Quantitative derivation of a two-phase porous media system from the one-velocity Baer-Nunziato and Kapila systems

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Abstract

We derive a novel two-phase flow system in porous media as a relaxation limit of compressible multi-fluid systems. Considering a one-velocity Baer-Nunziato system with friction forces, we first justify its pressure-relaxation limit toward a Kapila model in a uniform manner with respect to the time-relaxation parameter associated with the friction forces. Then, we show that the diffusely rescaled solutions of the damped Kapila system converge to the solutions of the new two-phase porous media system as the time-relaxation parameter tends to zero. In addition, we also prove the convergence of the Baer-Nunziato system to the same two-phase porous media system as both relaxation parameters tend to zero. For each relaxation limit, we exhibit sharp rates of convergence in a critical regularity setting.

Our proof is based on an elaborate low-frequency and high-frequency analysis via the Littlewood-Paley decomposition and includes three main ingredients: a refined spectral analysis of the linearized problem to determine the frequency threshold explicitly in terms of the time-relaxation parameter, the introduction of an effective flux in the low-frequency region to overcome the loss of parameters due to the *overdamping phenomenon*, and renormalized energy estimates in the high-frequency region to cancel higher-order nonlinear terms. To justify the convergence rates, we discover several *auxiliary unknowns* allowing us to recover crucial $\mathcal{O}(\varepsilon)$ bounds.

Keywords— Multi-fluid system, pressure-relaxation limit, overdamping phenomenon, critical regularity, two-phase flow in porous media, Kapila system, Baer-Nunziato system.

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1 Introduction

1.1 Models and motivations

Multi-phase flows have been used to simulate a wide range of physical mixing phenomenon, from engineering to biological systems (cf. [1, 11, 32, 46] and the references therein). In the present paper, we investigate an inviscid compressible one-velocity Baer-Nunziato system with two different pressure laws

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in presence of drag forces, which was discussed in the recent work [10] of Bresch and Hillairet:

$$\begin{cases} \partial_t \alpha_+ + u \cdot \nabla \alpha_+ = \dfrac{\alpha_+ \alpha_-}{\varepsilon} \big(P_+(\rho_+) - P_-(\rho_-) \big), \\[2mm] \partial_t (\alpha_\pm \rho_\pm) + \operatorname{div}(\alpha_\pm \rho_\pm u) = 0, \\[2mm] \partial_t (\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P + \dfrac{\rho u}{\tau} = 0, \qquad x \in \mathbb{R}^d, \quad t > 0, \end{cases} \tag{BN}$$

where the unknowns $\alpha_\pm = \alpha_\pm(t,x) \in [0,1]$, $\rho_\pm = \rho_\pm(t,x) \geq 0$ and $u = u(t,x) \in \mathbb{R}^d$ stand for the volume fractions, the densities and the common velocity of two fluids (denoted by $+$ and $-$), respectively, which satisfy

$$\alpha_+ + \alpha_- = 1, \qquad \rho = \alpha_+ \rho_+ + \alpha_- \rho_-, \qquad P = \alpha_+ P_+(\rho_+) + \alpha_- P_-(\rho_-).$$

The two positive constants ε and τ are (small) relaxation parameters associated to the pressure-relaxation and time-relaxation limits. Finally, the two pressures P_+ and P_- take the gamma-law forms

$$P_\pm(s) = A_\pm s^{\gamma_\pm} \quad \text{with constants} \quad A_\pm > 0, \qquad 1 \leq \gamma_- < \gamma_+. \tag{1.1}$$

The *Baer-Nunziato* terminology refers to the pressure-relaxation mechanism in the equations of volume fractions. Numerically, such relaxation procedure can simplify its resolution as it reduces the number of constraints by introducing new unknowns: *two pressures instead of one.* The readers can see [10] and references therein for more discussions on this pressure-relaxation process. Very recently, the one-dimensional version of System (BN) was rigorously derived by Bresch, Burtea and Lagoutiére in [6, 7].

There is an extensive literature on the mathematical analysis of multi-fluid systems. For example, in the one-velocity case, the global existence of weak solutions has been studied in [14, 37, 42, 45, 47, 52], and the global well-posedness and optimal time-decay rates of strong solutions has been established in the framework of Sobolev spaces [30, 51, 53, 54] and critical Besov spaces [15, 31, 36], etc. We also refer to [9, 12, 13, 26, 35] on the study of multi-fluid systems in the two-velocity case. Complete reviews on multi-fluid systems are presented in [8, 48]. Concerning the study of relaxation problems associated to systems of conservation laws, it can be traced back to the work [16] by Chen, Levermore and Liu. Recently, Giovangigli and Yong in [28, 29] studied a relaxation problem arising in the dynamics of perfect gases out of thermodynamic equilibrium.

At the formal level, the solution $(\alpha_\pm^{\varepsilon,\tau}, \rho_\pm^{\varepsilon,\tau}, u^{\varepsilon,\tau})$ of System (BN) tends, as $\varepsilon \to 0$, to some limit $(\alpha_\pm^\tau, \rho_\pm^\tau, u^\tau)$ that satisfies the so-called one-velocity Kapila system (cf. [34]):

$$\begin{cases} \partial_t (\alpha_\pm^\tau \rho_\pm^\tau) + \operatorname{div}(\alpha_\pm^\tau \rho_\pm^\tau u^\tau) = 0, \\[2mm] \partial_t (\rho^\tau u^\tau) + \operatorname{div}(\rho^\tau u^\tau \otimes u^\tau) + \nabla P^\tau + \dfrac{\rho^\tau u^\tau}{\tau} = 0, \\[2mm] P^\tau = P_+^\tau(\rho_+) = P_-^\tau(\rho_-), \end{cases} \tag{K}$$

with $\alpha_+^\tau + \alpha_-^\tau = 1$ and $\rho^\tau = \alpha_+^\tau \rho_+^\tau + \alpha_-^\tau \rho_-^\tau$. System (K) can be rewritten as classical two-phase fluid models of drift-flux type, see [25, 32, 47] and the references therein. For existence of finite energy weak solutions to System (K) with viscosities, refer to the recent works [14, 37, 42, 47].

Then, we further investigate the time-relaxation limit of System (K) as $\tau \to 0$. Inspired by the works [17, 33, 50] concerning the relaxation problems for the compressible Euler system with damping,

we introduce a large time-scale $\mathcal{O}(1/\tau)$ and define the following charge of variables

$$(\beta_{\pm}^{\tau}, \varrho_{\pm}^{\tau}, v^{\tau})(s, x) := \left(\alpha_{\pm}^{\tau}, \rho_{\pm}^{\tau}, \frac{u^{\tau}}{\tau}\right)\left(\frac{s}{\tau}, x\right). \tag{1.2}$$

Under the diffusive scaling (1.2), System (K) becomes

$$\begin{cases} \partial_s(\beta_{\pm}^{\tau}\varrho_{\pm}^{\tau}) + \operatorname{div}(\beta_{\pm}^{\tau}\varrho_{\pm}^{\tau}v^{\tau}) = 0, \\ \tau^2\partial_s(\varrho^{\tau}v^{\tau}) + \tau^2\operatorname{div}(\varrho^{\tau}v^{\tau} \otimes v^{\tau}) + \nabla\Pi^{\tau} + \varrho^{\tau}v^{\tau} = 0, \\ \beta_+^{\tau} + \beta_-^{\tau} = 1, \end{cases} \tag{K_τ}$$

with $\varrho^{\tau} = \beta_+^{\tau}\rho_+^{\tau} + \beta_-^{\tau}\varrho_-^{\tau}$ and $\Pi^{\tau} = P_+(\varrho_+^{\tau}) = P_-(\varrho_-^{\tau})$. As $\tau \to 0$, one then expects that $(\beta_{\pm}^{\tau}, \varrho_{\pm}^{\tau}, v^{\tau})$ converges to some limit $(\beta_{\pm}, \varrho_{\pm}, v)$ which is the solution of a new two-phase system

$$\begin{cases} \partial_s(\beta_{\pm}\varrho_{\pm}) + \operatorname{div}(\beta_{\pm}\varrho_{\pm}v) = 0, \\ \nabla\Pi + \varrho v = 0, \\ \beta_+ + \beta_- = 1, \end{cases} \tag{1.3}$$

with $\varrho = \beta_+\varrho_+ + \beta_-\varrho_-$ and $\Pi = P_+(\varrho_+) = P_-(\varrho_-)$. Inserting Darcy's law $(1.3)_2$ into $(1.3)_1$, we derive the following two-phase system in porous media:

$$\begin{cases} \partial_s\beta_+ + v \cdot \nabla\beta_+ = \dfrac{(\gamma_+ - \gamma_-)\beta_+\beta_-}{\gamma_+\beta_- + \gamma_-\beta_+} \operatorname{div}\left(\dfrac{\nabla\Pi}{\beta_+\varrho_+ + \beta_-\varrho_-}\right), \\ \partial_s\Pi + v \cdot \nabla\Pi = \dfrac{\gamma_+\gamma_-\Pi}{\gamma_+\beta_- + \gamma_-\beta_+}\operatorname{div}\left(\dfrac{\nabla\Pi}{\beta_+\varrho_+ + \beta_-\varrho_-}\right), \\ \beta_+ + \beta_- = 1, \\ \Pi = P_+(\varrho_+) = P_-(\varrho_-). \end{cases} \tag{PM}$$

The present paper is a follow-up to the paper [15] by Burtea, Crin-Barat and Tan where the authors justified the pressure-relaxation limit for the viscous version of System (BN) to System (K). In [15], the smallness condition on initial data employed to justify their global well-posedness result depends on $\min\{\tau, \frac{1}{\tau}\}$ (due to the overdamping phenomenon that will be explained below) and therefore does not allow to further investigate the limit when $\tau \to 0$.

The main results of this article are the quantitative justification of the pressure-relaxation limit from System (BN) *to System* (K) *as $\varepsilon \to 0$ uniformly in τ and the time-relaxation limit from System* (K_τ) *to System* (PM) *as $\tau \to 0$.* Consequently, a new two-phase flow system in porous media (PM) is rigorously derived from Systems (K_τ) and (BN_τ), which implies that Kapila and Baer-Nunziato systems considered in our paper can be viewed, for ε and τ small enough, as hyperbolic approximations of (PM).

For both relaxation limits, we will focus on global-in-time strong solutions being small perturbations of constant equilibrium states. In other words, we consider solutions $(\alpha_{\pm}^{\varepsilon,\tau}, \rho_{\pm}^{\varepsilon,\tau}, u^{\varepsilon,\tau})$ to System (BN) (resp. $(\alpha_{\pm}^{\tau}, \rho_{\pm}^{\tau}, u^{\tau})$ to System (K)) with positive densities and volume fractions which, as $|x| \to \infty$, tend to some thermodynamically stable equilibrium state $(\bar{\alpha}_{\pm}, \bar{\rho}_{\pm}, 0)$ fulfilling

$$0 < \bar{\alpha}_{\pm} < 1, \qquad \bar{\alpha}_+ + \bar{\alpha}_- = 1, \qquad \bar{\rho}_{\pm} > 0, \qquad P_+(\bar{\rho}_+) = P_-(\bar{\rho}_-). \tag{1.4}$$

3

For convenience, we also define the corresponding equilibrium state for the total density and the total pressure as

$$\bar{\rho} := \bar{\alpha}_+ \bar{\rho}_+ + \bar{\alpha}_- \bar{\rho}_-, \qquad \bar{P} := P_+(\bar{\rho}_+) = P_-(\bar{\rho}_-). \tag{1.5}$$

To achieve our goals, we prove uniform in ε and τ (such that $\varepsilon \leq \tau$) a priori estimate for System (BN) which improves the analysis performed in [15] that did not provide uniform-in-τ estimate. Such estimate allows us to justify a global well-posedness for a class of non-symmetric partially dissipative hyperbolic systems with rough coefficients in the context of overdamping phenomenon, which is not covered by the recent lecture of Danchin [24]. Indeed, our proof generalizes the techniques developed in [5, 18, 19, 21] which cannot be directly applied to System (BN) due to the complex forms of the total pressure and the lack of symmetry.

