{4.4}$$

where $h(\rho)$ is the enthalpy defined in (1.11). From (1.8) and (4.4), the equations of $(\delta\rho,\delta u)$ read

$$\begin{cases}\partial_t\delta\rho-P'(\bar\rho)\Delta\delta\rho+\bar\rho\delta\rho=\operatorname{div}(-\rho^\varepsilon z^\varepsilon+\varepsilon\rho^\varepsilon u^\varepsilon\times\bar B+\delta F),\\ \delta u=z^\varepsilon-\nabla(h(\rho^\varepsilon)-h(\rho^*))-\delta E,\end{cases}\tag{4.5}$$

where

$$\delta F:=(P'(\rho^\varepsilon)-P'(\rho^*))\nabla\rho^\varepsilon+(P'(\rho^*)-P'(\bar\rho))\nabla\delta\rho+\delta\rho E^\varepsilon+(\rho^*-\bar\rho)\delta E.$$

Due to $E^*=\nabla(-\Delta)^{-1}(\rho^*-\bar\rho)$, Darcy's law (1.10) and the fact that $\nabla\operatorname{div}=\nabla\times\nabla\times+\Delta$, one has

$$\begin{aligned}\partial_tE^*&=-\nabla(-\Delta)^{-1}\operatorname{div}(\rho^*u^*)\\&=\rho^*u^*+\nabla\times B^{1,*}\\&=-\rho^*E^*-\nabla P(\rho^*)+\nabla\times B^{1,*},\end{aligned}\tag{4.6}$$

with the term

$$B^{1,*}=-(-\Delta)^{-1}\nabla\times(\rho^*u^*).$$

In order to handle the last term on the right-hand side of (4.6), we introduce the modified error of the magnetic induction

$$\delta\mathcal{B}:=\delta B+\varepsilon B^{1,*}.$$

Then, by (4.4), (4.6) and $B^*=\bar B$, we obtain the equations of $(\delta E,\delta\mathcal{B})$ as follows

$$\begin{cases}\partial_t\delta E-\dfrac1\varepsilon\nabla\times\delta\mathcal{B}+\bar\rho\delta E=\rho^\varepsilon z^\varepsilon-\varepsilon\rho^\varepsilon u^\varepsilon\times\bar B-P'(\bar\rho)\nabla\delta\rho-\delta F,\\ \partial_t\delta\mathcal{B}+\dfrac1\varepsilon\nabla\times\delta E=\varepsilon\partial_tB^{1,*},\\ \operatorname{div}\delta E=-\delta\rho,\qquad\operatorname{div}\delta\mathcal{B}=0.\end{cases}\tag{4.7}$$

- **Step 1: Estimates in high frequencies.**

Recall that J_ε is given by (2.1). According to the property of high-frequency norms in (2.2) and the fact that $E^*=\nabla(-\Delta)^{-1}(\rho^*-\bar\rho)$, the uniform bounds (2.7) and (4.2) give directly

$$\begin{cases}\|\delta\rho\|^h_{\widetilde L^\infty_t(\dot B^{\frac d2-1})\cap\widetilde L^2_t(\dot B^{\frac d2})}\lesssim\varepsilon\|(\rho^\varepsilon-\bar\rho,\rho^*-\bar\rho)\|^h_{\widetilde L^\infty_t(\dot B^{\frac d2})\cap\widetilde L^2_t(\dot B^{\frac d2+1})}\lesssim(\alpha_0+\alpha_1)\varepsilon,\\ \|\delta E\|^h_{\widetilde L^\infty_t(\dot B^{\frac d2-1})\cap\widetilde L^2_t(\dot B^{\frac d2-1})}\lesssim\varepsilon\|(E^\varepsilon,E^*)\|^h_{\widetilde L^\infty_t(\dot B^{\frac d2})\cap\widetilde L^2_t(\dot B^{\frac d2})}\lesssim(\alpha_0+\alpha_1)\varepsilon,\\ \|\delta B\|^h_{\widetilde L^\infty_t(\dot B^{\frac d2-1})\cap\widetilde L^2_t(\dot B^{\frac d2-1})}\lesssim\varepsilon\|B^\varepsilon-\bar B\|^h_{\widetilde L^\infty_t(\dot B^{\frac d2})\cap\widetilde L^2_t(\dot B^{\frac d2})}\lesssim\alpha_0\varepsilon\end{cases}\tag{4.8}$$

Hence, it suffices to estimate $(\delta\rho,\delta E,\delta B)$ in low and medium frequencies as follows.

- **Step 2: Estimates of $\delta\rho$ in low frequencies.**

In the low-frequency region, one needs to perform $\dot{B}^{\frac{d}{2}-1}$-order estimate of $\delta\rho$ so as to concide with the regularity of z^ε. Making use of Lemma A.9 applied to $(4.5)_1$ with $f_1=-\mathrm{div}\,(\rho^\varepsilon z^\varepsilon)$, $f_2=\varepsilon\mathrm{div}\,(\rho^\varepsilon u^\varepsilon\times\bar{B})$ and $f_3=\mathrm{div}\,(\delta F)$, we have

$$\begin{aligned}&\|\delta\rho\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})}+\|\delta\rho\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1,\frac{d}{2}})}\\&\lesssim\|\rho^\varepsilon_0-\rho^*_0\|^\ell_{\dot{B}^{\frac{d}{2}-1}}+\|\mathrm{div}\,(\rho^\varepsilon z^\varepsilon)\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1})}+\varepsilon\|\mathrm{div}\,(\rho^\varepsilon u^\varepsilon\times\bar{B})\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-2})}+\|\mathrm{div}\,\delta F_1\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}\\&\lesssim\|\rho^\varepsilon_0-\rho^*_0\|^\ell_{\dot{B}^{\frac{d}{2}-1}}+\|\rho^\varepsilon z^\varepsilon\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}})}+\varepsilon\|\rho^\varepsilon u^\varepsilon\times\bar{B}\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}+\|\delta F_1\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}.\end{aligned}\tag{4.9}$$

We recall that z^ε has the decay estimate (2.8). Using (2.2), (2.7), (2.8) and (A.2), we obtain

$$\begin{aligned}\|\rho^\varepsilon z^\varepsilon\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}})}&\le\|\bar{\rho}z^\varepsilon\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}})}+\|(\rho^\varepsilon-\bar{\rho})z^\varepsilon\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}})}\\&\lesssim\alpha_0\varepsilon+\|(\rho^\varepsilon-\bar{\rho})(z^\varepsilon)^\ell\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}})}+\|(\rho^\varepsilon-\bar{\rho})(z^\varepsilon)^m\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1})}\\&\quad+\|(\rho^\varepsilon-\bar{\rho})(z^\varepsilon)^h\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1})}\\&\lesssim\alpha_0\varepsilon+\|\rho^\varepsilon-\bar{\rho}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})}\|z^\varepsilon\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}})}+\|\rho^\varepsilon-\bar{\rho}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})}\|z^\varepsilon\|^m_{L^1_t(\dot{B}^{\frac{d}{2}-1})}\\&\quad+\|\rho^\varepsilon-\bar{\rho}\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}\|z^\varepsilon\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}\\&\lesssim\alpha_0\varepsilon.\end{aligned}\tag{4.10}$$

By virtue of (2.7) and (A.2), it also holds that

$$\begin{aligned}\varepsilon\|\rho^\varepsilon u^\varepsilon\times\bar{B}\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}&\le\varepsilon\|\bar{\rho}u^\varepsilon\times\bar{B}\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}+\varepsilon\|(\rho^\varepsilon-\bar{\rho})u^\varepsilon\times\bar{B}\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}\\&\lesssim\varepsilon(1+\|\rho^\varepsilon-\bar{\rho}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})})\|u^\varepsilon\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}\lesssim\alpha_0\varepsilon.\end{aligned}\tag{4.11}$$

Then, we estimate the term δF. Using (2.7), (A.2) and (A.5) yields

$$\begin{aligned}\|(P'(\rho^\varepsilon)-P'(\rho^*))\nabla\rho^\varepsilon\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}&\lesssim\|(P'(\rho^\varepsilon)-P'(\rho^*))\nabla\rho^\varepsilon\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}\\&\lesssim\|P'(\rho^\varepsilon)-P'(\rho^*)\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}\|\rho^\varepsilon\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})}\lesssim\alpha_0\|\delta\rho\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}.\end{aligned}$$

Similarly, it follows that

$$\begin{aligned}\|(P'(\rho^*)-P'(\bar{\rho}))\nabla\delta\rho\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}&\lesssim\|(P'(\rho^*)-P'(\bar{\rho}))\nabla\delta\rho\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-2})}\\&\lesssim\|P'(\rho^*)-P'(\bar{\rho})\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}\|\nabla\delta\rho\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-2})}\lesssim\alpha_0\|\delta\rho\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})},\end{aligned}$$

and

$$\|\delta\rho E^\varepsilon\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}\lesssim\|\delta\rho E^\varepsilon\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}\lesssim\|\delta\rho\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}\|E^\varepsilon\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}\lesssim\alpha_0\|\delta\rho\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}.$$