On the other hand, it is natural to ask what happens for System (BN) as τ tends to 0 first. To investigate this process, we introduce a diffusive scaling similar to (1.2) as follows

$$(\beta_\pm^{\varepsilon,\tau}, \varrho_\pm^{\varepsilon,\tau}, v^{\varepsilon,\tau})(s, x) := \left(\alpha_\pm^{\varepsilon,\tau}, \rho_\pm^{\varepsilon,\tau}, \frac{u^{\varepsilon,\tau}}{\tau} \right) \left(\frac{s}{\tau}, x \right). \tag{1.6}$$

Under such scaling (1.6), System (BN) becomes

$$\begin{cases} \partial_s \beta_+^{\varepsilon,\tau} + v^{\varepsilon,\tau} \cdot \nabla \beta_+^{\varepsilon,\tau} = -\dfrac{\beta_+^{\varepsilon,\tau} \beta_-^{\varepsilon,\tau}}{\varepsilon\tau} \left(P_+(\varrho_+^{\varepsilon,\tau}) - P_-(\varrho_-^{\varepsilon,\tau}) \right), \\ \partial_s (\beta_\pm^{\varepsilon,\tau} \varrho_\pm^{\varepsilon,\tau}) + \mathrm{div}\, (\beta_\pm^{\varepsilon,\tau} \varrho_\pm^{\varepsilon,\tau} v^{\varepsilon,\tau}) = 0, \\ \tau^2 \partial_s (\varrho^{\varepsilon,\tau} v^{\varepsilon,\tau}) + \tau^2 \mathrm{div}\, (\varrho^{\varepsilon,\tau} v^{\varepsilon,\tau} \otimes v^{\varepsilon,\tau}) + \nabla \Pi^{\varepsilon,\tau} + \varrho^{\varepsilon,\tau} v^{\varepsilon,\tau} = 0, \end{cases} \tag{BN$_\tau$}$$

with $\beta_+^{\varepsilon,\tau} + \beta_-^{\varepsilon,\tau} = 1$, $\varrho^{\varepsilon,\tau} = \beta_+^{\varepsilon,\tau} \varrho_+^{\varepsilon,\tau} + \beta_-^{\varepsilon,\tau} \varrho_-^{\varepsilon,\tau}$ and $\Pi^{\varepsilon,\tau} = \beta_+^{\varepsilon,\tau} P_+(\varrho_+^{\varepsilon,\tau}) + \beta_-^{\varepsilon,\tau} P_-(\varrho_-^{\varepsilon,\tau})$. The crucial observation is that the parameter τ now also appears under the pressure-relaxation term in the equation of the volume fractions. This reveals that as $\tau \to 0$, the two pressures in System (BN$_\tau$) should converge to a common pressure, and the solutions of System (BN) should converge to the solutions of System (K) regardless of ε. Additionally, in the sequel of the paper we are only able to justify the limit in the case $\varepsilon \leq \tau$ which corresponds to the situation that the time-scale of the pressure-relaxation is small than the time-scale of the diffusive relaxation. The condition $\varepsilon \leq \tau$ appears in the spectral analysis of the system (see Section 1.4) and is essential for us to close the uniform a-priori estimate in both low and high frequencies (See Sections 3.1-3.2 for the details). But in a formal way, the condition $\varepsilon \leq \tau$ seems not necessary in the limit process $\tau \to 0$, so the case $\varepsilon > \tau$ remains an interesting open problem. The Figure 1 summarizes the limit processes that we tackle in this article.

1.2 Outline of the paper

The rest of the paper is organized as follows. Our main results are stated in Section 1.3. In Section 1.4, we first recall a reformulation of System (BN) from [15] and present an explicit spectral analysis for the associated linear system, then the difficulties and strategies of proof are discussed. Section 2 is devoted to some notations and properties of Besov spaces and Littlewood-Paley decomposition, and the regularity estimates for some linear problems are stated. In Section 3, we establish uniform a priori estimate for the linearized problem. Next, in Section 4, we prove the global existence and uniqueness
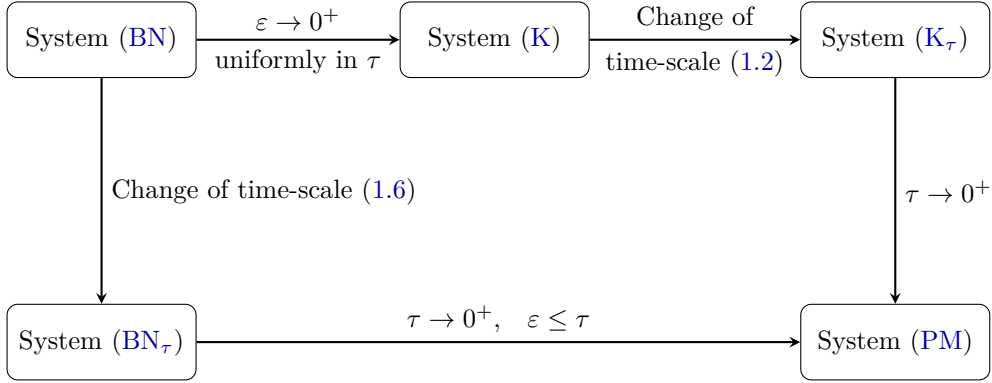
Figure 1: Relaxation limits diagram.

results of solutions for Systems (BN), (K) and (PM), respectively. Section 5 is devoted to the justification of the relaxation limits with explicit convergence rates.

Notations. We end this section by presenting a few notations. As usual, we denote by C (and sometimes with subscripts) harmless positive constants that may change from line to line, and $A \lesssim B$ ($A \gtrsim B$) means that both $A \leq CB$ ($A \geq CB$), while $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$. For X a Banach space, $p \in [1, \infty]$ and $T > 0$, the notation $L^p(0, T; X)$ or $L^p_T(X)$ designates the set of measurable functions $f : [0, T] \to X$ with $t \mapsto \|f(t)\|_X$ in $L^p(0, T)$, endowed with the norm $\|\cdot\|_{L^p_T(X)} := \|\|\cdot\|_X\|_{L^p(0,T)}$. We agree that $\mathcal{C}_b([0, T]; X)$ denotes the set of continuous and bounded (uniformly in T) functions from $[0, T]$ to X. Sometimes, we use the notation $L^p(X)$ to designate the space $L^p(\mathbb{R}_+; X)$ and $\|\cdot\|_{L^p(X)}$ for the associated norm. We will keep the same notations for multi-component functions, namely for $f : [0, T] \to X^m$ with $m \in \mathbb{N}$. $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for the Fourier transform and its inverse, and define the operator $\Lambda^\sigma := \mathcal{F}^{-1}(|\xi|^\sigma \mathcal{F}(\cdot))$. Finally, let $o(1)$ denote a generic constant which can be sufficiently small.

1.3 Main results

Our first theorem concerns the uniform, in both relaxation parameters ε and τ, global well-posedness of System (BN) in a critical regularity framework.

Theorem 1.1. *Let $d \geq 2$ and $0 < \varepsilon \leq \tau \leq 1$. Given the constants $\bar{\alpha}_\pm, \bar{\rho}_\pm$ verifying (1.4)-(1.5), assume that the initial data $(\alpha_{\pm,0}, \rho_{\pm,0}, u_0)$ satisfies $(\alpha_{\pm,0} - \bar{\alpha}_\pm, \rho_{\pm,0} - \bar{\rho}_\pm, u_0) \in \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}$. There exists a positive constant c_0 independent of τ and ε such that if*

$$\|(\alpha_{\pm,0} - \bar{\alpha}_\pm, \rho_{\pm,0} - \bar{\rho}_\pm, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}} \leq c_0, \tag{1.7}$$

then the Cauchy problem of System (BN) with the initial data $(\alpha_{\pm,0}, \rho_{\pm,0}, u_0)$ has a unique global solution

$(\alpha_{\pm}^{\varepsilon,\tau}, \rho_{\pm}^{\varepsilon,\tau}, u^{\varepsilon,\tau})$ *satisfying*

$$
\begin{cases}
(\alpha_{\pm}^{\varepsilon,\tau} - \bar{\alpha}_{\pm}, \rho_{\pm}^{\varepsilon,\tau} - \bar{\rho}_{\pm}, u^{\varepsilon,\tau}) \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}), \\
P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau}) \in L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}), \\
P^{\varepsilon,\tau} - \bar{P} \in L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}+1}) \cap L^2(\mathbb{R}_+; \dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}), \\
u^{\varepsilon,\tau} \in L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}) \cap L^2(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}).
\end{cases}
\tag{1.8}
$$

Moreover, the following uniform estimate holds:

$$
\begin{aligned}
&\|(\alpha_{\pm}^{\varepsilon,\tau} - \bar{\alpha}_{\pm}, \rho_{\pm}^{\varepsilon,\tau} - \bar{\rho}_{\pm}, u^{\varepsilon,\tau})\|_{L^{\infty}(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} + \|(\partial_t \alpha_{\pm}^{\varepsilon,\tau}, \partial_t \rho_{\pm}^{\varepsilon,\tau}, \partial_t u^{\varepsilon,\tau})\|_{L^1(\dot{B}^{\frac{d}{2}})} \\
&\quad + \frac{1}{\varepsilon} \|P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\|_{L^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})} + \frac{1}{\sqrt{\varepsilon}} \|P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\|_{L^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} \\
&\quad + \tau \|P^{\varepsilon,\tau} - \bar{P}\|_{L^1(\dot{B}^{\frac{d}{2}+1})} + \sqrt{\tau} \|P^{\varepsilon,\tau} - \bar{P}\|_{L^2(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} \\
&\quad + \|u^{\varepsilon,\tau}\|_{L^1(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} + \frac{1}{\sqrt{\tau}} \|u^{\varepsilon,\tau}\|_{L^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} \\
&\quad + \left\| \frac{\rho^{\varepsilon,\tau} u^{\varepsilon,\tau}}{\tau} + \nabla P^{\varepsilon,\tau} \right\|_{L^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})} \leq C \|(\alpha_{\pm,0} - \bar{\alpha}_{\pm}, \rho_{\pm,0} - \bar{\rho}_{\pm}, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}},
\end{aligned}
\tag{1.9}
$$

where $C > 0$ is a generic constant.

Remark 1.1. *It should be emphasized that the regularity and decay-in-τ properties of the effective flux $\dfrac{\rho^{\varepsilon,\tau} u^{\varepsilon,\tau}}{\tau} + \nabla P^{\varepsilon,\tau}$ is better than the one verified by the solution $(\alpha_{\pm}^{\varepsilon,\tau}, \rho_{\pm}^{\varepsilon,\tau}, u^{\varepsilon,\tau})$. This is consistent with Darcy's law and plays key role in the justification of the time-relaxation limit.*

By classical compactness arguments and the uniform estimate (1.9), we obtain the following global well-posedness theorems for Systems (K) and (PM) in the critical regularity framework.

Theorem 1.2. *Let $d \geq 2$ and $0 < \tau \leq 1$. Given the constants $\bar{\alpha}_{\pm}, \bar{\rho}_{\pm}$ verifying (1.4)-(1.5), assume that the initial data $(\alpha_{\pm,0}, \rho_{\pm,0}, u_0)$ satisfies $(\alpha_{\pm,0} - \bar{\alpha}_{\pm}, \rho_{\pm,0} - \bar{\rho}_{\pm}, u_0) \in \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}$. There exists a positive constant c_1 independent of τ such that if*

$$
\|(\alpha_{\pm,0} - \bar{\alpha}_{\pm}, \rho_{\pm,0} - \bar{\rho}_{\pm}, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}} \leq c_1,
\tag{1.10}
$$

then the Cauchy problem of System (K) with the initial data $(\alpha_{\pm,0}, \rho_{\pm,0}, u_0)$ admits a unique global solution $(\alpha_{\pm}^{\tau}, \rho_{\pm}^{\tau}, u^{\tau})$ satisfying

$$
\begin{cases}
(\alpha_{\pm}^{\tau} - \bar{\alpha}_{\pm}, \rho_{\pm}^{\tau} - \bar{\rho}_{\pm}, u^{\tau}) \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}), \\
P^{\tau} - \bar{P} \in L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}+1}) \cap L^2(\mathbb{R}_+; \dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}), \\
u^{\tau} \in L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}) \cap L^2(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}).
\end{cases}
\tag{1.11}
$$

Moreover, the following uniform estimate holds:

$$
\begin{aligned}
&\|(\alpha_{\pm}^{\tau} - \bar{\alpha}_{\pm}, \rho_{\pm}^{\tau} - \bar{\rho}_{\pm}, u^{\tau})\|_{L^{\infty}(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} + \|(\partial_t \alpha_{\pm}^{\tau}, \partial_t \rho_{\pm}^{\tau}, \partial_t u^{\tau})\|_{L^1(\dot{B}^{\frac{d}{2}})} + \tau \|P^{\tau} - \bar{P}\|_{L^1(\dot{B}^{\frac{d}{2}+1})} \\
&\quad + \sqrt{\tau} \|P^{\tau} - \bar{P}\|_{L^2(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} + \|u^{\tau}\|_{L^1(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} + \frac{1}{\sqrt{\tau}} \|u^{\tau}\|_{L^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} \\
&\quad + \left\| \frac{\rho^{\varepsilon,\tau} u^{\tau}}{\tau} + \nabla P^{\tau} \right\|_{L^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})} \leq C \|(\alpha_{\pm,0} - \bar{\alpha}_{\pm}, \rho_{\pm,0} - \bar{\rho}_{\pm}, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}},
\end{aligned}
\tag{1.12}
$$

where $C > 0$ is a generic constant.

Theorem 1.3. *Let $d \geq 2$. Given the constants $\bar{\alpha}_\pm, \bar{\rho}_\pm$ verifying (1.4)-(1.5), assume that the initial data $(\beta_{\pm,0}, \varrho_{\pm,0})$ satisfies $(\beta_{\pm,0} - \bar{\alpha}_\pm, \varrho_{\pm,0} - \bar{\rho}_\pm) \in \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}$ and*

$$\|(\beta_{\pm,0} - \bar{\alpha}_\pm, \varrho_{\pm,0} - \bar{\rho}_\pm)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}} \leq c_2, \tag{1.13}$$

for a positive constant c_2, then the Cauchy problem of System (PM) with the initial data $(\beta_{\pm,0}, \varrho_{\pm,0})$ admits a unique global solution $(\beta_\pm, \varrho_\pm)$, which satisfies

$$\begin{cases} \beta_\pm - \bar{\alpha}_\pm \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}), \\ \varrho_\pm - \bar{\rho}_\pm \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}+1}). \end{cases} \tag{1.14}$$

Moreover, the following uniform estimate holds:

$$\|(\beta_\pm - \bar{\alpha}_\pm, \varrho_\pm - \bar{\rho}_\pm)\|_{L^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} + \|(\partial_t \beta_\pm, \partial_t \varrho_\pm)\|_{L^1(\dot{B}^{\frac{d}{2}})} + \|\varrho_\pm - \bar{\rho}_\pm\|_{L^1(\dot{B}^{\frac{d}{2}+1} \cap \dot{B}^{\frac{d}{2}+3})} \tag{1.15}$$

$$\leq C\|(\beta_{\pm,0} - \bar{\alpha}_\pm, \varrho_{\pm,0} - \bar{\rho}_\pm)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}}, \tag{1.16}$$

where $C > 0$ is a generic constant.

Next, we present the rigorous justifications of the pressure-relaxation limit for System (BN) to System (K) as $\varepsilon \to 0$ uniform with respect to τ, and further the time-relaxation limit for System (K$_\tau$) to System (PM) as $\tau \to 0$, with explicit convergence rates.

Theorem 1.4. *Let $d \geq 2$ and $0 < \varepsilon \leq \tau \leq 1$. Given the constants $\bar{\alpha}_\pm, \bar{\rho}_\pm$ verifying (1.4)-(1.5), let $(\alpha_\pm^{\varepsilon,\tau}, \rho_\pm^{\varepsilon,\tau}, u^{\varepsilon,\tau})$, $(\alpha_\pm^\tau, \rho_\pm^\tau, u^\tau)$ and $(\beta_\pm, \varrho_\pm)$ be the global solutions to the Cauchy problems of Systems (BN), (K) and (PM) obtained from Theorems 1.1-1.3 associated to their corresponding initial data $(\alpha_{\pm,0}^{\varepsilon,\tau}, \rho_{\pm,0}^{\varepsilon,\tau}, u_0^{\varepsilon,\tau})$, $(\alpha_{\pm,0}^\tau, \rho_{\pm,0}^\tau, u_0^\tau)$ and $(\beta_{\pm,0}, \varrho_{\pm,0})$, respectively.*

- *Let the initial quantities $P_0^{\varepsilon,\tau} - P_0^\tau$ and $Y_0^{\varepsilon,\tau} - Y_0^\tau$ be denoted by (5.1) and (5.3), respectively. If $d \geq 3$ and*

$$\left\|\left(P_+(\rho_{+,0}^{\varepsilon,\tau}) - P_-(\rho_{-,0}^{\varepsilon,\tau}), Y_0^{\varepsilon,\tau} - Y_0^\tau, P_0^{\varepsilon,\tau} - P_0^\tau, u_0^{\varepsilon,\tau} - u_0^\tau\right)\right\|_{\dot{B}^{\frac{d}{2}-2} \cap \dot{B}^{\frac{d}{2}-1}} \leq \sqrt{\varepsilon\tau}, \tag{1.17}$$

  then there exists a universal constant C_1 such that the following estimate holds:

$$\|(\alpha_\pm^{\varepsilon,\tau} - \alpha_\pm^\tau, \rho_\pm^{\varepsilon,\tau} - \rho_\pm^\tau, u^{\varepsilon,\tau} - u^\tau)\|_{L^\infty(\dot{B}^{\frac{d}{2}-2} \cap \dot{B}^{\frac{d}{2}-1})}$$
$$+ \sqrt{\tau}\|\rho_\pm^{\varepsilon,\tau} - \rho_\pm^\tau\|_{L^2(\dot{B}^{\frac{d}{2}-1})} + \frac{1}{\sqrt{\tau}}\|u^{\varepsilon,\tau} - u^\tau\|_{L^2(\dot{B}^{\frac{d}{2}-2} \cap \dot{B}^{\frac{d}{2}-1})} \tag{1.18}$$
$$+ \|u^{\varepsilon,\tau} - u^\tau\|_{L^1(\dot{B}^{\frac{d}{2}-1})} \leq C_1\sqrt{\varepsilon\tau}.$$

- *Furthermore, define $(\beta_\pm^\tau, \varrho_\pm^\tau, v^\tau)$ by the diffusive scaling (1.2) and v by Darcy's law (1.3)$_2$. Let the initial quantity $Z_0^\tau - Z_0$ be denoted by (5.32). If*

$$\|Z_0^\tau - Z_0\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}}} + \|\varrho_{\pm,0}^\tau - \varrho_{\pm,0}\|_{\dot{B}^{\frac{d}{2}-1}} \leq \tau, \tag{1.19}$$

  then there exists a universal constant C_2 such that the following estimate holds:

$$\|(\beta_\pm^\tau - \beta_\pm, \varrho_\pm^\tau - \varrho_\pm)\|_{L^\infty(\dot{B}^{\frac{d}{2}-1})} + \|\varrho_\pm^\tau - \varrho_\pm\|_{L^1(\dot{B}^{\frac{d}{2}+1})} + \|v^\tau - v\|_{L^1(\dot{B}^{\frac{d}{2}})} \leq C_2\tau. \tag{1.20}$$

Finally, Theorem 1.4 implies the relaxation limit for System (BN_τ) to System (PM) as both $\varepsilon, \tau \to 0$.

Corollary 1.1. *Let $d \geq 3$, $0 < \varepsilon \leq \tau \leq 1$, and $(\beta_\pm^{\varepsilon,\tau}, \varrho_\pm^{\varepsilon,\tau}, v^{\varepsilon,\tau})$ be defined by (1.6). Then under the assumptions of Theorem 1.4, there is a generic constant C_3 such that*

$$\|(\beta_\pm^{\varepsilon,\tau} - \beta_\pm, \varrho_\pm^{\varepsilon,\tau} - \varrho_\pm)\|_{L^\infty(\dot{B}^{\frac{d}{2}-1})} \leq C_3(\sqrt{\varepsilon\tau} + \tau).$$

1.4 Difficulties and strategies

The first difficulty concerning the study of System (BN) are its lack of dissipativity and symmetrizability. Indeed, the linearization of (BN) admits the eigenvalue 0 and therefore does not satisfy the well-known "Shizuta-Kawashima" stability condition for partially dissipative hyperbolic systems (cf. [44]). Additionally, System (BN) cannot be written in a conservative form and the entropy naturally associated to (BN) is not positive definite, therefore the notion of entropic variables does not make sense in this case. Therefore, the first crucial step in our analysis is to partially symmetrize System (BN), by hands. We refer to [13, 27] for the treatment of non-conservative systems in similar contexts. In our setting, as explained in [15], we define the new unknowns

$$\begin{cases} y^{\varepsilon,\tau} := \dfrac{\alpha_+^{\varepsilon,\tau} \rho_+^{\varepsilon,\tau}}{\alpha_+^{\varepsilon,\tau} \rho_+^{\varepsilon,\tau} + \alpha_-^{\varepsilon,\tau} \rho_-^{\varepsilon,\tau}} - \dfrac{\bar{\alpha}_+ \bar{\rho}_+}{\bar{\alpha}_+ \bar{\rho}_+ + \bar{\alpha}_- \bar{\rho}_-}, \\[3mm] w^{\varepsilon,\tau} := \dfrac{\alpha_+^{\varepsilon,\tau} \alpha_-^{\varepsilon,\tau}}{\gamma_+ \alpha_-^{\varepsilon,\tau} + \gamma_- \alpha_+^{\varepsilon,\tau}} \big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big), \\[3mm] r^{\varepsilon,\tau} := P^{\varepsilon,\tau} - \bar{P} - (\gamma_+ - \gamma_-)w^{\varepsilon,\tau}, \end{cases} \tag{1.21}$$

and the corresponding initial data

$$\begin{cases} y_0 := \dfrac{\alpha_{+,0} \rho_{+,0}}{\alpha_{+,0} \rho_{+,0} + \alpha_{-,0} \rho_{-,0}} - \dfrac{\bar{\alpha}_+ \bar{\rho}_+}{\bar{\alpha}_+ \bar{\rho}_+ + \bar{\alpha}_- \bar{\rho}_-}, \\[3mm] w_0 := \dfrac{\alpha_{+,0} \alpha_{-,0}}{\gamma_+ \alpha_{-,0} + \gamma_- \alpha_{+,0}} \big(P_+(\rho_{+,0}) - P_-(\rho_{-,0})\big), \\[3mm] r_0 := \alpha_{+,0} P_+(\rho_{+,0}) + \alpha_{-,0} P_-(\rho_{-,0}) - \bar{P} - (\gamma_+ - \gamma_-)w_0, \end{cases} \tag{1.22}$$

so that the Cauchy problem of System (BN) subject to the initial data $(\alpha_{\pm,0}, \rho_{\pm,0}, u_0)$ is reformulated as

$$\begin{cases} \partial_t y^{\varepsilon,\tau} + u^{\varepsilon,\tau} \cdot \nabla y^{\varepsilon,\tau} = 0, \\[2mm] \partial_t w^{\varepsilon,\tau} + u^{\varepsilon,\tau} \cdot \nabla w^{\varepsilon,\tau} + (\bar{F}_1 + G_1^{\varepsilon,\tau})\mathrm{div}\, u^{\varepsilon,\tau} + (\bar{F}_2 + G_2^{\varepsilon,\tau})\dfrac{w^{\varepsilon,\tau}}{\varepsilon} = 0, \\[2mm] \partial_t r^{\varepsilon,\tau} + u^{\varepsilon,\tau} \cdot \nabla r^{\varepsilon,\tau} + (\bar{F}_3 + G_3^{\varepsilon,\tau})\mathrm{div}\, u^{\varepsilon,\tau} = F_4^{\varepsilon,\tau} \dfrac{(w^{\varepsilon,\tau})^2}{\varepsilon}, \\[2mm] \partial_t u^{\varepsilon,\tau} + u^{\varepsilon,\tau} \cdot \nabla u^{\varepsilon,\tau} + \dfrac{u^{\varepsilon,\tau}}{\tau} + (\bar{F}_0 + G_0^{\varepsilon,\tau})\nabla r^{\varepsilon,\tau} + (\gamma_+ - \gamma_-)(\bar{F}_0 + G_0^{\varepsilon,\tau})\nabla w^{\varepsilon,\tau} = 0, \\[2mm] (y^{\varepsilon,\tau}, w^{\varepsilon,\tau}, r^{\varepsilon,\tau}, u^{\varepsilon,\tau})(0,x) = (y_0, w_0, r_0, u_0)(x), \end{cases} \tag{1.23}$$

where $F_i^{\varepsilon,\tau} = F_i^{\varepsilon,\tau}(y, w, r)$ $(i = 0, 1, 2, 3, 4)$ are the nonlinear terms

$$
\begin{cases}
F_0^{\varepsilon,\tau} := \dfrac{1}{\alpha_+^{\varepsilon,\tau} \rho_+^{\varepsilon,\tau} + \alpha_-^{\varepsilon,\tau} \rho_-^{\varepsilon,\tau}}, \\[2mm]
F_1^{\varepsilon,\tau} := \dfrac{(\gamma_+ - \gamma_-)\alpha_+^{\varepsilon,\tau} \alpha_-^{\varepsilon,\tau}}{\gamma_+ \alpha_-^{\varepsilon,\tau} + \gamma_- \alpha_+^{\varepsilon,\tau}} (\bar{P} + r^{\varepsilon,\tau}) + \dfrac{\gamma_+^2 \alpha_-^{\varepsilon,\tau} + \gamma_-^2 \alpha_+^{\varepsilon,\tau}}{\gamma_+ \alpha_-^{\varepsilon,\tau} + \gamma_- \alpha_+^{\varepsilon,\tau}} w^{\varepsilon,\tau}, \\[2mm]
F_2^{\varepsilon,\tau} := (\gamma_+ \alpha_-^{\varepsilon,\tau} + \gamma_- \alpha_+^{\varepsilon,\tau})(\bar{P} + r^{\varepsilon,\tau}) - \dfrac{(\gamma_+ - \gamma_+^2)(\alpha_-^{\varepsilon,\tau})^2 - (\gamma_- - \gamma_-^2)(\alpha_+^{\varepsilon,\tau})^2}{\alpha_+^{\varepsilon,\tau} \alpha_-^{\varepsilon,\tau}} w^{\varepsilon,\tau}, \\[2mm]
F_3^{\varepsilon,\tau} := \dfrac{\gamma_+ \gamma_-}{\gamma_+ \alpha_-^{\varepsilon,\tau} + \gamma_- \alpha_+^{\varepsilon,\tau}} P^{\varepsilon,\tau}, \\[2mm]
F_4^{\varepsilon,\tau} := \dfrac{\gamma_+ \gamma_-}{\alpha_+^{\varepsilon,\tau} \alpha_-^{\varepsilon,\tau}} (1 - \gamma_+ \alpha_-^{\varepsilon,\tau} - \gamma_- \alpha_+^{\varepsilon,\tau}),
\end{cases}
\tag{1.24}
$$

$\bar{F}_i$ $(i = 0, 1, 2, 3)$ are the constants

$$
\begin{cases}
\bar{F}_0 := \dfrac{1}{\bar{\alpha}_+ \bar{\rho}_+ + \bar{\alpha}_- \bar{\rho}_-} > 0, \\[2mm]
\bar{F}_1 := \dfrac{(\gamma_+ - \gamma_-)\bar{\alpha}_+ \bar{\alpha}_-}{\gamma_+ \bar{\alpha}_- + \gamma_- \bar{\alpha}_+} \bar{P} > 0, \\[2mm]
\bar{F}_2 := (\gamma_+ \bar{\alpha}_- + \gamma_- \bar{\alpha}_+)\bar{P} > 0, \\[2mm]
\bar{F}_3 := \dfrac{\gamma_+ \gamma_-}{\gamma_+ \bar{\alpha}_- + \gamma_- \bar{\alpha}_+} \bar{P} > 0,
\end{cases}
\tag{1.25}
$$

and $G_i^{\varepsilon,\tau} = G_i^{\varepsilon,\tau}(y, w, r)$ $(i = 0, 1, 2, 3)$ are the coefficients

$$
G_i^{\varepsilon,\tau} := F_i^{\varepsilon,\tau} - \bar{F}_i.
\tag{1.26}
$$

In this formulation, the equation $(1.23)_1$ is purely transport and the linear part of subsystem $(1.23)_2$-$(1.23)_4$ is partially dissipative and satisfies the "Shizuta-Kawashima" stability condition. Thus, we will estimate the undamped unknown $y^{\varepsilon,\tau}$ and the dissipative components $(w^{\varepsilon,\tau}, r^{\varepsilon,\tau}, u^{\varepsilon,\tau})$ separately. We emphasize here that due to the double parameters ε, τ and the lack of time-integrability of $G_i^{\varepsilon,\tau}$, the dissipative structures of subsystem $(1.23)_2$-$(1.23)_4$ does not fit into the general theorems that can be found in [18, 19, 24, 44, 49, 50], and a new analysis is needed to be developed to obtain the uniform estimates with respect to the two relaxation parameters ε, τ.