Considering the term involving δE in δF, we have

$$\begin{aligned}&\|(\rho^*-\bar{\rho})\delta E\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}\\&\lesssim\|(\rho^*-\bar{\rho})\delta E^\ell\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}+\|(\rho^*-\bar{\rho})\delta E^m\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}+\|(\rho^*-\bar{\rho})\delta E^h\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}\\&\lesssim\|\rho^*-\bar{\rho}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})}(\|\delta E\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}+\|\delta E\|^m_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}+\|\delta E\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})})\\&\lesssim\alpha_1(\|\delta E\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}+\|\delta E\|^m_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}+\varepsilon),\end{aligned}$$

where we used (4.2) and (4.8). Therefore, we arrive at

$$\begin{aligned}\|\delta F\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}&\lesssim\|(P'(\rho^\varepsilon)-P'(\rho^*))\nabla\rho^\varepsilon\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}+\|(P'(\rho^*)-P'(\bar{\rho}))\nabla\delta\rho\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}})}\\&\quad+\|\delta\rho E^\varepsilon\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}+\|(\rho^*-\bar{\rho})\delta E\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}\\&\lesssim(\alpha_0+\alpha_1)(\|\delta\rho\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}+\|\delta E\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}+\|\delta E\|^m_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}+\varepsilon).\end{aligned}\tag{4.12}$$

Inserting (4.10), (4.11) and (4.12) into (4.9) leads to

$$
\begin{aligned}
&\|\delta\rho\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-1})}+\|\delta\rho\|^{\ell}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1,\frac{d}{2}})}\\
&\quad\lesssim\|\rho_0^{\varepsilon}-\rho_0^{*}\|_{\dot{B}^{\frac{d}{2}-1}}+(\alpha_0+\alpha_1)(\|\delta\rho\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})}+\|\delta E\|^{\ell}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}})}+\|\delta E\|^{m}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})}+\varepsilon).
\end{aligned}\tag{4.13}
$$

- **Step 3: Estimates of $\delta\rho$ in medium frequencies.**

According to the regularity of z^{ε} in (2.8), we perform $\dot{B}^{\frac{d}{2}-2}$-order estimate of $\delta\rho$ in medium frequencies. Similarly, applying Lemma A.8 to $(4.5)_1$ we have

$$
\begin{aligned}
&\|\delta\rho\|^{m}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-2})}+\|\delta\rho\|^{m}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})}+\|\delta\rho\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}})}\\
&\quad\lesssim\|\rho_0^{\varepsilon}-\rho_0^{*}\|^{m}_{\dot{B}^{\frac{d}{2}-2}}+\|\rho^{\varepsilon}z^{\varepsilon}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}+\varepsilon\|\rho^{\varepsilon}u^{\varepsilon}\times\bar{B}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}+\|\delta F\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}.
\end{aligned}\tag{4.14}
$$

Using (2.7), (2.8) and (A.2), we have

$$
\begin{aligned}
\|\rho^{\varepsilon}z^{\varepsilon}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}&\leq\|\bar{\rho}z^{\varepsilon}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}+\|(\rho^{\varepsilon}-\bar{\rho})(z^{\varepsilon})^{\ell}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}})}\\
&\quad+\|(\rho^{\varepsilon}-\bar{\rho})(z^{\varepsilon})^{m}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}+\|(\rho^{\varepsilon}-\bar{\rho})(z^{\varepsilon})^{h}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}\\
&\lesssim\alpha_0\varepsilon+\|\rho^{\varepsilon}-\bar{\rho}\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\|z^{\varepsilon}\|^{\ell}_{L^{1}_t(\dot{B}^{\frac{d}{2}})}+\|\rho^{\varepsilon}-\bar{\rho}\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\|z^{\varepsilon}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}\\
&\quad+\|\rho^{\varepsilon}-\bar{\rho}\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}})}\|z^{\varepsilon}\|^{h}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})}\\
&\lesssim\alpha_0\varepsilon.
\end{aligned}\tag{4.15}
$$

And it also holds that

$$
\begin{aligned}
\varepsilon\|\rho^{\varepsilon}u^{\varepsilon}\times\bar{B}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}&\leq\varepsilon\|\bar{\rho}u^{\varepsilon}\times\bar{B}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}+\varepsilon\|(\rho^{\varepsilon}-\bar{\rho})u^{\varepsilon}\times\bar{B}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}\\
&\lesssim\varepsilon\|u^{\varepsilon}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}+\varepsilon\|\rho^{\varepsilon}-\bar{\rho}\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}})}\|u^{\varepsilon}\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})}\lesssim\alpha_0\varepsilon.
\end{aligned}\tag{4.16}
$$

We now estimate the norm of δF. From (4.2), (A.2) and (A.5) one has

$$
\begin{aligned}
\|(P'(\rho^{\varepsilon})-P'(\rho^{*}))\nabla\rho^{\varepsilon}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}&\lesssim\|P'(\rho^{\varepsilon})-P'(\rho^{*})\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})}\|\rho^{\varepsilon}\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}})}\\
&\lesssim\alpha_0\|\delta\rho\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})},
\end{aligned}
$$

and

$$
\|\delta\rho E^{\varepsilon}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}\lesssim\|\delta\rho\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})}\|E^{\varepsilon}\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}})}\lesssim\alpha_0\|\delta\rho\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})}.
$$

In addition, using (2.2), (4.2), (4.8), (A.2), (A.4), we get

$$
\begin{aligned}
&\|(P'(\rho^{*})-P'(\bar{\rho}))\nabla\delta\rho\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}\\
&\lesssim\|(P'(\rho^{*})-P'(\bar{\rho}))\nabla\delta\rho^{\ell}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}+\|(P'(\rho^{*})-P'(\bar{\rho}))\nabla\delta\rho^{m}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}\\
&\quad+\|(P'(\rho^{*})-P'(\bar{\rho}))\nabla\delta\rho^{h}\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}\\
&\lesssim\|P'(\rho^{*})-P'(\bar{\rho})\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})}\|\delta\rho\|^{\ell}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}})}+\|P'(\rho^{*})-P'(\bar{\rho})\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}})}(\|\delta\rho\|^{m}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})}+\|\delta\rho\|^{h}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})})\\
&\lesssim\alpha_0(\|\delta\rho\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})}+\varepsilon).
\end{aligned}
$$

A similar computation yields

$$
\|(\rho^{*}-\bar{\rho})\delta E\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}\lesssim\alpha_1(\|\delta E\|^{\ell}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}})}+\|\delta E\|^{m}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})}+\varepsilon).
$$

It thus holds that

$$
\|\delta F\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-1})}\lesssim(\alpha_0+\alpha_1)(\|\delta\rho\|^{\ell}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}})}+\|\delta\rho\|^{m}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})}+\|\delta E\|^{\ell}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}})}+\|\delta E\|^{m}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})}+\varepsilon).\tag{4.17}
$$

The combination of (2.8) and (4.14)-(4.17) gives rise to

$$
\begin{aligned}
&\|\delta\rho\|^{m}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-2})}+\|\delta\rho\|^{m}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})}+\|\delta\rho\|^{m}_{L^{1}_t(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}})}\\
&\quad\lesssim\|\rho_0^{\varepsilon}-\rho_0^{*}\|^{m}_{\dot{B}^{\frac{d}{2}-2}}+(\alpha_0+\alpha_1)(\|\delta\rho\|_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})}+\|\delta E\|^{\ell}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}})}+\|\delta E\|^{m}_{\widetilde{L}^{2}_t(\dot{B}^{\frac{d}{2}-1})}+\varepsilon).
\end{aligned}\tag{4.18}
$$

- **Step 4: Estimates of $(\delta E, \delta\mathcal{B})$ in low frequencies.**

We now perform a hycoercive argument to estimate $(\delta E, \delta\mathcal{B})$ in the low-frequency $\dot{B}^{\frac{d}{2}}$-regularity level. From (4.7), we have the basic localized energy inequality

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\delta E_j,\delta\mathcal{B}_j)\|_{L^2}^2+\bar{\rho}\|\delta E_j\|_{L^2}\\
&\leq \|\dot{\Delta}_j(\rho^\varepsilon z^\varepsilon-\varepsilon\rho^\varepsilon u^\varepsilon\times\bar{B}-P'(\bar{\rho})\nabla\delta\rho-\delta F)\|_{L^2}\|\delta E_j\|_{L^2}\\
&\quad+\varepsilon\|\partial_t B_j^{1,*}\|_{L^2}\|\delta\mathcal{B}_j\|_{L^2},
\end{aligned}
\tag{4.19}
$$

and the cross inequality

$$
\begin{aligned}
&-\frac{d}{dt}\int\varepsilon\delta E_j\cdot\nabla\times\delta\mathcal{B}_j\,dx+\|\nabla\times\mathcal{B}_j\|_{L^2}^2+\bar{\rho}\int\delta E_j\cdot\nabla\times\delta\mathcal{B}_j\,dx-\|\nabla\times E_j\|_{L^2}^2\\
&\leq\varepsilon\|\dot{\Delta}_j(\rho^\varepsilon z^\varepsilon-\varepsilon\rho^\varepsilon u^\varepsilon\times\bar{B}-P'(\bar{\rho})\nabla\delta\rho-\delta F)\|_{L^2}\|\nabla\times\delta\mathcal{B}_j\|_{L^2}\\
&\quad+\varepsilon^2\|\partial_t B_j^{1,*}\|_{L^2}\|\nabla\times\delta E_j\|_{L^2}.
\end{aligned}
\tag{4.20}
$$

Define the functional

$$
\delta\mathcal{L}_{\ell,j}(t):=\frac{1}{2}\|(\delta E_j,\delta\mathcal{B}_j)\|_{L^2}^2+\eta_{*\ell}\int\varepsilon\delta E_j\cdot\nabla\times\delta\mathcal{B}_j\,dx.
$$

Then for all $j\leq 0$ and some sufficiently small constant $\eta_{*\ell}$, one deduces from (4.19) and (4.20) that $\delta\mathcal{L}_{\ell,j}(t)\sim\|(\delta E_j,\delta\mathcal{B}_j)\|_{L^2}^2$ and

$$
\begin{aligned}
&\frac{d}{dt}\delta\mathcal{L}_{\ell,j}(t)+\|\delta E_j\|_{L^2}^2+2^{2j}\|\delta\mathcal{B}_j\|_{L^2}^2\\
&\leq C(\|\dot{\Delta}_j(\rho^\varepsilon z^\varepsilon)\|_{L^2}+\varepsilon\|\partial_t B_j^{1,*}\|_{L^2})\sqrt{\delta\mathcal{L}_{\ell,j}}\\
&\quad+(\varepsilon\|\dot{\Delta}_j(\rho^\varepsilon u^\varepsilon\times\bar{B})\|_{L^2}+\|\nabla\delta\rho_j\|_{L^2}+\|\delta F_j\|_{L^2})(\|\delta E_j\|_{L^2}+2^j\|\mathcal{B}_j\|_{L^2}).
\end{aligned}
$$

Applying Lemma A.7 (2) to the above inequality leads to

$$
\begin{aligned}
&\|(\delta E_j,\delta\mathcal{B}_j)\|_{L^\infty_t(L^2)}+\|\delta E_j\|_{L^2_t(L^2)}+2^j\|\delta\mathcal{B}_j\|_{L^2_t(L^2)}\\
&\lesssim\|(\delta E_j,\delta\mathcal{B}_j)(0)\|_{L^2}+\|\dot{\Delta}_j(\rho^\varepsilon z^\varepsilon)\|_{L^1_t(L^2)}+\varepsilon\|\dot{\Delta}_j(\rho^\varepsilon u^\varepsilon\times\bar{B})\|_{L^2_t(L^2)}+\|\delta\rho_j\|_{L^2_t(L^2)}\\
&\quad+\|\delta F_j\|_{L^2_t(L^2)}+\varepsilon\|\partial_t B_j^{1,*}\|_{L^1_t(L^2)}.
\end{aligned}
\tag{4.21}
$$

Here one has used $\|\nabla\delta\rho_j\|_{L^2(L^2)}\lesssim\|\delta\rho_j\|_{L^2(L^2)}$ due to $j\leq 0$. Multiplying (4.21) by $2^{j\frac{d}{2}}$ and summing it over $j\leq 0$, we get the low-frequency estimate

$$
\begin{aligned}
&\|(\delta E,\delta\mathcal{B})\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})}+\|\delta E\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}+\|\delta\mathcal{B}\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}\\
&\lesssim\|(E_0^\varepsilon-E_0^*,B_0^\varepsilon-\bar{B},\varepsilon B^{1,*}(0))\|^\ell_{\dot{B}^{\frac{d}{2}}}+\|\rho^\varepsilon z^\varepsilon\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}})}+\varepsilon\|\rho^\varepsilon u^\varepsilon\times\bar{B}\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}+\|\delta\rho\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}\\
&\quad+\|\delta F\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}+\varepsilon\|\partial_t B^{1,*}\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}})}.
\end{aligned}
\tag{4.22}
$$

Before estimating (4.22), we need to give some necessary bounds of $B^{1,*}$.

Lemma 4.2. *Let $B^{1,*}=-(-\Delta)^{-1}\nabla\times(\rho^*u^*)$. Then it holds that*

$$
\begin{cases}
\|B^{1,*}(0)\|_{\dot{B}^{\frac{d}{2}}}\lesssim\|\rho_0^*-\bar{\rho}\|^2_{\dot{B}^{\frac{d}{2}-1}},\\
\|B^{1,*}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})\cap\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}\lesssim\|\rho_0^*-\bar{\rho}\|^2_{\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}}},\\
\|\partial_t B^{1,*}\|_{L^1_t(\dot{B}^{\frac{d}{2}})}\lesssim\|\rho_0^*-\bar{\rho}\|^2_{\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}}}.
\end{cases}
\tag{4.23}
$$

Proof. It follows from $E^*=\nabla(-\Delta)^{-1}\rho^*$ that

$$
B^{1,*}=(-\Delta)^{-1}\nabla\times\big(\nabla P(\rho^*)+\rho^*E^*\big)=(-\Delta)^{-1}\nabla\times\big((\rho^*-\bar{\rho})\nabla(-\Delta)^{-1}\rho^*\big).
$$

Hence, for the initial datum $B^{1,*}(0)$ of $B^{1,*}$, by employing the product law (A.2) we arrive at

$$
\begin{aligned}
\|B^{1,*}(0)\|_{\dot{B}^{\frac{d}{2}}}&\lesssim\|(\rho_0^*-\bar{\rho})\nabla(-\Delta)^{-1}\rho_0^*\|_{\dot{B}^{\frac{d}{2}-1}}\\
&\lesssim\|\rho_0^*-\bar{\rho}\|_{\dot{B}^{\frac{d}{2}-1}}\|\nabla(-\Delta)^{-1}\rho_0^*\|_{\dot{B}^{\frac{d}{2}}}\lesssim\|\rho_0^*-\bar{\rho}\|^2_{\dot{B}^{\frac{d}{2}-1}}.
\end{aligned}
$$

Concerning the estimates of $B^{1,*}$, a similar computation gives

$$\|B^{1,*}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})\cap\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})} \lesssim \|\rho^* u^*\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})\cap\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})} \lesssim \|\rho^*-\bar{\rho}\|^2_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})\cap\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})} \lesssim \|\rho_0^*-\bar{\rho}\|^2_{\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}}},$$

where we have used (4.2). Finally, using $(1.9)_1$ and (4.2), one gets the estimate of the time deterivative $\partial_t\rho^*$:

$$\|\partial_t\rho^*\|_{L^1_t(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})} \lesssim \|\rho^*-\bar{\rho}\|_{L^1_t(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}+2})} \lesssim \|\rho_0^*-\bar{\rho}\|_{\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}}}.$$

Hence, one also has

$$\begin{aligned}\|\partial_t B^{1,*}\|_{L^1_t(\dot{B}^{\frac{d}{2}})} &\lesssim \|\partial_t\rho^*\nabla(-\Delta)^{-1}\rho^*\|_{L^1_t(\dot{B}^{\frac{d}{2}-1})} + \|(\rho^*-\bar{\rho})\nabla(-\Delta)^{-1}\partial_t\rho^*\|_{L^1_t(\dot{B}^{\frac{d}{2}-1})}\\ &\lesssim \|\partial_t\rho^*\|_{L^1_t(\dot{B}^{\frac{d}{2}-1})}\|\nabla(-\Delta)^{-1}\rho^*\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})} + \|\rho^*-\bar{\rho}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})}\|\nabla(-\Delta)^{-1}\partial_t\rho^*\|_{L^1_t(\dot{B}^{\frac{d}{2}})}\\ &\lesssim \|\rho^*-\bar{\rho}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})}\|\partial_t\rho^*\|_{L^1_t(\dot{B}^{\frac{d}{2}-1})} \lesssim \|\rho_0^*-\bar{\rho}\|^2_{\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}}}.\end{aligned}$$