In order to understand the behaviors of the solution to (1.23) with respect to ε, τ, we perform a spectral analysis of the linear system for (1.23). For simplicity we set $\bar{F}_i = 1$ $(i = 0, 2, 3)$ and $\bar{F}_1 = \gamma_+ - \gamma_-$. In terms of Hodge decomposition, we denote the compressible part $m = \Lambda^{-1} \mathrm{div}\, u$ and the incompressible part $\Omega = \Lambda^{-1} \nabla \times u$ and rewrite the linear system of (1.23) as

$$
\partial_t \begin{pmatrix} w \\ r \\ m \end{pmatrix} = \mathbb{A} \begin{pmatrix} w \\ r \\ m \end{pmatrix}, \quad
\mathbb{A} := \begin{pmatrix} -\dfrac{1}{\varepsilon} & 0 & -(\gamma_+ - \gamma_-)\Lambda \\ 0 & 0 & -\Lambda \\ (\gamma_+ - \gamma_-)\Lambda & \Lambda & -\dfrac{1}{\tau} \end{pmatrix}, \quad
\partial_t \Omega + \dfrac{1}{\tau} \Omega = 0.
$$

The eigenvalues of the matrix $\widehat{\mathbb{A}}(\xi)$ satisfy

$$
|\widehat{\mathbb{A}}(\xi) - \lambda \mathbb{I}_{3 \times 3}| = \lambda^3 + \left(\frac{1}{\tau} + \frac{1}{\varepsilon} \right) \lambda^2 + \left[\frac{1}{\varepsilon\tau} + (|\gamma_+ - \gamma_-|^2 + 1)|\xi|^2 \right] \lambda + \frac{1}{\varepsilon} |\xi|^2 = 0.
$$

Under the condition $0 < \varepsilon << \tau$, the behaviors of λ_i $(i = 1, 2, 3)$ can be analyzed as follows:

9

- In the low-frequency region $|\xi| << \frac{1}{\tau}$, by Taylor's expansion near $|\tau\xi| << 1$ as in [41], all the eigenvalues are real, and we have $\lambda_1 = -\frac{1}{\varepsilon} + \frac{1}{\tau}\mathcal{O}(|\tau\xi|^2)$, $\lambda_2 = -\tau|\xi|^2 + \frac{1}{\tau}\mathcal{O}(|\tau\xi|^3)$ and $\lambda_3 = -\frac{1}{\tau} + \frac{1}{\tau}\mathcal{O}(|\tau\xi|^2)$.

- In the medium-frequency region $\frac{1}{\tau} << |\xi| << \frac{1}{\varepsilon}$, according to Cardano's formula, λ_1 is real and λ_i ($i = 2, 3$) are conjugated complex, and $\mathrm{Re}\,\lambda_i \lesssim -\frac{1}{\tau}$ holds for all $i = 1, 2, 3$.

- In the high-frequency region $|\xi| >> \frac{1}{\varepsilon}$, by Taylor's expansion near $|\varepsilon\xi|^{-1} << 1$, the real eigenvalue λ_1 and the conjugated complex eigenvalues λ_i ($i = 2, 3$) satisfy $\lambda_1 = -\frac{1}{|\gamma_+ - \gamma_-|^2 + 1}\frac{1}{\varepsilon} + \frac{1}{\varepsilon}\mathcal{O}\left(\frac{1}{|\varepsilon\xi|^2}\right)$ and $\lambda_{2,3} = -\frac{1}{2\tau} - \frac{|\gamma_+ - \gamma_-|^2}{|\gamma_+ - \gamma_-|^2 + 1}\frac{1}{2\varepsilon} \pm \sqrt{|\gamma_+ - \gamma_-|^2 + 1}|\xi|i + (\frac{1}{\tau} + \frac{|\gamma_+ - \gamma_-|^2}{\varepsilon})\mathcal{O}\left(\frac{1}{|\varepsilon\xi|}\right)$.

The above spectral analysis suggests us to separate the whole frequencies into two parts $|\xi| \lesssim \frac{1}{\tau}$ and $|\xi| \gtrsim \frac{1}{\tau}$ so as to capture the qualitative properties of solutions for System (1.23). Indeed, the time-decay rates (determined by λ_2) achieve the fastest rate in the low-frequency region $|\xi| \lesssim \frac{1}{\tau}$. Moreover this region recover the whole frequency-space when $\tau \to 0$, as expected from the well-known overdamping phenomenon which will be mentioned below. To this end, the threshold J_τ between these two regions is used in the definition of the hybrid Besov spaces in next section.

It should be noted that λ_2 and λ_3 exhibit similar behaviors to the eigenvalues of the compressible Euler equations with damping. Indeed, to study System (1.23), one considers the following simplified system of damped Euler type with rough coefficients:

$$\begin{cases} \partial_t r^\tau + (1 + G_3^\tau)\mathrm{div}\,u^\tau = 0, \\ \partial_t u^\tau + (1 + G_0^\tau)\nabla r^\tau + \dfrac{u^\tau}{\tau} = 0. \end{cases} \tag{1.27}$$

The well-known spectral analysis for the linear Euler part of System (1.27) implies that the frequency space shall be separated into the low-frequency region $|\xi| \lesssim \frac{1}{\tau}$ and the high-frequency region $|\xi| \gtrsim \frac{1}{\tau}$ to recover the uniform estimates and optimal regularity of solutions. Formally, this implies that as $\tau \to 0$, the low-frequency region covers the whole frequency space and is therefore be dominant at the limit. We observe here the classical *overdamping phenomenon*: as the friction coefficient $\frac{1}{\tau}$ gets larger, the decay rates of r^τ do not necessarily increase and on the contrary follow $\min\{\tau, \frac{1}{\tau}\}$, cf. Figure 2. For more discussion on the overdamping phenomenon, see Zuazua's sildes [55].

Recently, in [18, 19], the issue concerning the relaxation limit from compressible Euler system with damping toward the porous media equation has been rigorously justified in critical space $\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}$. The readers also can refer to the work [20] about the relaxation limit for a hyperbolic-parabolic chemotaxis system to a parabolic-elliptic Keller-Segel model. The regularity index $\frac{d}{2} + 1$ is called critical for initial data of general hyperbolic systems since $\dot{B}^{\frac{d}{2}+1}$ is embedded in the set of globally Lipschitz functions. Indeed, It has been observed by many authors that controlling the Lipschitz regularity of solutions for general hyperbolic systems can prevent blow-up in finite time, see e.g., [40, 49]. We also refer to [38, 39] about the ill-posedness results for hyperbolic systems in H^s with $s < \frac{d}{2} + 1$.

Nevertheless, the methods developed in [18–20] are not applicable in the current situation to derive estimates which are uniform with respect to the relaxation parameter τ. This is mainly due to the *complex form of the total pressure $P^{\varepsilon,\tau}$* in the velocity equation $(\mathrm{BN})_3$ and the fact that one can not expect any
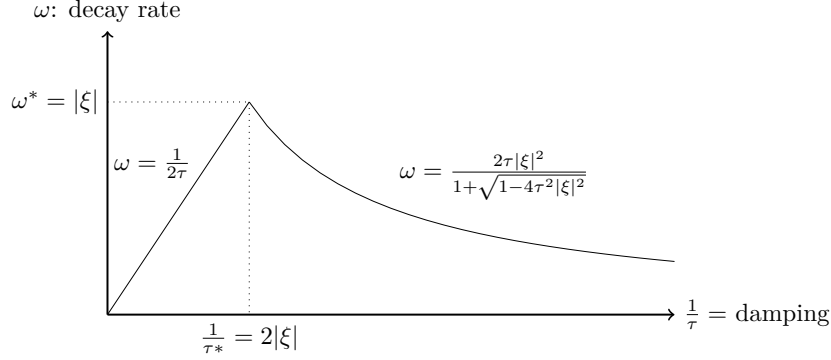
Figure 2: A graph of overdamping phenomenon for System (1.27).

time integrability property on $\mathbb{R}_+$ for the purely transported unknown $y^{\varepsilon,\tau}$, which generally leads to a lack of time integrability on $\mathbb{R}_+$ for $(\alpha_\pm^{\varepsilon,\tau} - \bar{\alpha}_\pm, \rho_\pm^{\varepsilon,\tau} - \bar{\rho}_\pm)$ (see Remarks 3.2-3.3), and thus for G_3^τ, G_0^τ in System (1.27). In addition, we can not find a a rescaling to reduce the proof to the case $\tau = 1$ and then recover the corresponding uniform estimates with respect to τ thanks to the homogeneity of the Besov norms as in [18, 19]. To overcome these new difficulties, we will keep track of the dependence of ε, τ and perform elaborate energy estimates with mixed L^1-time and L^2-time type dissipation. More precisely, in the low-frequency region, we introduce a purely damped mode (effective flux)

$$u^{\varepsilon,\tau} + \frac{\tau}{\rho^{\varepsilon,\tau}}\nabla P^{\varepsilon,\tau}$$

corresponding to Darcy's law $(1.3)_2$ in the low-frequency setting to partially diagonalize the system and capture maximal dissipative structures. In addition, we derive some uniform estimates at a lower regularity level compared to [18–20] (see Lemma 3.1). In the high-frequency setting, due to the lack of symmetry, we need to cancel higher-order terms so as not to lose derivatives. For that, the construction of a Lyapunov functional in the spirit of Beauchard and Zuazua as in [3] with additional nonlinear weights allows us to capture the L^1-time dissipation properties in high frequencies (cf. Lemma 3.2). Moreover, we also establish the uniform L^2-in-time estimates at $\dot{B}^{\frac{d}{2}+1}$-regularity level to recover the necessary bounds of parameters (refer to Lemma 3.3). Applying these ideas, we obtain uniform estimates in terms of the parameters ε, τ satisfying $0 < \varepsilon \leq \tau < 1$ for the linearized problem (see Proposition 3.1), which is crucial for our later nonlinear analysis.

Let us finally sketch the proof of the justifications of the strong relaxation limits. In fact, to obtain convergence rates, we will not estimate the differences of solutions between systems directly. The reason shares similarities with the proof of global uniform well-posedness. Roughly speaking, since both pressure-relaxation limit and time-relaxation limit are singular limits, there are singular terms which are only uniformly bounded but not necessarily vanishing in the equations satisfied by the difference of solutions. To overcome these difficulties, we discover some *auxiliary unknowns* associated with the difference systems, which reveal better structures (cancellations), and then perform error estimates on them for each relaxation limit. More details are presented in Sections 5.1 and 5.2.

2 Functional framework and tools

In this section, we recall the notations of the Littlewood-Paley decomposition and Besov spaces. The reader can refer to [2][Chapter 2] for a complete overview. Choose a smooth radial non-increasing function $\chi(\xi)$ with compact supported in $B(0, \frac{4}{3})$ and $\chi(\xi) = 1$ in $B(0, \frac{3}{4})$ such that

$$\varphi(\xi) := \chi(\xi/2) - \chi(\xi), \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \cdot) = 1, \quad \text{Supp } \varphi \subset \left\{ \xi \in \mathbb{R}^d \; \middle| \; \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \right\}.$$

For any $j \in \mathbb{Z}$, the homogeneous dyadic blocks $\dot{\Delta}_j$ and the low-frequency cut-off operator $\dot{S}_j$ are defined by

$$\dot{\Delta}_j u := \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} u), \qquad \dot{S}_j u := \mathcal{F}^{-1}(\chi(2^{-j} \cdot) \mathcal{F} u).$$

From now on, we use the shorthand notation

$$\dot{\Delta}_j u = u_j.$$

Let $\mathcal{S}'_h$ be the set of tempered distributions on $\mathbb{R}^d$ such that every $u \in \mathcal{S}'_h$ satisfies $u \in \mathcal{S}'$ and $\lim_{j \to -\infty} \|\dot{S}_j u\|_{L^\infty} = 0$. Then it follows that

$$u = \sum_{j \in \mathbb{Z}} u_j \quad \text{in } \mathcal{S}', \qquad \dot{S}_j u = \sum_{j' \leq j-1} u_{j'}, \quad \forall u \in \mathcal{S}'_h,$$

With the help of these dyadic blocks, the homogeneous Besov space $\dot{B}^s$ for $s \in \mathbb{R}$ is defined by

$$\dot{B}^s := \left\{ u \in \mathcal{S}'_h \; \middle| \; \|u\|_{\dot{B}^s} := \sum_{j \in \mathbb{Z}} 2^{js} \|u_j\|_{L^2} < \infty \right\}.$$

We denote the Chemin-Lerner type space $\widetilde{L}^\varrho(0, T; \dot{B}^s)$ for $s \in \mathbb{R}$ and $T > 0$:

$$\widetilde{L}^\varrho(0, T; \dot{B}^s) := \left\{ u \in L^\varrho(0, T; \mathcal{S}'_h) \; \middle| \; \|u\|_{\widetilde{L}^\varrho_T(\dot{B}^s)} := \sum_{j \in \mathbb{Z}} 2^{js} \|u_j\|_{L^\varrho_T(L^2)} < \infty \right\}.$$

By the Minkowski inequality, it holds that

$$\|u\|_{L^\varrho_T(\dot{B}^s)} \leq \|u\|_{\widetilde{L}^\varrho_T(\dot{B}^s)} \quad \varrho > 1, \qquad \|u\|_{L^1_T(\dot{B}^s)} = \|u\|_{\widetilde{L}^1_T(\dot{B}^s)},$$

where $\|\cdot\|_{L^\varrho_T(\dot{B}^s)}$ is the usual Lebesgue-Besov norm.

In order to perform our analysis on the low and high frequencies regions, we set the threshold

$$J_\tau := -[\log_2 \tau] + k, \tag{2.1}$$

for suitable negative integer k (to be determined). Denote the following notations for $p \in [1, \infty]$ and $s \in \mathbb{R}$:

$$\|u\|^\ell_{\dot{B}^s} := \sum_{j \leq J_\tau} 2^{js} \|u_j\|_{L^2}, \qquad \|u\|^h_{\dot{B}^s} := \sum_{j \geq J_\tau - 1} 2^{js} \|u_j\|_{L^\varrho_T(L^2)},$$

$$\|u\|^\ell_{\widetilde{L}^\varrho_T(\dot{B}^s)} := \sum_{j \leq J_\tau} 2^{js} \|u_j\|_{L^2}, \qquad \|u\|^h_{\widetilde{L}^\varrho_T(\dot{B}^s)} := \sum_{j \geq J_\tau - 1} 2^{js} \|u_j\|_{L^\varrho_T(L^2)}.$$

For any $u \in \mathcal{S}'_h$, we also define the low-frequency part u^ℓ and the high-frequency part u^h by

$$u^\ell := \sum_{j \le J_\tau - 1} u_j, \qquad u^h := u - u^\ell = \sum_{j \ge J_\tau} u_j.$$

It is easy to check for any $s' > 0$ that

$$\begin{cases} \|u^\ell\|_{\dot{B}^s} \le \|u\|^\ell_{\dot{B}^s} \le 2^{J_\tau s'} \|u\|^\ell_{\dot{B}^{s-s'}} \le 2^{s'} (2^k \tau^{-1})^{s'} \|u\|^\ell_{\dot{B}^{s-s'}}, \\ \|u^h\|_{\dot{B}^s} \le \|u\|^h_{\dot{B}^s} \le 2^{-(J_\tau - 1)s'} \|u\|^h_{\dot{B}^{s+s'}} \le 2^{s'} (2^{-k} \tau)^{s'} \|u\|^h_{\dot{B}^{s+s'}}. \end{cases} \tag{2.2}$$

Next, we state some properties of Besov spaces and related estimates which will be repeatedly used in the rest of paper. The reader can refer to [2, Chapters 2-3] for more details. Below, all the properties of Besov norms can be easily extended to the Chemin-Lerner norms.

The first lemma pertains to the so-called Bernstein's inequalities.

Lemma 2.1. *Let* $0 < r < R$, $1 \le p \le q \le \infty$ *and* $k \in \mathbb{N}$. *For any function* $u \in L^p$ *and* $\lambda > 0$, *it holds that*

$$\begin{cases} \text{Supp } \mathcal{F}(u) \subset \{\xi \in \mathbb{R}^d \mid |\xi| \le \lambda R\} \Rightarrow \|D^k u\|_{L^q} \lesssim \lambda^{k + d(\frac{1}{p} - \frac{1}{q})} \|u\|_{L^p}, \\ \text{Supp } \mathcal{F}(u) \subset \{\xi \in \mathbb{R}^d \mid \lambda r \le |\xi| \le \lambda R\} \Rightarrow \|D^k u\|_{L^p} \sim \lambda^k \|u\|_{L^p}. \end{cases}$$

Due to the Bernstein inequalities, the Besov spaces have many useful properties:

Lemma 2.2. *The following properties hold*:

- *For any* $s \in \mathbb{R}$ *and* $q \ge 2$, *we have the following continuous embeddings*:

$$\dot{B}^s \hookrightarrow \dot{H}^s, \qquad \dot{B}^{\frac{d}{2} - \frac{d}{q}} \hookrightarrow L^q.$$

- $\dot{B}^{\frac{d}{2}}$ *is continuously embedded in the set of continuous functions decaying to* 0 *at infinity.*

- *For any* $\sigma \in \mathbb{R}$, *the operator* Λ^σ *is an isomorphism from* $\dot{B}^s$ *to* $\dot{B}^{s-\sigma}$.

- *Let* $s_1 \in \mathbb{R}$ *and* $s_2 \le \frac{d}{2}$. *Then the space* $\dot{B}^{s_1} \cap \dot{B}^{s_2}$ *is a Banach space and satisfies weak compact and Fatou properties: If* u_k *is a uniformly bounded sequence of* $\dot{B}^{s_1} \cap \dot{B}^{s_2}$, *then an element* u *of* $\dot{B}^{s_1} \cap \dot{B}^{s_2}$ *and a subsequence* u_{n_k} *exist such that*

$$\lim_{k \to \infty} u_{n_k} = u \quad in \quad \mathcal{S}' \quad and \quad \|u\|_{\dot{B}^{s_1} \cap \dot{B}^{s_2}} \lesssim \liminf_{n_k \to \infty} \|u_{n_k}\|_{\dot{B}^{s_1} \cap \dot{B}^{s_2}}.$$

The following Morse-type product estimates in Besov spaces play a fundamental role in our analysis of nonlinear terms.

Lemma 2.3. *The following statements hold:*

- *Let* $s > 0$. *Then* $\dot{B}^s \cap L^\infty$ *is a algebra and*

$$\|uv\|_{\dot{B}^s} \lesssim \|u\|_{L^\infty} \|v\|_{\dot{B}^s} + \|v\|_{L^\infty} \|u\|_{\dot{B}^s}. \tag{2.3}$$

- *Let* s_1, s_2 *satisfy* $s_1, s_2 \le \frac{d}{2}$ *and* $s_1 + s_2 > 0$. *Then there holds*

$$\|uv\|_{\dot{B}^{s_1 + s_2 - \frac{d}{2}}} \lesssim \|u\|_{\dot{B}^{s_1}} \|v\|_{\dot{B}^{s_2}}. \tag{2.4}$$

13

The following commutator estimates will used to control some nonlinearities in high frequencies.

Lemma 2.4. *Let $p \in [1, \infty]$ and $-\frac{d}{2} - 1 < s \le \frac{d}{2} + 1$. Then it holds that*

$$\sum_{j \in \mathbb{Z}} 2^{js} \|[v, \dot{\Delta}_j] \partial_i u\|_{L^2} \lesssim \|\nabla v\|_{\dot{B}^{\frac{d}{2}}} \|u\|_{\dot{B}^s}, \quad i = 1, ..., d, \tag{2.5}$$

for the commutator $[A, B] := AB - BA$.

We prove the following lemma about the continuity for composition of multi-component functions. It should be noted that (2.7) will be used to deal with the two-dimensional case in $\dot{B}^0$.

Lemma 2.5. *Let $m \in \mathbb{N}$, $s > 0$, and $G \in C^\infty(\mathbb{R}^m)$ satisfy $G(0, ..., 0) = 0$. Then for any $f_i \in \dot{B}^s \cap L^\infty$ $(i = 1, ..., m)$, there exists a constant $C_f > 0$ depending on $\sum_{i=1}^m \|f_i\|_{L^\infty}$, F, s, m and d such that*

$$\|G(f_1, ..., f_m)\|_{\dot{B}^s} \le C_f \sum_{i=1}^m \|f_i\|_{\dot{B}^s}. \tag{2.6}$$

In the case $s > -\frac{d}{2}$ and $f_i \in \dot{B}^s \cap \dot{B}^{\frac{d}{2}}$, it holds that

$$\|G(f_1, ..., f_m)\|_{\dot{B}^s} \le C_f \Big(1 + \sum_{i=1}^m \|f_i\|_{\dot{B}^{\frac{d}{2}}}\Big) \sum_{i=1}^m \|f_i\|_{\dot{B}^s}. \tag{2.7}$$

Furthermore, for any $f_i^1, f_i^2 \in \dot{B}^s \cap \dot{B}^{\frac{d}{2}}$, we have

$$\|G(f_1^1, ..., f_m^1) - G(f_1^1, ..., f_m^1)\|_{\dot{B}^s} \le C_f^* \Big(1 + \sum_{i=1}^m \|f_i\|_{\dot{B}^s \cap \dot{B}^{\frac{d}{2}}}\Big) \sum_{i=1}^m \|f_i^1 - f_i^2\|_{\dot{B}^s \cap \dot{B}^{\frac{d}{2}}}. \tag{2.8}$$

Here $C_f^* > 0$ depends on $\sum_{i=1}^m \|(f_i^1, f_i^2)\|_{L^\infty}$, F, s, m and d.

Proof. The estimate (2.6) can be found in [43][Pages 387-388]. Then for $-\frac{d}{2} < s \le \frac{d}{2}$, Taylor's formula implies that there exists a sequence $\widetilde{H}_i(f_1, ..., f_m)$ satisfying $\widetilde{H}_i(0, ..., 0) = 0$ and

$$G(f_1, ..., f_m) = \sum_{i=1}^m \big(\partial_{f_i} G(0, ..., 0) + \widetilde{H}_i(f_1, ..., f_m)\big) f_i.$$

This together with the product law (2.4) and the estimate (2.6) yields (2.7).

Moreover, we note that

$$G(f_1^1, ..., f_m^1) - G(f_1^1, ..., f_m^1) = \int_0^1 \frac{d}{ds} G\big(f_1^1 + s(f_1^1 - f_2^1), ..., f_m^1 + s(f_m^1 - f_m^1)\big) ds$$

$$= \sum_{i=1}^m (f_i^1 - f_i^2) \partial_{f_i} G(0, ..., 0)$$

$$+ \sum_{i=1}^m (f_i^1 - f_i^2) \int_0^1 \Big(\partial_{f_i} G\big(f_1^1 + s(f_1^1 - f_2^1), ..., f_m^1 + s(f_m^1 - f_m^1)\big) - \partial_{f_i} G(0, ..., 0)\Big) ds.$$

Therefore, applying (2.3), (2.6) and the embedding $\dot{B}^{\frac{d}{2}} \hookrightarrow L^\infty$, we get (2.8). $\qquad \square$

Finally, we give optimal regularity estimates of some linear equations. We mention that such estimates on usual Besov norms can be easily extended to the norms restricted in low or high frequencies. We recall the estimates of the heat equation as follows (cf. [2][Page 157] for example).

14

Lemma 2.6. *Let $\mu_* > 0$, $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. For given time $T > 0$, assume $u_0 \in \dot{B}^s$ and $f \in \widetilde{L}^p(0,T; \dot{B}^{s-2+\frac{2}{p}})$. If u solves the problem*

$$
\begin{cases}
\partial_t u - \mu_* \Delta u = f, & x \in \mathbb{R}^d, \quad t > 0, \\
u(0,x) = u_0(x), & x \in \mathbb{R}^d,
\end{cases}
$$

then the following estimate is fulfilled:

$$
\|u\|_{\widetilde{L}_t^\infty(\dot{B}^s)} + \mu_*^{\frac{1}{p_1}} \|u\|_{\widetilde{L}_t^p(\dot{B}^{s+\frac{2}{p}})} \leq C\big(\|u_0\|_{\dot{B}^s} + \mu_*^{\frac{1}{p}-1} \|f\|_{\widetilde{L}_t^p(\dot{B}^{s-2+\frac{2}{p}})}\big), \quad t \in (0,T),
$$

where $C > 0$ is a constant independent of T and μ_*.