The proof of Lemma 4.2 is finished. □

Then, we estimate the right-hand side of (4.22). Using (4.23), we have

$$\varepsilon\|B^{1,*}(0)\|^\ell_{\dot{B}^{\frac{d}{2}}} \lesssim \varepsilon\alpha_1, \tag{4.24}$$

and

$$\varepsilon\|\partial_t B^{1,*}\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}})} \lesssim \varepsilon\|\partial_t B^{1,*}\|_{L^1_t(\dot{B}^{\frac{d}{2}})} \lesssim \alpha_1\varepsilon. \tag{4.25}$$

Putting (4.10)-(4.13), (4.24) and (4.25) into (4.22) yields

$$\begin{aligned}&\|(\delta E,\delta\mathcal{B})\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})} + \|\delta E\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})} + \|\delta\mathcal{B}\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}\\ &\quad\lesssim \|(E_0^\varepsilon-E_0^*, B_0^\varepsilon-\bar{B})\|^\ell_{\dot{B}^{\frac{d}{2}}} + (\alpha_0+\alpha_1)\varepsilon\\ &\qquad + (\alpha_0+\alpha_1)(\|\delta\rho\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})} + \|\delta E\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})} + \|\delta E\|^m_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})}).\end{aligned} \tag{4.26}$$

- **Step 5: Estimates of $(\delta E,\delta\mathcal{B})$ in medium frequencies.**

For $-1\le j\le J_\varepsilon$, we define the functional

$$\delta\mathcal{L}_{m,j}(t) := \frac{1}{2}\|(\delta E_j,\delta\mathcal{B}_j)\|^2_{L^2} + \eta_{*m}2^{-2j}\int \varepsilon\delta E_j\cdot\nabla\times\delta\mathcal{B}_j\,dx.$$

By (4.19) and (4.20), there is a suitable small constant η_{*m} such that $\delta\mathcal{L}_{m,j}(t)\sim\|(\delta E_j,\delta\mathcal{B}_j)\|^2_{L^2}$ and

$$\begin{aligned}&\frac{d}{dt}\delta\mathcal{L}_{m,j}(t) + \|(\delta E_j,\delta\mathcal{B}_j)\|^2_{L^2}\\ &\le C(\|\dot{\Delta}_j(\rho^\varepsilon z^\varepsilon)\|_{L^2} + \varepsilon\|\dot{\Delta}_j(\rho^\varepsilon u^\varepsilon\times\bar{B})\|_{L^2} + \varepsilon\|\partial_t B_j^{1,*}\|_{L^2} + \|\nabla\delta\rho_j\|_{L^2} + \|\delta F_j\|_{L^2})\sqrt{\delta\mathcal{L}_{\ell,j}},\end{aligned}$$

which together with Lemma A.7 leads to

$$\begin{aligned}&\|(\delta E_j,\delta\mathcal{B}_j)\|_{\widetilde{L}^\infty_t(L^2)} + \|(\delta E_j,\delta\mathcal{B}_j)\|_{L^1_t(L^2)} + \|(\delta E_j,\delta\mathcal{B}_j)\|_{L^2_t(L^2)}\\ &\quad\lesssim \|(\delta E_j,\delta\mathcal{B}_j)(0)\|_{L^2} + \|\dot{\Delta}_j(\rho^\varepsilon z^\varepsilon)\|_{L^1_t(L^2)} + \varepsilon\|\dot{\Delta}_j(\rho^\varepsilon u^\varepsilon\times\bar{B})\|_{L^1_t(L^2)} + \|\nabla\delta\rho_j\|_{L^1_t(L^2)}\\ &\qquad + \|\delta F_j\|_{L^1_t(L^2)} + \varepsilon\|\partial_t B_j^{1,*}\|_{L^1_t(L^2)}.\end{aligned} \tag{4.27}$$

Therefore, we derive

$$\begin{aligned}&\|(\delta E,\delta\mathcal{B})\|^m_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})} + \|(\delta E,\delta\mathcal{B})\|^m_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})} + +\|(\delta E,\delta\mathcal{B})\|^m_{L^1_t(\dot{B}^{\frac{d}{2}-1})}\\ &\quad\lesssim \|(E_0^\varepsilon-E_0^*, B_0^\varepsilon-B^*, \varepsilon B^{1,*}(0))\|^m_{\dot{B}^{\frac{d}{2}-1}} + \|\rho^\varepsilon z^\varepsilon\|^m_{L^1_t(\dot{B}^{\frac{d}{2}-1})} + \|\rho^\varepsilon u^\varepsilon\times\bar{B}\|^m_{L^1_t(\dot{B}^{\frac{d}{2}-1})}\\ &\qquad + \|\delta\rho\|^m_{L^1_t(\dot{B}^{\frac{d}{2}})} + \|\delta F\|^m_{L^1_t(\dot{B}^{\frac{d}{2}-1})} + \varepsilon\|\partial_t B_j^{1,*}\|^m_{L^1_t(\dot{B}^{\frac{d}{2}-1})}.\end{aligned} \tag{4.28}$$

Due to (4.23), there holds that

$$\|B^{1,*}(0)\|^m_{\dot{B}^{\frac{d}{2}-1}} \lesssim \|B^{1,*}(0)\|^m_{\dot{B}^{\frac{d}{2}}} \lesssim \alpha_1, \tag{4.29}$$

and

$$\|\partial_t B_j^{1,*}\|^m_{L^1_t(\dot B^{\frac d2-1})} \lesssim \|\partial_t B^{1,*}\|_{L^1_t(\dot B^{\frac d2})} \lesssim \alpha_1. \tag{4.30}$$

We thence deduce from (4.15)-(4.18) and (4.28)-(4.30) that

$$\begin{aligned}
&\|(\delta E,\delta\mathcal{B})\|^m_{\widetilde L^\infty_t(\dot B^{\frac d2-1})}+\|(\delta E,\delta\mathcal{B})\|^m_{\widetilde L^2_t(\dot B^{\frac d2-1})}+\|(\delta E,\delta\mathcal{B})\|^m_{L^1_t(\dot B^{\frac d2-1})}\\
&\lesssim \|(E_0^\varepsilon-E_0^*,B_0^\varepsilon-B^*)\|^m_{\dot B^{\frac d2-1}}+(\alpha_0+\alpha_1)\varepsilon\\
&\quad+(\alpha_0+\alpha_1)(\|\delta\rho\|_{\widetilde L^2_t(\dot B^{\frac d2-1})}+\|\delta E\|^\ell_{\widetilde L^2_t(\dot B^{\frac d2})}+\|\delta E\|^m_{\widetilde L^2_t(\dot B^{\frac d2-1})}).
\end{aligned}\tag{4.31}$$

- **Step 6: Strong convergence**

Recalling that $\delta\mathcal{B}=\delta B+\varepsilon B^{1,*}$, we recover the estimate of δB as follows

$$\begin{aligned}
&\|\delta B\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac d2})}\lesssim \|\delta\mathcal{B}\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac d2})}+\varepsilon\|B^{1,*}\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac d2})},\\
&\|\delta B\|^\ell_{\widetilde L^2_t(\dot B^{\frac d2+1})}\lesssim \|\delta\mathcal{B}\|^\ell_{\widetilde L^2_t(\dot B^{\frac d2+1})}+\varepsilon\|B^{1,*}\|^\ell_{\widetilde L^2_t(\dot B^{\frac d2})},\\
&\|\delta B\|^m_{\widetilde L^\infty_t(\dot B^{\frac d2-1})}\lesssim \|\delta\mathcal{B}\|^m_{\widetilde L^\infty_t(\dot B^{\frac d2-1})}+\varepsilon\|B^{1,*}\|^m_{\widetilde L^\infty_t(\dot B^{\frac d2})},\\
&\|\delta B\|^m_{\widetilde L^2_t(\dot B^{\frac d2-1})}\lesssim \|\delta\mathcal{B}\|^m_{\widetilde L^2_t(\dot B^{\frac d2-1})}+\varepsilon\|B^{1,*}\|^m_{\widetilde L^2_t(\dot B^{\frac d2})}.
\end{aligned}$$

Hence, combining (4.8), (4.18), (4.23), (4.26) and (4.28) together, we have

$$\begin{aligned}
&\|\delta\rho\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1,\frac d2-2})}+\|\delta\rho\|_{\widetilde L^2_t(\dot B^{\frac d2-1})}\\
&\quad+\|\delta E\|_{\widetilde L^\infty_t(\dot B^{\frac d2,\frac d2-1})}+\|\delta E\|_{\widetilde L^2_t(\dot B^{\frac d2,\frac d2-1})}+\|\delta B\|_{\widetilde L^\infty_t(\dot B^{\frac d2,\frac d2-1})}+\|\delta B\|_{\widetilde L^2_t(\dot B^{\frac d2+1,\frac d2-1})}\\
&\lesssim \|\rho_0^\varepsilon-\rho_0^*\|_{\dot B^{\frac d2-1,\frac d2-2}}+\|E_0^\varepsilon-E_0^*\|_{\dot B^{\frac d2,\frac d2-1}}+\|B_0^\varepsilon-B^*\|_{\dot B^{\frac d2,\frac d2-1}}+(\alpha_0+\alpha_1)\varepsilon\\
&\quad+(\alpha_0+\alpha_1)(\|\delta\rho\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1,\frac d2-2})}+\|\delta\rho\|_{\widetilde L^2_t(\dot B^{\frac d2-1})}+\|\delta E\|_{\widetilde L^2_t(\dot B^{\frac d2,\frac d2-1})}),
\end{aligned}$$

which is uniform with respect to all $\varepsilon\le\varepsilon_0$ and $t\in\mathbb{R}_+$. Using the smallness of α_0,α_1 and (2.15), we conclude

$$\begin{aligned}
&\|\delta\rho\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1,\frac d2-2})}+\|\delta\rho\|_{\widetilde L^2_t(\dot B^{\frac d2-1})}\\
&\quad+\|\delta E\|_{\widetilde L^\infty_t(\dot B^{\frac d2,\frac d2-1})}+\|\delta E\|_{\widetilde L^2_t(\dot B^{\frac d2,\frac d2-1})}+\|\delta B\|_{\widetilde L^\infty_t(\dot B^{\frac d2,\frac d2-1})}+\|\delta B\|_{\widetilde L^2_t(\dot B^{\frac d2+1,\frac d2-1})}\\
&\lesssim \|E_0^\varepsilon-E_0^*\|_{\dot B^{\frac d2,\frac d2-1}}+\|B_0^\varepsilon-B^*\|_{\dot B^{\frac d2,\frac d2-1}}+\varepsilon.
\end{aligned}\tag{4.32}$$

Therefore, under the condition (2.10), (4.32) implies the estimate (2.11). This completes the proof of Theorem 2.3. We will handle the well-prepared case in Section 4.2 below.

4.2. **Proof of Theorem 2.4.** Finally, under the stronger conditions (2.12) and (2.13), one deduces from (2.9) that

$$\|z^\varepsilon\|_{\widetilde L^2_t(\dot B^{\frac d2-1})}\lesssim\varepsilon. \tag{4.33}$$

Therefore, we are able to establish the enhanced error estimate (2.14). Applying Lemma A.9 to $(4.5)_1$, we obtain

$$\begin{aligned}
&\|\delta\rho\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1})}+\|\delta\rho\|_{\widetilde L^2_t(\dot B^{\frac d2-1,\frac d2})}\\
&\lesssim \|\rho_0^\varepsilon-\rho_0^*\|_{\dot B^{\frac d2-1}}+\|\mathrm{div}\,(\rho^\varepsilon z^\varepsilon)\|_{\widetilde L^2_t(\dot B^{\frac d2-2})}+\varepsilon\|\mathrm{div}\,(\rho^\varepsilon u^\varepsilon\times\bar B)\|_{\widetilde L^2_t(\dot B^{\frac d2-2})}+\|\mathrm{div}\,\delta F\|_{\widetilde L^2_t(\dot B^{\frac d2-2})}.
\end{aligned}\tag{4.34}$$

From (2.7), (4.33) and (A.2), we have

$$\|\mathrm{div}\,(\rho^\varepsilon z^\varepsilon)\|_{\widetilde L^2_t(\dot B^{\frac d2-2})}\lesssim\|\rho^\varepsilon z^\varepsilon\|_{\widetilde L^2_t(\dot B^{\frac d2-1})}\lesssim(1+\|\rho^\varepsilon-\bar\rho\|_{\widetilde L^\infty_t(\dot B^{\frac d2})})\|z^\varepsilon\|_{\widetilde L^2_t(\dot B^{\frac d2-1})}\lesssim(\alpha_o+1)\varepsilon,$$

and

$$\varepsilon\|\mathrm{div}\,(\rho^\varepsilon u^\varepsilon\times\bar B)\|_{\widetilde L^2_t(\dot B^{\frac d2-2})}\lesssim(1+\|\rho^\varepsilon-\bar\rho\|_{\widetilde L^\infty_t(\dot B^{\frac d2})})\|u^\varepsilon\|_{\widetilde L^2_t(\dot B^{\frac d2-1})}\lesssim\alpha_0\varepsilon. \tag{4.35}$$

It also follows from (2.7), (4.2), (A.2) and (A.4) that

$$\|\delta F\|_{\widetilde L^2_t(\dot B^{\frac d2-1})}\lesssim(\alpha_0+\alpha_1)(\|\delta\rho\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1})}+\|\delta\rho\|_{\widetilde L^2_t(\dot B^{\frac d2-1,\frac d2})}+\|\delta E\|_{\widetilde L^2_t(\dot B^{\frac d2})}). \tag{4.36}$$

Thus, by (4.1), (2.15) and (4.34)-(4.36) we arrive at

$$\begin{aligned}&\|\delta\rho\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}+\|\delta\rho\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1,\frac d2})}\\&\lesssim \varepsilon+(\alpha_0+\alpha_1)(\|\delta\rho\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}+\|\delta\rho\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1,\frac d2})}+\|\delta E\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}).\end{aligned}\tag{4.37}$$

Next, from $(4.5)_2$, (4.33) and (A.5) we have the estimate of δu:

$$\begin{aligned}\|\delta u\|_{\widetilde{L}^2_t(\dot B^{\frac d2,\frac d2-1})}&\lesssim \|z^\varepsilon\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}+\|h(\rho^\varepsilon)-h(\rho^*)\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}+\|\delta E\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}\\&\lesssim \varepsilon+\|\delta\rho\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}+\|\delta E\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}.\end{aligned}\tag{4.38}$$

Finally, we recall that δE and $\delta\mathcal{B}$ satisfy (4.7). For some suitable small constant η_*, define the functional

$$\delta\mathcal{L}_j(t):=\frac12\|(\delta E_j,\delta\mathcal{B}_j)\|_{L^2}^2+\eta_*\min\{1,2^{-2j}\}\int \varepsilon\delta E_j\cdot\nabla\times\delta\mathcal{B}_j\,dx\sim\|(\delta E_j,\delta\mathcal{B}_j)\|_{L^2}^2.$$

Here $\min\{1,2^{-2j}\}=1$ for $j\le 0$ and $\min\{1,2^{-2j}\}=2^{-2j}$ for $j\ge 1$. The inequalities (4.19) and (4.20) ensure that

$$\begin{aligned}&\frac{d}{dt}\delta\mathcal{L}_j(t)+\|\delta E_j\|_{L^2}^2+\min\{1,2^{2j}\}\|\delta\mathcal{B}_j\|_{L^2}^2\\&\lesssim \varepsilon\|\partial_t B^{1,*}_j\|_{L^2}\sqrt{\delta\mathcal{L}_{\ell,j}}\\&\quad+2^{-j}(\|\dot\Delta_j(\rho^\varepsilon z^\varepsilon)\|_{L^2}+\varepsilon\|\dot\Delta_j(\rho^\varepsilon u^\varepsilon\times\bar B)\|_{L^2}+\|\delta\rho_j\|_{L^2}+\|\delta F_j\|_{L^2})(\|\delta E_j\|_{L^2}+\min\{1,2^j\}\|\delta\mathcal{B}_j\|_{L^2}).\end{aligned}$$

Therefore, a use of Lemma A.7 gives

$$\begin{aligned}&\|(\delta E,\delta\mathcal{B})\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2})}+\|\delta E\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}+\|\delta\mathcal{B}\|_{\widetilde{L}^2_t(\dot B^{\frac d2+1,\frac d2})}\\&\lesssim \|(E_0^\varepsilon-E_0^*,B_0^\varepsilon-B^*,\varepsilon B^{1,*}(0))\|_{\dot B^{\frac d2}}+\varepsilon\|\partial_t B^{1,*}_j\|_{L^1_t(\dot B^{\frac d2})}+\|\rho^\varepsilon z^\varepsilon\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\\&\quad+\|\rho^\varepsilon u^\varepsilon\times\bar B\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}+\|\delta\rho\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}+\|\delta F\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1})}\\&\lesssim \varepsilon+\|\delta\rho\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}\\&\lesssim \varepsilon+(\alpha_0+\alpha_1)(\|\delta\rho\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2-1})}+\|\delta\rho\|_{\widetilde{L}^2_t(\dot B^{\frac d2-1,\frac d2})}+\|\delta E\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}),\end{aligned}\tag{4.39}$$

where we have employed (4.23) and (4.34)-(4.37). Furthermore, in view of (4.23) one can recover the estimate of δB as follows

$$\begin{aligned}&\|\delta B\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2})}+\|\delta B\|_{\widetilde{L}^2_t(\dot B^{\frac d2+1,\frac d2})}\\&\lesssim \|\delta\mathcal{B}\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2})}+\|\delta\mathcal{B}\|_{\widetilde{L}^2_t(\dot B^{\frac d2+1,\frac d2})}+\varepsilon\|B^{1,*}\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2})}+\varepsilon\|B^{1,*}\|_{\widetilde{L}^2_t(\dot B^{\frac d2})}\\&\lesssim \|\delta\mathcal{B}\|_{\widetilde{L}^\infty_t(\dot B^{\frac d2})}+\|\delta\mathcal{B}\|_{\widetilde{L}^2_t(\dot B^{\frac d2+1,\frac d2})}+\varepsilon.\end{aligned}\tag{4.40}$$

Combining (2.12) and (4.37)-(4.40) together with the smallness of α_0 and α_1 concludes the proof of Theorem 2.4.