We have the regularity estimates of the damped transport equation. Since it can be directly shown by the commutator estimates (2.5) and Grönwall's inequality as in [15, 23], we omit the proof for brevity.

Lemma 2.7. *Let $\lambda_* \geq 0$, $p = 1$ or $\lambda_* > 0$, $1 \leq p \leq \infty$. For $-\frac{d}{2} < s \leq \frac{d}{2} + 1$ and given time $T > 0$, assume that $u_0 \in \dot{B}^s$, $v \in L^1(0,T; \dot{B}^{\frac{d}{2}+1})$ and $f \in \widetilde{L}^p(0,T; \dot{B}^s)$. If u solves the problem*

$$
\begin{cases}
\partial_t u + v \cdot \nabla u + \lambda_* u = f, & x \in \mathbb{R}^d, \quad t > 0, \\
u(0,x) = u_0(x), & x \in \mathbb{R}^d,
\end{cases}
$$

then it holds that

$$
\|u\|_{\widetilde{L}_t^\infty(\dot{B}^s)} + \lambda_*^{\frac{1}{p}} \|u\|_{\widetilde{L}_t^p(\dot{B}^s)} \leq C\exp\big(C\|v\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})}\big)\big(\|u_0\|_{\dot{B}^s} + \lambda_*^{\frac{1}{p}-1} \|f\|_{\widetilde{L}_t^p(\dot{B}^s)}\big), \quad t \in (0,T),
$$

where $C > 0$ is a constant independent of T and λ_*.

3 Analysis of the linearized system

We now consider the linearized problem associated to (1.23), which reads

$$
\begin{cases}
\partial_t y + v \cdot \nabla y = 0, \\
\partial_t w + v \cdot \nabla w + (h_1 + H_1)\operatorname{div} u + (h_2 + H_2)\dfrac{w}{\varepsilon} = S_1, \\
\partial_t r + v \cdot \nabla r + (h_3 + H_3)\operatorname{div} u = S_2, \\
\partial_t u + v \cdot \nabla u + \dfrac{u}{\tau} + (h_4 + H_4)\nabla r + (h_5 + H_5)\nabla w = S_3, \\
(y, w, r, u)(0,x) = (y_0, w_0, r_0, u_0)(x),
\end{cases}
\tag{3.1}
$$

where h_i $(i = 1, ..., 5)$ are given positive constants and $H_i = H_i(t,x)$ $(i = 1, ..., 5)$, $S_i = S_i(t,x)$ $(i = 1, 2, 3)$ are given smooth functions.

We first establish the following a-priori estimate for solutions of the linear problem (3.1) uniformly with respect to the parameters ε, τ, which improves the result in [15] without the uniformity with respect to τ. As explained before, the threshold J_τ between low and high frequencies given by (2.1) is the key to our analysis.

Proposition 3.1. *Let $d \geq 2$, $0 < \varepsilon \leq \tau < 1$, $T > 0$, and the threshold J_τ be given by (2.1). Assume that $(w_0, r_0, u_0) \in \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}$, $S_1, S_2, S_3 \in L^1(0, T; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})$, $H_i \in C([0, T]; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})$ and $\partial_t H_i \in L^1(0, T; \dot{B}^{\frac{d}{2}})$ for $i = 1, 2, \ldots 5$. There exists a constant $c > 0$ independent of T, ε and τ such that if*

$$\mathcal{Z}(t) := \sum_{i=1}^{5} \|H_i\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} \leq c, \qquad t \in (0, T), \tag{3.2}$$

then for $t \in (0, T)$, the solution (y, w, r, u) of the Cauchy problem (3.1) satisfies

$$
\begin{aligned}
\mathcal{X}(t) := &\|(y, w, r, u)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} + \|(\partial_t y, \partial_t w, \partial_t r, \partial_t u)\|_{L_t^1(\dot{B}^{\frac{d}{2}})} \\
&+ \frac{1}{\varepsilon}\|w\|_{L_t^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})} + \frac{1}{\sqrt{\varepsilon}}\|w\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} \\
&+ \tau\|r\|^\ell_{L_t^1(\dot{B}^{\frac{d}{2}+1} \cap \dot{B}^{\frac{d}{2}+2})} + \|r\|^h_{L_t^1(\dot{B}^{\frac{d}{2}+1})} + \tau\|r\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})} + \sqrt{\tau}\|r\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} \\
&+ \|u\|_{L_t^1(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} + \frac{1}{\sqrt{\tau}}\|u\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} \\
&+ \frac{1}{\tau}\|u + \tau(h_4 + H_4)\nabla r\|_{L_t^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})} \\
\leq &\ C_0 \exp\left(C_0 \int_0^t \mathcal{V}(s)ds\right)\left(\|(y_0, w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}} + \|(S_1, S_2, S_3)\|_{L_t^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})}\right),
\end{aligned}
\tag{3.3}
$$

where $C_0 > 1$ is a universal constant, and $\mathcal{V}(t)$ is denoted by

$$\mathcal{V}(t) := \|v(t)\|_{\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}} + \sum_{i=1}^{5} \|\partial_t H_i(t)\|_{\dot{B}^{\frac{d}{2}}}. \tag{3.4}$$

Proof. First, we deal with the purely transport unknown y. By the regularity estimate in Lemma 2.7 for the transport equation $(3.1)_1$, it follows that

$$\|y\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} \lesssim \exp\left(\int_0^t \|v(s)\|_{\dot{B}^{\frac{d}{2}+1}}ds\right)\|y_0\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}}. \tag{3.5}$$

And direct produce law (2.4) for the equation $(3.1)_1$ gives that

$$\|\partial_t y\|_{L_t^1(\dot{B}^{\frac{d}{2}})} \lesssim \int_0^t \|v(s)\|_{\dot{B}^{\frac{d}{2}}} \|y(s)\|_{\dot{B}^{\frac{d}{2}+1}}ds. \tag{3.6}$$

Similarly, we also get from (2.4) and $(3.1)_3$ that

$$\|\partial_t r\|_{L_t^1(\dot{B}^{\frac{d}{2}})} \lesssim \int_0^t \|v(s)\|_{\dot{B}^{\frac{d}{2}}} \|r(s)\|_{\dot{B}^{\frac{d}{2}+1}}ds + (1 + \|H_3\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})})\|u\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})} + \|S_2\|_{L_t^1(\dot{B}^{\frac{d}{2}})}, \tag{3.7}$$

and

$$
\begin{aligned}
\|(\partial_t w, \partial_t u)\|_{L_t^1(\dot{B}^{\frac{d}{2}})} \lesssim &\int_0^t \|v(s)\|_{\dot{B}^{\frac{d}{2}}} \|(w, u)(s)\|_{\dot{B}^{\frac{d}{2}+1}}ds + \frac{1}{\tau}\|u + \tau(h_5 + H_5)\nabla r\|_{L_t^1(\dot{B}^{\frac{d}{2}})} \\
&+ \left(1 + \sum_{i=1}^{5} \|H_i\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})}\right)\left(\frac{1}{\varepsilon}\|w\|_{L_t^1(\dot{B}^{\frac{d}{2}})} + \|u\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})}\right) + \|(S_1, S_3)\|_{L_t^1(\dot{B}^{\frac{d}{2}})}.
\end{aligned}
\tag{3.8}
$$

The conclusion of the proof will follow from Lemmas 3.1-3.3 given and proven in the next three subsections. Indeed, combining (3.5)-(3.8) and the uniform estimates of (w, r, u) from Lemmas 3.1-3.3 together and taking the constant $\eta > 0$ suitable small in Lemma 3.3, we obtain

$$\mathcal{X}(t) \lesssim \|(S_1, S_2, S_3)\|_{L_t^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} + \left(\sqrt{\mathcal{Z}(t)} + \mathcal{Z}(t)\right)\mathcal{X}(t) + \int_0^t \mathcal{V}(s)\mathcal{X}(s)ds$$
$$+ \|(y_0, w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}}.$$

Then making use of the Grönwall inequality and the smallness assumption (3.2) of $\mathcal{Z}(t)$, we obtain the uniform a-priori estimate (3.3). $\qquad\square$

3.1 Low-frequency analysis

Motivated by Darcy's law $(1.3)_2$, we introduce the following effective flux

$$z := u + \tau(h_4 + H_4)\nabla r, \tag{3.9}$$

which undergoes a purely damped effect in the low-frequency region $|\xi| \leq \frac{2^k}{\tau}$ and allows us to diagonalize the subsystem $(3.1)_2$-$(3.1)_4$ up to some higher-order terms that can be absorbed. Indeed, substituting (3.9) into (3.1), we obtain

$$\begin{cases} \partial_t w + \dfrac{h_2}{\varepsilon}w = L_1 + R_1 + S_1, \\ \partial_t r - h_3 h_4 \tau \Delta r = L_2 + R_2 + S_2, \\ \partial_t z + \dfrac{z}{\tau} = L_3 + R_3 + S_3, \\ (w, r, z)(0, x) = (w_0, r_0, z_0)(x), \end{cases} \tag{3.10}$$

where the higher-order linear terms L_i $(i = 1, 2, 3)$ are denoted as

$$\begin{cases} L_1 := h_1\big(h_4\tau\Delta r - \operatorname{div} z\big), \\ L_2 := -h_3 \operatorname{div} z, \\ L_3 := h_3 h_4 \tau \nabla\big(h_4\tau\Delta r - \operatorname{div} z\big) - h_5\nabla w, \end{cases}$$

and the nonlinear terms R_i $(i = 1, 2, 3)$ are defined by

$$\begin{cases} R_1 := -v \cdot \nabla w - H_1 \operatorname{div} u + h_1\tau\operatorname{div}(H_4\nabla r) - \dfrac{1}{\varepsilon}H_2 w, \\ R_2 := -v \cdot \nabla r - H_3 \operatorname{div} u + h_3\tau\operatorname{div}(H_4\nabla r), \\ R_3 := -v \cdot \nabla u - H_5\nabla w - \tau\partial_t(H_4\nabla r) + h_4\tau\nabla R_2. \end{cases} \tag{3.11}$$

Now, to establish the $\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}}$-estimates in low frequencies to the solutions of System (3.1) uniformly with respect to both ε and τ, we understand the equations in (3.10) are decoupled. More precisely, we will treat the equations of w and z as damped equations and r as a heat equation, respectively. This viewpoint plays a key role in the proof of the following lemma.

Lemma 3.1. *Let $T > 0$, and the threshold J_τ be given by* (2.1). *Then for $t \in (0, T)$, the solution (w, r, u) to the linear problem* $(3.1)_2$-$(3.1)_4$ *satisfies*

$$
\begin{aligned}
&\|(w, r, u)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})}^\ell + \frac{1}{\varepsilon}\|w\|_{L_t^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})}^\ell + \frac{1}{\sqrt{\varepsilon}}\|w\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})}^\ell \\
&\quad + \tau\|r\|_{L_t^1(\dot{B}^{\frac{d}{2}+1} \cap \dot{B}^{\frac{d}{2}+2})}^\ell + \sqrt{\tau}\|r\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})}^\ell \\
&\quad + \|u\|_{L_t^1(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})}^\ell + \frac{1}{\sqrt{\tau}}\|u\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})}^\ell + \frac{1}{\tau}\|z\|_{L_t^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})}^\ell \\
&\lesssim \|(w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}}}^\ell + \|(S_1, S_2, S_3)\|_{L_t^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})}^\ell + \mathcal{Z}(t)\mathcal{X}(t) + \int_0^t \mathcal{V}(s)\mathcal{X}(s)\,ds,
\end{aligned}
\tag{3.12}
$$

where $\mathcal{Z}(t)$, $\mathcal{X}(t)$, $\mathcal{V}(t)$ and z are defined by (3.2), (3.3), (3.4) *and* (3.9), *respectively.*

Remark 3.1. *In* [15], *the authors obtained the low-frequency estimates by constructing a related Lyapunov functional. However, that method does not lead to the desired estimates which uniform with respect to τ. Moreover, it should be noted that the effective unknown z given by* (3.9) *enables us to capture the heat-like behavior of the unknown r in low frequencies directly, which is consistent with the parabolic nature of the limiting porous media equations.*

3.1.1 The $\dot{B}^{\frac{d}{2}}$-estimates

We first perform $\dot{B}^{\frac{d}{2}}$-estimates in low frequencies for the heat equation $(3.10)_2$. It follows from the regularity estimate in Lemma 2.6 that

$$
\begin{aligned}
\|r\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})}^\ell + \tau\|r\|_{L_t^1(\dot{B}^{\frac{d}{2}+2})}^\ell &\lesssim \|r_0\|_{\dot{B}^{\frac{d}{2}}}^\ell + \|L_2\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell + \|R_2\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell + \|S_2\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell \\
&\lesssim \|r_0\|_{\dot{B}^{\frac{d}{2}}}^\ell + 2^{J_\tau}\|z\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell + \|R_2\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell + \|S_2\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell.
\end{aligned}
\tag{3.13}
$$

Applying Lemma 2.7 to the damped equation $(3.10)_1$, we get

$$
\begin{aligned}
&\|w\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})}^\ell + \frac{1}{\varepsilon}\|w\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell \\
&\lesssim \|w_0\|_{\dot{B}^{\frac{d}{2}}}^\ell + \|L_1\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell + \|R_1\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell + \|S_1\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell \\
&\lesssim \|w_0\|_{\dot{B}^{\frac{d}{2}}}^\ell + \tau\|r\|_{L_t^1(\dot{B}^{\frac{d}{2}+2})}^\ell + 2^{J_\tau}\|z\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell + \|R_1\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell + \|S_1\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell \\
&\lesssim \|(w_0, r_0)\|_{\dot{B}^{\frac{d}{2}}}^\ell + 2^{J_\tau}\|z\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell + \|(R_1, R_2)\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell + \|(S_1, S_2)\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell,
\end{aligned}
\tag{3.14}
$$

where we used inequality (3.13) to control terms involving r in equation $(3.10)_1$.