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Appendix A. Technical lemmas

We recall some basic properties of Besov spaces and product estimates which are repeatedly used in this manuscript. The reader can refer to [1, Chapters 2-3] for more details. Remark that all the properties remain true for the Chemin–Lerner type spaces, up to the modification of the regularity exponent according to Hölder's inequality for the time variable.

The first lemma pertains to the so-called Bernstein inequalities.

Lemma A.1. *Let $0<r<R$, $1\leq p\leq q\leq\infty$ and $k\in\mathbb{N}$. For any function $u\in L^p$ and $\lambda>0$, it holds*

$$\begin{cases}\text{Supp } \mathcal{F}(u)\subset\{\xi\in\mathbb{R}^d\ :\ |\xi|\leq\lambda R\}\Rightarrow\|D^k u\|_{L^q}\lesssim\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p},\\ \text{Supp } \mathcal{F}(u)\subset\{\xi\in\mathbb{R}^d\ :\ \lambda r\leq|\xi|\leq\lambda R\}\Rightarrow\|D^k u\|_{L^p}\sim\lambda^{k}\|u\|_{L^p}.\end{cases}$$

Next, we state some properties related to homogeneous Besov spaces.

Lemma A.2. *The following properties hold*:

- *For any $s\in\mathbb{R}$ and $q\geq 2$, we have the following continuous embeddings*:
$$\dot{B}^s\hookrightarrow\dot{H}^s,\qquad \dot{B}^{\frac{d}{2}-\frac{d}{q}}\hookrightarrow L^q.$$
- *$\dot{B}^{\frac{d}{2}}$ is continuously embedded in the set of continuous functions decaying to 0 at infinity.*
- *For any $\sigma\in\mathbb{R}$, the operator Λ^σ is an isomorphism from $\dot{B}^s$ to $\dot{B}^{s-\sigma}$.*
- *Let $s_1\in\mathbb{R}$ and $s_2\leq\frac{d}{2}$. Then the space $\dot{B}^{s_1}\cap\dot{B}^{s_2}$ is a Banach space and satisfies weak compact and Fatou properties: If u_k is a uniformly bounded sequence of $\dot{B}^{s_1}\cap\dot{B}^{s_2}$, then an element u of $\dot{B}^{s_1}\cap\dot{B}^{s_2}$ and a subsequence u_{n_k} exist such that*
$$\lim_{k\to\infty}u_{n_k}=u\quad\text{in}\quad\mathcal{S}'\quad\text{and}\quad\|u\|_{\dot{B}^{s_1}\cap\dot{B}^{s_2}}\lesssim\liminf_{n_k\to\infty}\|u_{n_k}\|_{\dot{B}^{s_1}\cap\dot{B}^{s_2}}.$$

The following Morse-type product estimates in Besov spaces play a fundamental role in our analysis of nonlinear terms.

Lemma A.3. *The following statements hold:*

- *Let $s>0$. Then $\dot{B}^s\cap L^\infty$ is a algebra and*

$$\|uv\|_{\dot{B}^s}\lesssim\|u\|_{L^\infty}\|v\|_{\dot{B}^s}+\|v\|_{L^\infty}\|u\|_{\dot{B}^s}.\tag{A.1}$$

- *Let s_1,s_2 satisfy $s_1,s_2\leq\frac{d}{2}$ and $s_1+s_2>0$. Then there holds*

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{d}{2}}}\lesssim\|u\|_{\dot{B}^{s_1}}\|v\|_{\dot{B}^{s_2}}.\tag{A.2}$$

Next, we present a commutator estimate that is used to control some nonlinearities in high and medium frequencies.

Lemma A.4. *Let $s\in(-\frac{d}{2}-1,\frac{d}{2}+1]$. Then it holds*

$$\sum_{j\in\mathbb{Z}}2^{js}\|[u,\dot{\Delta}_j]\partial_{x_i}v\|_{L^2}\lesssim\|\nabla u\|_{\dot{B}^{\frac{d}{2}}}\|v\|_{\dot{B}^s},\qquad i=1,2,...d.\tag{A.3}$$

We recall the classical estimates about the composition of functions.

Lemma A.5. *Let $s>0$, and $F:I\to\mathbb{R}$ with I being an open interval of $\mathbb{R}$. Assume that $F(0)=0$ and that F' belongs to $W^{[s]+1,\infty}(I)$. Let $u,v\in\dot{B}^s\cap L^\infty$ have value in I. There exists a constant $C=C(s,d,I)$ such that*

$$\|F(f)\|_{\dot{B}^s}\leq C(1+\|f\|_{L^\infty})^{[s]+1}\|F'\|_{W^{[s]+1,\infty}(I)}\|f\|_{\dot{B}^s}.\tag{A.4}$$

In addition, if F'' belongs to $W^{[s]+1,\infty}(I)$, then

$$\begin{aligned}\|F(f_1)-F(f_2)\|_{\dot{B}^s}\leq F'(0)\|f_1-f_2\|_{\dot{B}^s}+C(1+\|(f_1,f_2)\|_{L^\infty})^{[s]+1}\|F'\|_{W^{[s]+1,\infty}(I)}\\ \times\big(\|f_1-f_2\|_{\dot{B}^s}\|(f_1,f_2)\|_{L^\infty}+\|f_1-f_2\|_{L^\infty}\|(f_1,f_2)\|_{\dot{B}^s}\big).\end{aligned}\tag{A.5}$$

In order to control the nonlinear term $\Phi(n)$, we need the following lemma concerning the composition of quadratic functions. The proof can be found in [10].

Lemma A.6. *Let $s>0$, J be a given integer, and $F: I\to\mathbb{R}$ with I being an open interval of $\mathbb{R}$. Assume that F'' belongs to $W^{[s]+1,\infty}(I)$. Then there exists a constant $C=C(s,p,r,d,I)$ such that it holds for $\sigma\geq 0$ that*

$$\begin{aligned}&\sum_{j\leq J}2^{js}\|\dot{\Delta}_j(F(f)-F(0)-F'(0)f)\|_{L^2}\\&\leq C\|F''\|_{W^{[s]+1,\infty}(I)}(1+\|f\|_{L^\infty})^{[s]+1}\|f\|_{L^\infty}\big(\sum_{j\leq J}2^{js}\|\dot{\Delta}_jf\|_{L^2}+2^{J(s-\sigma)}\sum_{j\geq J-1}2^{j\sigma}\|\dot{\Delta}_jf\|_{L^2}\big),\end{aligned}\tag{A.6}$$

and for any $\sigma\in\mathbb{R}$ that

$$\begin{aligned}&\sum_{j\geq J-1}2^{js}\|\dot{\Delta}_j(F(f)-F(0)-F'(0)f)\|_{L^2}\\&\leq C\|F''\|_{W^{[s]+1,\infty}(I)}(1+\|f\|_{L^\infty})^{[s]+1}\|f\|_{L^\infty}\big(2^{J(s-\sigma)}\sum_{j\leq J}2^{j\sigma}\|\dot{\Delta}_jf\|_{L^2}+\sum_{j\geq J-1}2^{js}\|\dot{\Delta}_jf\|_{L^2}\big).\end{aligned}\tag{A.7}$$

Lemma A.7. *Let $T>0$ be given time, $E_1(t),E_2(t)$ and $E_3(t)$ be three nonnegative and absolutely continuous functions on $[0,T)$. Suppose that there exists a functional $\mathcal{L}(t)\sim E_1(t)+E_2(t)+E_3(t)$ such that*

$$\frac{d}{dt}\mathcal{L}(t)+a_1E_1(t)+a_2E_2(t)+a_3E_3(t)\leq g_1(t)\sqrt{\mathcal{L}(t)}+g_2(t)\sqrt{E_1(t)},\qquad t\in(0,T),\tag{A.8}$$

where a_1, a_2,a_3 are positive constants. Then, there exists a constant $C>0$ independent of T and a_1, a_2,a_3 such that