Similarly, by virtue of inequality (3.13) and Lemmas 2.6-2.7, we have for equation $(3.10)_3$ that

$$
\begin{aligned}
&\|z\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})}^\ell + \frac{1}{\tau}\|z\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell \\
&\lesssim \|z_0\|_{\dot{B}^{\frac{d}{2}}}^\ell + \|L_3\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell + \|(R_3, S_3)\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell \\
&\lesssim \|z_0\|_{\dot{B}^{\frac{d}{2}}}^\ell + 2^{J_\tau}\|w\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell + \tau^2 2^{J_\tau}\|r\|_{L_t^1(\dot{B}^{\frac{d}{2}+2})}^\ell + \tau 2^{2J_\tau}\|z\|_{L_t^1(\dot{B}^{\frac{d}{2}+2})}^\ell + \|(R_3, S_3)\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell.
\end{aligned}
\tag{3.15}
$$

Since the threshold J_τ satisfies condition (2.1), thus $\tau 2^{J_\tau} \sim 2^k << 1$ for suitable negative integer k. Due to the condiiton $\varepsilon \le \tau$ so that $2^{J_\tau}\|w\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell \le \frac{2^{k+1}}{\varepsilon}\|w\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell$, we have by the inequalities (3.13)-(3.15)

that

$$\|(w,r,z)\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})} + \tau\|r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+2})} + \frac{1}{\varepsilon}\|w\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})} + \frac{1}{\tau}\|z\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}$$
$$\lesssim \|(w_0,r_0,z_0)\|^{\ell}_{\dot{B}^{\frac{d}{2}}} + \|(R_1,R_2,R_3)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})} + \|(S_1,S_2,S_3)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}. \tag{3.16}$$

The terms on the right-hand side of (3.16) can be estimated as follows. First, one derives from inequality (2.2) and product law (2.4) and the composition estimate (2.6) that

$$\|z_0\|^{\ell}_{\dot{B}^{\frac{d}{2}}} \lesssim \|(r_0,u_0)\|^{\ell}_{\dot{B}^{\frac{d}{2}}} + \|H_4(0)\|_{\dot{B}^{\frac{d}{2}}}\|r_0\|_{\dot{B}^{\frac{d}{2}}} \lesssim \|(r_0,u_0)\|_{\dot{B}^{\frac{d}{2}}}. \tag{3.17}$$

By the product law (2.4) again, we also get

$$\begin{cases} \|v\cdot\nabla w\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})} \lesssim \int_0^t \|v(s)\|_{\dot{B}^{\frac{d}{2}}}\|w(s)\|_{\dot{B}^{\frac{d}{2}+1}}ds, \\[2mm] \|H_1\mathrm{div}\,u\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})} \lesssim \|H_1\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\|u\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})}, \\[2mm] \frac{1}{\varepsilon}\|H_2w\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})} \lesssim \|H_2\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\frac{1}{\varepsilon}\|w\|_{L^1_t(\dot{B}^{\frac{d}{2}})}. \end{cases} \tag{3.18}$$

According to (2.2) and (2.3)-(2.4), the tricky nonlinear term $H_4\nabla r$ in (3.11) can be estimated as

$$\tau\|\mathrm{div}\,(H_4\nabla r)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}$$
$$\lesssim \tau\|H_4\nabla r^{\ell}\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \tau\|H_4\nabla r^h\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1})}$$
$$\lesssim \tau\|H_4\nabla r^{\ell}\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \|H_4\nabla r^h\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})} \tag{3.19}$$
$$\lesssim \|H_4\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\tau\|r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+2})} + \|H_4\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}+1})}\tau\|r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \|H_4\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\|r\|^{h}_{L^1_t(\dot{B}^{\frac{d}{2}+1})}.$$

Remark 3.2. *The above estimate* (3.19) *for $H_4\nabla r$ arising from two pressures implies that one needs uniform $\dot{B}^{\frac{d}{2}-1}$-estimates for low frequencies. Indeed, as H_4 does not have the either L^1-in-time or L^2-in-time integrability property, the product law* (2.3) *in $\dot{B}^{\frac{d}{2}+1}$ indicates us to discover the control of $\tau\|r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1})}$, which can not be obtained from the $\dot{B}^{\frac{d}{2}}$-estimates in this section.*

Remark 3.3. *It is also one of the reasons why we need to perform the $\dot{B}^{\frac{d}{2}+1}$-estimates in the both low and high frequencies in the later Section 3.3. Indeed, in the low-frequency setting, the uniform $\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})$-norm is not enough to produce the uniform $\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}+1})$-estimates required in* (3.19) *due to the inclusion* (2.2).

Now, one derives from inequalities (3.17)-(3.19) that

$$\|R_1\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})} \lesssim \|v\cdot\nabla w\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})} + \|H_1\mathrm{div}\,u\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})}$$
$$+ \tau\|H_4\nabla r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \frac{1}{\varepsilon}\|H_2w\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})} \tag{3.20}$$
$$\lesssim \mathcal{Z}(t)\mathcal{X}(t) + \int_0^t \mathcal{V}(s)\mathcal{X}(s)ds.$$

Similarly, we have

$$\|R_2\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})} \lesssim \|v\cdot\nabla r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})} + \|H_3\mathrm{div}\,u\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})} + \tau\|H_4\nabla r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1})}$$
$$\lesssim \mathcal{Z}(t)\mathcal{X}(t) + \int_0^t \mathcal{V}(s)\mathcal{X}(s)ds. \tag{3.21}$$

To estimate R_3, we notice that (2.2) together with (2.4) implies

$$\tau\|\partial_t(H_4\nabla r)\|^\ell_{L^1_t(\dot B^{\frac d2})} \lesssim \|\partial_t(H_4\nabla r)\|^\ell_{L^1_t(\dot B^{\frac d2-1})}$$
$$\lesssim \int_0^t \|\partial_t H_4(s)\|_{\dot B^{\frac d2}}\|r(s)\|_{\dot B^{\frac d2}}ds + \|H_4\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}\|\partial_t r\|_{L^1_t(\dot B^{\frac d2})}, \tag{3.22}$$

and

$$\|H_5\nabla w\|^\ell_{L^1_t(\dot B^{\frac d2})} \lesssim \frac1\tau\|H_5\nabla w\|^\ell_{L^1_t(\dot B^{\frac d2-1})} \lesssim \|H_5\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}\frac1\varepsilon\|w\|_{L^1_t(\dot B^{\frac d2})},$$

where we used the assumption $\varepsilon \le \tau$. Thus, it holds that

$$\|R_3\|^\ell_{L^1_t(\dot B^{\frac d2})} \lesssim \int_0^t \|v(s)\|_{\dot B^{\frac d2}}\|u(s)\|_{\dot B^{\frac d2+1}}ds + \tau\|H_4\nabla r\|^\ell_{L^1_t(\dot B^{\frac d2+1})} + \|H_5\nabla w\|^\ell_{L^1_t(\dot B^{\frac d2})}$$
$$+ \tau\|\partial_t(H_4\nabla r)\|^\ell_{L^1_t(\dot B^{\frac d2-1})} + \|R_2\|^\ell_{L^1_t(\dot B^{\frac d2-1})} \tag{3.23}$$
$$\lesssim \mathcal Z(t)\mathcal X(t) + \int_0^t \mathcal V(s)\mathcal X(s)ds.$$

We substitute inequalities (3.17), (3.20)-(3.21) and (3.23) into inequality (3.16) and use standard interpolation to get

$$\|(w,r,z)\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac d2})} + \tau\|r\|^\ell_{L^1_t(\dot B^{\frac d2+2})} + \sqrt\tau\|r\|^\ell_{\widetilde L^2_t(\dot B^{\frac d2+1})}$$
$$+ \frac1\varepsilon\|w\|^\ell_{L^1_t(\dot B^{\frac d2})} + \frac1{\sqrt\varepsilon}\|w\|^\ell_{\widetilde L^2_t(\dot B^{\frac d2})} + \frac1\tau\|z\|^\ell_{L^1_t(\dot B^{\frac d2})} \tag{3.24}$$
$$\lesssim \|(w_0,r_0,u_0)\|_{\dot B^{\frac d2}} + \|(S_1,S_2,S_3)\|^\ell_{L^1_t(\dot B^{\frac d2})} + \mathcal Z(t)\mathcal X(t) + \int_0^t \mathcal V(s)\mathcal X(s)ds.$$

Thence, we rewrite the form (3.9) and use inequalities (2.2) and (3.19) to obtain the $L^1_t(\dot B^{\frac d2})$-estimate of u as follows:

$$\|u\|^\ell_{L^1_t(\dot B^{\frac d2+1})} \lesssim \|z\|^\ell_{L^1_t(\dot B^{\frac d2+1})} + \tau\|\nabla r\|^\ell_{L^1_t(\dot B^{\frac d2+1})} + \tau\|H_4\nabla r\|^\ell_{L^1_t(\dot B^{\frac d2+1})}$$
$$\lesssim \frac1\tau\|z\|^\ell_{L^1_t(\dot B^{\frac d2})} + \tau\|r\|^\ell_{L^1_t(\dot B^{\frac d2+2})} + \|H_4\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}\tau\|r\|^\ell_{L^1_t(\dot B^{\frac d2+2})}$$
$$+ \|H_4\|_{\widetilde L^\infty_t(\dot B^{\frac d2+1})}\tau\|r\|^\ell_{L^1_t(\dot B^{\frac d2+1})} + \|H_4\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}\|r\|^h_{L^1_t(\dot B^{\frac d2+1})}.$$

Similarly, we have

$$\|u\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac d2})} \lesssim \|(z,r)\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac d2})} + \|H_4\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}\|r\|_{\widetilde L^\infty_t(\dot B^{\frac d2})},$$

and

$$\frac1{\sqrt\tau}\|u\|^\ell_{\widetilde L^2_t(\dot B^{\frac d2})} \lesssim \frac1{\sqrt\tau}\|z\|^\ell_{\widetilde L^2_t(\dot B^{\frac d2})} + \sqrt\tau\|r\|^\ell_{\widetilde L^2_t(\dot B^{\frac d2+1})} + \|H_4\|_{\widetilde L^\infty_t(\dot B^{\frac d2})}\sqrt\tau\|r\|_{\widetilde L^2_t(\dot B^{\frac d2+1})}.$$

We thus obtain from inequality (3.24) that

$$\|u\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac d2})} + \frac1{\sqrt\tau}\|u\|^\ell_{\widetilde L^2_t(\dot B^{\frac d2})} + \|u\|^\ell_{L^1_t(\dot B^{\frac d2+1})}$$
$$\lesssim \|(w_0,r_0,u_0)\|_{\dot B^{\frac d2}} + \|(S_1,S_2,S_3)\|^\ell_{L^1_t(\dot B^{\frac d2})} + \mathcal Z(t)\mathcal X(t) + \int_0^t \mathcal V(s)\mathcal X(s)ds. \tag{3.25}$$

3.1.2 The $\dot{B}^{\frac{d}{2}-1}$-estimates

We perform the $\dot{B}^{\frac{d}{2}-1}$-estimates so as to control $\tau\|r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1})}$, as explained in Remark 3.2. Arguing similarly as for inequalities (3.13)-(3.16), we have

$$
\begin{aligned}
&\|(w,r,z)\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-1})} + \tau\|r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \frac{1}{\varepsilon}\|w\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1})} + \frac{1}{\tau}\|z\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1})} \\
&\lesssim \|(w_0,r_0,u_0)\|_{\dot{B}^{\frac{d}{2}-1}} + \|(R_1,R_2,R_3)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1})} + \|(S_1,S_2,S_3)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1})}.
\end{aligned} \tag{3.26}
$$

Direct calculations give

$$
\begin{aligned}
&\|(R_1,R_2)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1})} \\
&\lesssim \int_0^t \|v(s)\|_{\dot{B}^{\frac{d}{2}}}\|(w,r)(s)\|_{\dot{B}^{\frac{d}{2}}}ds + \|(H_1,H_3)\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\|u\|_{L^1_t(\dot{B}^{\frac{d}{2}})} \\
&\quad + \|H_4\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\tau\|r\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \|H_2\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\frac{1}{\varepsilon}\|w\|_{L^1_t(\dot{B}^{\frac{d}{2}-1})} \\
&\lesssim \mathcal{Z}(t)\mathcal{X}(t) + \int_0^t \mathcal{V}(s)\mathcal{X}(s)ds.
\end{aligned} \tag{3.27}
$$