- (1) *If $g_1(t)\in L^1(0,T)$ and $g_2(t)\in L^1(0,T)$, then we have*

$$\begin{aligned}&\sup_{t\in[0,T]}(E_1(t)+E_2(t)+E_3(t))\\&\quad+\min\{a_1,a_2,a_3\}\|(E_1,E_2,E_3)(t)\|_{L^1(0,T)}\\&\quad+\sqrt{a_1}\|E_1\|_{L^2(0,T)}+\sqrt{a_2}\|E_2\|_{L^2(0,T)}+\sqrt{a_3}\|E_3\|_{L^2(0,T)}\\&\leq E_1(0)+E_2(0)+E_3(0)+\|(g_1,g_2)\|_{L^1(0,T)}.\end{aligned}\tag{A.9}$$

- (2) *If $g_1(t)\in L^1(0,T)$ and $g_2(t)\in L^2(0,T)$, then we have*

$$\begin{aligned}&\sup_{t\in[0,T]}(E_1(t)+E_2(t)+E_3(t))\\&\quad+\sqrt{a_1}\|E_1\|_{L^2(0,T)}+\sqrt{a_2}\|E_2\|_{L^2(0,T)}+\sqrt{a_3}\|E_3\|_{L^2(0,T)}\\&\leq E_1(0)+E_2(0)+E_3(0)+\|g_1\|_{L^1(0,T)}+\frac{1}{\sqrt{a_1}}\|g_2\|_{L^2(0,T)}.\end{aligned}\tag{A.10}$$

We consider the following Cauchy problem for the damped heat equation:

$$\begin{cases}\partial_tu-c_1\Delta u+c_2u=f, & x\in\mathbb{R}^d,\quad t>0,\\ u(0,x)=u_0(x), & x\in\mathbb{R}^d,\end{cases}\tag{A.11}$$

We have the following L^1-in-time maximal regularity estimates for (A.11).

Lemma A.8. *Let $s\in\mathbb{R}$, $T>0$ be given time, and $c_i\geq 0$ $(i=1,2)$ be positive constants. Assume $u_0\in\dot{B}^s$ and $f\in L^1(0,T;\dot{B}^s)$. If u is the solution to the Cauchy problem (A.11) for $t\in(0,T)$, then u satisfies*

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty_t(\dot{B}^s)}+c_1\|u\|_{L^1_t(\dot{B}^{s+2})}+c_2\|u\|_{L^1_t(\dot{B}^s)}+\|\partial_tu\|_{L^1_t(\dot{B}^s)}+\sqrt{c_1}\|u\|_{\widetilde{L}^2_t(\dot{B}^{s+1})}+\sqrt{c_2}\|u\|_{\widetilde{L}^2_t(\dot{B}^s)}\\&\qquad\leq C(\|u_0\|_{\dot{B}^s}+\|f\|_{L^1_t(\dot{B}^s)}),\end{aligned}\tag{A.12}$$

where $C>0$ is a constant independent of c_i $(i=1,2)$ and T.

Proof. Taking the L^2 inner product of (A.11) with u_j, we have

$$\frac{d}{dt}\|u_j\|_{L^2}^2+c_1\|\nabla u_j\|^2+c_2\|u_j\|_{L^2}^2\leq\|u_j\|_{L^2}\,\|f_j\|_{L^2}.\tag{A.13}$$

Dividing the two sides of (A.13) by $(\|u_j\|_{L^2}^2+\eta)^{\frac{1}{2}}$, integrating the resulting equation over $[0,t]$, and then taking the limit as $\eta\to 0$, we have

$$\|u_j\|_{L^\infty_t(L^2)}+c_1 2^{2j}\int_0^t\|u_j\|_{L^2}d\tau+c_2\int_0^t\|u_j\|_{L^2}d\tau\lesssim\|u_j(0)\|_{L^2}+\int_0^t\|f_j\|_{L^2}d\tau,$$

which yields

$$\|u\|_{\widetilde{L}^\infty_t(\dot{B}^s)}+c_1\|u\|_{L^1_t(\dot{B}^{s+2})}+c_2\|u\|_{L^1_T(\dot{B}^s)}\lesssim\|u_0\|_{\dot{B}^s}+\|f\|_{L^1_T(\dot{B}^s)}. \tag{A.14}$$

According to (A.14) and the equation $(\text{A.11})_1$, one also has

$$\|\partial_t u\|_{L^1_T(\dot{B}^s)}\lesssim c_1\|u\|_{L^1_t(\dot{B}^{s+2})}+c_2\|u\|_{L^1_T(\dot{B}^s)}+\|f\|_{L^1_T(\dot{B}^s)}\lesssim\|u_0\|_{\dot{B}^s}+\|f\|_{L^1_T(\dot{B}^s)}.$$

Furthermore, integrating (A.16) over $[0,t]$, taking the square root and summing it over $j\in\mathbb{Z}$, we get

$$\begin{aligned}\sqrt{c_1}\|u\|_{\widetilde{L}^2_t(\dot{B}^{s+1})}+\sqrt{c_2}\|u\|_{\widetilde{L}^2_t(\dot{B}^s)}&\lesssim\|u_0\|_{\dot{B}^s}+\|u\|^{\frac{1}{2}}_{\widetilde{L}^\infty_t(\dot{B}^s)}\|f\|^{\frac{1}{2}}_{L^1_T(\dot{B}^s)}\\&\lesssim\|u_0\|_{\dot{B}^s}+\|f\|_{L^1_T(\dot{B}^s)}\end{aligned}$$

where we used (A.14). Combining the above estimates yields (A.12). □

Additionally, we have L^2-in-time estimates for the solutions of (A.11).

Lemma A.9. *Let $s\in\mathbb{R}$, $T>0$ be given time, and $c_i\geq 0$ $(i=1,2)$ be positive constants. Assume $u_0\in\dot{B}^s$, and $f=f_1+f_2+f_3$ with f_i $(i=1,2,3)$ satisfying $f_1\in L^1(0,T;\dot{B}^s)$, $f_2\in\widetilde{L}^2(0,T;\dot{B}^{s-1})$, $f_3\in\widetilde{L}^2(0,T;\dot{B}^s)$ and $f_i=0$ when $c_i=0$ $(i=2,3)$. If u is the solution to the Cauchy problem (A.11), then u satisfies*

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty_t(\dot{B}^s)}+\sqrt{c_1}\|u\|_{\widetilde{L}^2_t(\dot{B}^{s+1})}+\sqrt{c_2}\|u\|_{\widetilde{L}^2_t(\dot{B}^s)}\\&\leq C(\|u_0\|_{\dot{B}^s}+\|f_1\|_{L^1_t(\dot{B}^s)}+\frac{1}{\sqrt{c_1}}\|f_2\|_{\widetilde{L}^2_t(\dot{B}^{s-1})}+\frac{1}{\sqrt{c_2}}\|f_3\|_{\widetilde{L}^2_t(\dot{B}^s)}),\quad t\in(0,T),\end{aligned} \tag{A.15}$$

where $C>0$ is a constant independent of c_i $(i=1,2)$ and T.

Proof. It follows by (A.13) and Young's inequality that

$$\frac{d}{dt}\|u_j\|_{L^2}^2+c_1 2^{2j}\|u_j\|^2+c_2\|u_j\|_{L^2}^2\leq\|u_j\|_{L^2}\|\dot{\Delta}_j f_1\|_{L^2}+\frac{2^{-2j}}{c_1}\|\dot{\Delta}_j f_2\|_{L^2}^2+\frac{1}{c_2}\|\dot{\Delta}_j f_3\|_{L^2}^2. \tag{A.16}$$

Integrating (A.16) over $[0,t]$, we have

$$\begin{aligned}&\|u_j\|^2_{L^\infty_t(L^2)}+c_1 2^{2j}\int_0^t\|u_j\|^2d\tau+c_2\int_0^t\|u_j\|^2d\tau\\&\lesssim\|u_j(0)\|_{L^2}^2+\int_0^t\|f_j\|_{L^2}d\tau\|u_j\|_{L^\infty_t(L^2)}+\frac{2^{-2j}}{c_1}\int_0^t\|\dot{\Delta}_j f_2\|_{L^2}^2d\tau+\frac{1}{c_2}\int_0^t\|\dot{\Delta}_j f_3\|_{L^2}^2d\tau.\end{aligned} \tag{A.17}$$

By (A.17) and Young's inequality again, there holds

$$\begin{aligned}&\|u_j\|_{L^\infty_t(L^2)}+\sqrt{c_1}2^j\|u_j\|_{L^2_t(L^2)}+\sqrt{c_2}\|u_j\|_{L^2_t(L^2)}\\&\lesssim\|u_j(0)\|_{L^2}+\|f_j\|_{L^1_t(L^2)}+\frac{2^{-j}}{\sqrt{c_1}}\|\dot{\Delta}_j f_2\|_{L^2_t(L^2)}+\frac{1}{\sqrt{c_2}}\|\dot{\Delta}_j f_3\|_{L^2_t(L^2)}\end{aligned}$$

which gives (A.15). □

Appendix B. Proof of Proposition 3.1: The pointwise estimates

Finally, we prove Proposition 3.1 concerning the pointwise estimates (3.5) and (3.6). Applying the Fourier transform to (3.4), we have

$$\begin{cases} \partial_t \widehat{n} + P'(\bar{\rho}) i\xi \widehat{u} = 0, \\ \varepsilon^2 \partial_t \widehat{u} + i\xi \widehat{n} + \widehat{E} + \widehat{u} + \varepsilon \widehat{u} \times \bar{B} = 0, \\ \varepsilon \partial_t \widehat{E} - i\xi \times \widehat{H} - \varepsilon \bar{\rho} \widehat{u} = 0, \\ \varepsilon \partial_t \widehat{H} + i\xi \times \widehat{E} = 0, \\ i\xi \widehat{E} = -K\widehat{n}, \qquad i\xi \widehat{H} = 0, \end{cases} \tag{B.1}$$

where we recall that $K = \dfrac{\bar{\rho}}{P'(\bar{\rho})}$. Taking the Hermitian scalar product of (B.1) with $\widehat{n}, P'(\bar{\rho})\widehat{u}, \frac{1}{K}\widehat{E}$ and $\frac{1}{K}\widehat{H}$, adding the resulting equalities together and then taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\big(|\widehat{n}|^2 + P'(\bar{\rho})\varepsilon^2|\widehat{u}|^2 + \frac{1}{K}|\widehat{E}|^2 + \frac{1}{K}|\widehat{H}|^2\big) + P'(\bar{\rho})|\widehat{u}|^2 = 0. \tag{B.2}$$

To capture dissipation for n, we do the following computation

$$\begin{aligned} -\varepsilon^2\frac{d}{dt}\mathrm{Re} < \widehat{u}, i\xi\widehat{n} > + |\xi|^2|\widehat{n}|^2 + K|\widehat{n}|^2 &= \mathrm{Re} < \widehat{u} + \varepsilon\widehat{u} \times \bar{B}, i\xi\widehat{n} > + P'(\bar{\rho})\varepsilon^2|\xi\cdot\widehat{u}|^2 \\ &\leq \frac{1}{2}|\xi|^2|\widehat{n}|^2 + C(1+\varepsilon^2|\xi|^2)|\widehat{u}|^2. \end{aligned} \tag{B.3}$$

Then, multiplying (B.3) by $\frac{1}{1+\varepsilon^2|\xi|^2}$, we obtain

$$-\frac{d}{dt}\frac{\varepsilon^2\mathrm{Re} < \widehat{u}, i\xi\widehat{n} >}{1+\varepsilon^2|\xi|^2} + \frac{|\xi|^2}{2(1+\varepsilon^2|\xi|^2)}|\widehat{n}|^2 + \frac{K}{1+\varepsilon^2|\xi|^2}|\widehat{n}|^2 \leq C|\widehat{u}|^2. \tag{B.4}$$

By the Hermitian scalar product of $(\mathrm{B.1})_2$ with $\widehat{E}$ (associated with the skew-symmetric part of the relaxation matrix), we capture dissipation for $\widehat{E}$ as follows:

$$\begin{aligned} &\varepsilon^2\frac{d}{dt}\mathrm{Re} < \widehat{u}, \widehat{E} > + |\widehat{E}|^2 + \frac{1}{K}|\xi\cdot\widehat{E}|^2 \\ &= -\mathrm{Re} < \widehat{u} + \varepsilon\widehat{u}\times\bar{B}, \widehat{E} > + \varepsilon\mathrm{Re} < i\xi\times\widehat{H}, \widehat{u} > + \varepsilon^2\bar{\rho}|\widehat{u}|^2 \\ &\leq \frac{1}{2}|\widehat{E}|^2 + \frac{C(1+\varepsilon^2|\xi|^2)}{\sqrt{\eta}}|\widehat{u}|^2 + \frac{C\sqrt{\eta}|\xi|^2}{1+|\xi|^2}|\widehat{H}|^2 \end{aligned} \tag{B.5}$$

for some small constant $\eta \in (0,1)$ to be chosen later. In order to be consistent with the dissipation obtained for u in (B.2), we multiply (B.5) by $\frac{1}{1+\varepsilon^2|\xi|^2}$ and obtain

$$\frac{d}{dt}\frac{\varepsilon^2\mathrm{Re} < \widehat{u}, \widehat{E} >}{1+\varepsilon^2|\xi|^2} + \frac{1}{2(1+\varepsilon^2|\xi|^2)}|\widehat{E}|^2 \leq \frac{C}{\sqrt{\eta}}|\widehat{u}|^2 + \frac{C\sqrt{\eta}|\xi|^2}{(1+\varepsilon^2|\xi|^2)(1+|\xi|^2)}|\widehat{H}|^2. \tag{B.6}$$

To derive, dissipation for $\widehat{H}$, using $|\xi|^2|\widehat{H}|^2 = |\xi\times\widehat{H}|^2$ due to $\xi\cdot\widehat{H} = 0$ yields

$$\begin{aligned} \varepsilon\frac{d}{dt}\mathrm{Re} < \widehat{E}, -i\xi\times\widehat{H} > + |\xi|^2|\widehat{H}|^2 &= |\xi\times\widehat{E}|^2 - \bar{\rho}\varepsilon\mathrm{Re} < \widehat{u}, i\xi\times\widehat{H} > \\ &\leq \frac{1}{2}|\xi|^2|\widehat{H}|^2 + C|\xi|^2|\widehat{E}|^2 + C|\widehat{u}|^2. \end{aligned} \tag{B.7}$$

In view of the dissipation of $\widehat{E}$ in (B.6), we have

$$\frac{d}{dt}\frac{\varepsilon\mathrm{Re} < \widehat{E}, -i\xi\times\widehat{H} >}{(1+\varepsilon^2|\xi|^2)(1+|\xi|^2)} + \frac{|\xi|^2}{2(1+\varepsilon^2|\xi|^2)(1+|\xi|^2)}|\widehat{H}|^2 \leq C|\widehat{u}|^2 + \frac{|\xi|^2}{(1+\varepsilon^2|\xi|^2)(1+|\xi|^2)}|\widehat{E}|^2. \tag{B.8}$$

Then, we define the Lyapunov functional

$$\begin{aligned} \mathcal{L}_\xi(t) :=& \frac{1}{2}\Big(|\widehat{n}|^2 + P'(\bar{\rho})\varepsilon^2|\widehat{u}|^2 + \frac{1}{K}|\widehat{E}|^2 + \frac{1}{K}|\widehat{H}|^2\Big) \\ &- \eta\frac{\varepsilon^2\mathrm{Re} < \widehat{u}, i\xi\widehat{n} >}{1+\varepsilon^2|\xi|^2} + \eta\frac{\varepsilon^2\mathrm{Re} < \widehat{u}, \widehat{E} >}{1+\varepsilon^2|\xi|^2} + \eta^{\frac{5}{4}}\frac{\varepsilon\mathrm{Re} < \widehat{E}, -i\xi\times\widehat{H} >}{(1+\varepsilon^2|\xi|^2)(1+|\xi|^2)}. \end{aligned} \tag{B.9}$$

It follows from (B.2), (B.4), (B.6) and (B.8) that

$$\begin{aligned}\frac{d}{dt}\mathcal{L}_\xi(t)+(P'(\bar\rho)-C\eta-C\sqrt{\eta})|\widehat{u}|^2+\frac{\eta(|\xi|^2+1)}{1+\varepsilon^2|\xi|^2}|\widehat{n}|^2\\ +(\frac12-\eta^{\frac14})\eta\frac{1}{1+\varepsilon^2|\xi|^2}|\widehat{E}|^2+\eta^{\frac54}(\frac12-\eta^{\frac14})(\frac{|\xi|^2}{(1+\varepsilon^2|\xi|^2)(1+|\xi|^2)}|\widehat{H}|^2\le 0.\end{aligned} \tag{B.10}$$

Choosing a suitable small constant η, we get $\mathcal{L}_\xi(t)\sim|(\widehat{n},\varepsilon\widehat{u},\widehat{E},\widehat{H})|^2$ and (3.5). In particular, it holds that

$$\frac{d}{dt}\mathcal{L}_\xi(t)+c_0\lambda_\varepsilon(\xi)\mathcal{L}_\xi(t)\le 0. \tag{B.11}$$

Applying Grönwall's inequality to (B.11), we conclude (3.6).

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