By inequalities (3.22), (3.27) and product law (2.4) for $d \geq 2$, the term R_3 can be bounded by

$$
\begin{aligned}
\|R_3\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1})} &\lesssim \int_0^t \|v(s)\|_{\dot{B}^{\frac{d}{2}}}\|u(s)\|_{\dot{B}^{\frac{d}{2}}}ds + \|H_5\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\frac{1}{\varepsilon}\|w\|_{L^1_t(\dot{B}^{\frac{d}{2}})} \\
&\quad + \tau\|\partial_t(H_4\nabla r)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1})} + \|R_2\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1})} \\
&\lesssim \mathcal{Z}(t)\mathcal{X}(t) + \int_0^t \mathcal{V}(s)\mathcal{X}(s)ds.
\end{aligned} \tag{3.28}
$$

Inserting (3.27) and (3.28) into (3.26) and taking advantage of interpolation, we obtain

$$
\begin{aligned}
&\|(w,r,z)\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-1})} + \tau\|r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \sqrt{\tau}\|r\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})} \\
&\quad + \frac{1}{\varepsilon}\|w\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1})} + \frac{1}{\sqrt{\varepsilon}}\|w\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})} + \frac{1}{\tau}\|z\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1})} \\
&\lesssim \|(w_0,r_0,u_0)\|_{\dot{B}^{\frac{d}{2}-1}} + \|(S_1,S_2,S_3)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1})} \\
&\quad + \mathcal{Z}(t)\mathcal{X}(t) + \int_0^t \mathcal{V}(s)\mathcal{X}(s)ds.
\end{aligned} \tag{3.29}
$$

This together with inequality (2.2) and the fact that $u = z - \tau(h_4 + H_4)\nabla r$ leads to

$$
\|u\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-1})} \lesssim \|(z,r)\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-1})} + \|H_4\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\|r\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-1})}.
$$

Similarly, one gets

$$
\frac{1}{\sqrt{\tau}}\|u\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})} \lesssim \frac{1}{\sqrt{\tau}}\|z\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})} + \sqrt{\tau}\|r\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})} + \|H_4\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\sqrt{\tau}\|r\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})},
$$

and

$$
\|u\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})} \lesssim \frac{1}{\tau}\|z\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1})} + \tau\|r\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \|H_4\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}})}\tau\|r\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})}.
$$

Combining the above three estimates, we are led to

$$\|u\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-1})} + \frac{1}{\sqrt{\tau}}\|u\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})} + \|u\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}})} \lesssim \|(w_0,r_0,u_0)\|_{\dot{B}^{\frac{d}{2}-1}} + \|(S_1,S_2,S_3)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1})}$$
$$+ \mathcal{Z}(t)\mathcal{X}(t) + \int_0^t \mathcal{V}(s)\mathcal{X}(s)ds. \tag{3.30}$$

Putting the estimates (3.12), (3.24), (3.25) and (3.29), (3.30) together, we complete the proof of Lemma 3.1.

3.2   High-frequency analysis

In this section, we establish some uniform high-frequency estimates of solutions to the linear problem (3.1) in terms of the Lyapunov functional. More precisely, we establish the $\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})$-estimates, and furthermore obtain the control of higher-order $L^1_t(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})$-norms.

Lemma 3.2. *Let $T > 0$, and the threshold J_{τ} be given by (2.1). Then for any $t \in (0,T)$, the solution (w,r,u) to the linear problem $(3.1)_2$-$(3.1)_4$ satisfies*

$$\|(w,r,u)\|^h_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})} + \|(w,r,u)\|^h_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \frac{1}{\varepsilon}\|w\|^h_{L^1_t(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})} + \frac{1}{\sqrt{\varepsilon}}\|w\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})}$$
$$+ \sqrt{\tau}\|r\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})} + \frac{1}{\tau}\|z\|^h_{L^1_t(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})} \tag{3.31}$$
$$\lesssim \|(w_0,r_0,u_0)\|^h_{\dot{B}^{\frac{d}{2}+1}} + \|(S_1,S_2,S_3)\|^h_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \mathcal{Z}(t)\mathcal{X}(t) + \int_0^t \mathcal{V}(s)\mathcal{X}(s)ds.$$

Proof. To prove of Lemma 3.2, we localize in frequencies for the equations $(3.1)_2$-$(3.1)_4$ as

$$\begin{cases} \partial_t w_j + v \cdot \nabla w_j + (h_1 + H_1)\mathrm{div}\, u_j + (h_2 + H_2)\dfrac{w_j}{\varepsilon} = \dot{\Delta}_j S_1 + T_j^1, \\[2mm] \partial_t r_j + v \cdot \nabla r_j + (h_3 + H_3)\mathrm{div}\, u_j = \dot{\Delta}_j S_2 + T_j^2, \\[2mm] \partial_t u_j + v \cdot \nabla u_j + \dfrac{u_j}{\tau} + (h_4 + H_4)\nabla r_j + (h_5 + H_5)\nabla w_j = \dot{\Delta}_j S_3 + T_j^3, \end{cases} \tag{3.32}$$

with the commutator terms

$$\begin{cases} T_j^1 := [v, \dot{\Delta}_j]\nabla w + [H_1, \dot{\Delta}_j]\mathrm{div}\, u + \dfrac{1}{\varepsilon}[H_2, \dot{\Delta}_j]w, \\[2mm] T_j^2 := [v, \dot{\Delta}_j]\nabla r + [H_3, \dot{\Delta}_j]\mathrm{div}\, u, \\[2mm] T_j^3 := [v, \dot{\Delta}_j]\nabla u + [H_4, \dot{\Delta}_j]\nabla r + [H_5, \dot{\Delta}_j]\nabla w. \end{cases} \tag{3.33}$$

Multiplying $(3.32)_3$ by u_j and integrating the resulting equation by parts, we get

$$\frac{d}{dt}\int_{\mathbb{R}^d} \frac{1}{2}|u_j|^2 dx + \int_{\mathbb{R}^d} \frac{1}{\tau}|u_j|^2 dx$$
$$- \int_{\mathbb{R}^d}\big((h_4+H_4)r_j\mathrm{div}\, u_j + (h_5+H_5)w_j\mathrm{div}\, u_j\big)dx \tag{3.34}$$
$$\lesssim \|\mathrm{div}\, v\|_{L^\infty}\|u_j\|^2_{L^2} + (\|\dot{\Delta}_j S_3\|_{L^2} + \|T_j^3\|_{L^2})\|u_j\|_{L^2} + \|\nabla H_4\|_{L^\infty}\|(w_j,r_j)\|_{L^2}\|u_j\|_{L^2}.$$

Thence, we multiply $(3.32)_1$ by $\frac{h_5 + H_5}{h_1 + H_1} w_j$ and integrate the resulting equation by parts to show

$$
\begin{aligned}
\frac{d}{dt} &\int_{\mathbb{R}^d} \frac{1}{2} \frac{h_5 + H_5}{h_1 + H_1} |w_j|^2 dx \\
&+ \int_{\mathbb{R}^d} \left((h_5 + H_5) w_j \operatorname{div} u_j + \frac{(h_2 + H_2)(h_5 + H_5)}{\varepsilon(h_1 + H_1)} |w_j|^2 \right) dx \\
&\lesssim \left(\left\| \partial_t \left(\frac{h_5 + H_5}{h_1 + H_1} \right) \right\|_{L^\infty} + \left\| \frac{h_5 + H_5}{h_1 + H_1} \right\|_{L^\infty} \|\operatorname{div} v\|_{L^\infty} + \left\| \nabla \left(\frac{h_5 + H_5}{h_1 + H_1} \right) \right\|_{L^\infty} \|v\|_{L^\infty} \right) \|w_j\|_{L^2}^2 \\
&+ \left\| \frac{h_5 + H_5}{h_1 + H_1} \right\|_{L^\infty} \left(\|\dot{\Delta}_j S_1\|_{L^2} + \|T_j^1\|_{L^2} \right) \|w_j\|_{L^2}.
\end{aligned}
\tag{3.35}
$$

Similarly, direct computations on $(3.32)_2$ yield

$$
\begin{aligned}
\frac{d}{dt} &\int_{\mathbb{R}^d} \frac{1}{2} \frac{h_4 + H_4}{h_3 + H_3} |r_j|^2 dx + \int_{\mathbb{R}^d} (h_4 + H_4) r_j \operatorname{div} u_j dx \\
&\leq \left(\left\| \partial_t \left(\frac{h_4 + H_4}{h_3 + H_3} \right) \right\|_{L^\infty} + \left\| \frac{h_4 + H_4}{h_3 + H_3} \right\|_{L^\infty} \|\operatorname{div} v\|_{L^\infty} + \left\| \nabla \left(\frac{h_4 + H_4}{h_3 + H_3} \right) \right\|_{L^\infty} \|v\|_{L^\infty} \right) \|r_j\|^2 \\
&+ \left\| \frac{h_4 + H_4}{h_3 + H_3} \right\|_{L^\infty} \left(\|\dot{\Delta}_j S_2\|_{L^2} + \|T_j^2\|_{L^2} \right) \|r_j\|_{L^2}.
\end{aligned}
\tag{3.36}
$$

To derive the cross estimate and capture the dissipative property of r_j, we gain by taking the L^2-inner product of $(3.32)_3$ with ∇r_j that

$$
\begin{aligned}
\int_{\mathbb{R}^d} \partial_t u_j \cdot \nabla r_j dx &+ \int_{\mathbb{R}^d} (h_4 + H_4) |\nabla r_j|^2 dx \\
&+ \int_{\mathbb{R}^d} \left((h_5 + H_5) \nabla w_j \cdot \nabla r_j + \frac{1}{\tau} u_j \cdot \nabla r_j \right) dx \\
&\lesssim \left(\|v\|_{L^\infty} \|\nabla u_j\|_{L^2} + \|\dot{\Delta}_j S_3\|_{L^2} + \|T_j^3\|_{L^2} \right) \|\nabla r_j\|_{L^2},
\end{aligned}
\tag{3.37}
$$

and taking the L^2-inner product of $(3.32)_2$ with $\operatorname{div} u_j$ that

$$
\begin{aligned}
\int_{\mathbb{R}^d} u_j \cdot \nabla \partial_t r_j dx &- \int_{\mathbb{R}^d} (h_3 + H_3) |\operatorname{div} u_j|^2 dx \\
&\lesssim \left(\|v\|_{L^\infty} \|\nabla r_j\|_{L^2} + \|\dot{\Delta}_j S_2\|_{L^2} + \|T_j^2\|_{L^2} \right) \|\operatorname{div} u_j\|_{L^2}.
\end{aligned}
\tag{3.38}
$$

In the spirit of the work [3] by Beauchard and Zuazua, for a small constant $\eta_* > 0$ to be determined, we define the following Lyapunov functional with nonlinear weights as

$$
\mathcal{L}_j(t) := \int_{\mathbb{R}^d} \frac{1}{2} \left(\frac{h_5 + H_5}{h_1 + H_1} |w_j|^2 + \frac{h_4 + H_4}{h_3 + H_3} |r_j|^2 + |u_j|^2 \right) dx + \frac{\eta_*}{\tau} 2^{-2j} \int_{\mathbb{R}^d} u_j \cdot \nabla r_j dx,
$$

and its dissipation rate

$$
\begin{aligned}
\mathcal{H}_j(t) := &\int_{\mathbb{R}^d} \left(\frac{1}{\tau} |u_j|^2 + \frac{(h_2 + H_2)(h_5 + H_5)}{\varepsilon(h_1 + H_1)} |w_j|^2 \right) dx \\
&+ \frac{\eta_*}{\tau} 2^{-2j} \int_{\mathbb{R}^d} \left((h_4 + H_4) |\nabla r_j|^2 + (h_5 + H_5) \nabla w_j \cdot \nabla r_j + \frac{1}{\tau} u_j \cdot \nabla r_j \right) dx.
\end{aligned}
$$

One derives from assumption (3.2) and the embedding $\dot{B}^{\frac{d}{2}} \hookrightarrow L^\infty$ that

$$
\|H_i\|_{L_t^\infty(L^\infty)} + \|\nabla H_i\|_{L_t^\infty(L^\infty)} \lesssim \|H_i\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} \lesssim c << 1,
\tag{3.39}
$$

23

which together with estimates (3.34)-(3.38) and the fact that $2^{-j} \lesssim \tau \leq 1$ for any $j \geq J_\tau - 1$ yields the following Lyapunov inequality:

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}_j(t) + \mathcal{H}_j(t) &\lesssim (\|\operatorname{div} v\|_{L^\infty} + \|v\|_{L^\infty} + \sum_{i=1}^{5}\|\partial_t H_i\|_{L^\infty})\|(r_j, w_j, u_j)\|_{L^2}^2 \\
&+ \left(\sum_{i=1}^{5}\|\partial_t H_i\|_{L^\infty} + \|\dot{\Delta}_j(S_1, S_2, S_3)\|_{L^2} + \|(T_j^1, T_j^2, T_j^3)\|_{L^2}\right)\|(r_j, w_j, u_j)\|_{L^2}.
\end{aligned}
\tag{3.40}
$$

It follows from the smallness condition (3.39), Bernstein's inequality in Lemma 2.1 and the fact $2^{-j} \lesssim \tau$ that

$$
(1 - \eta_*)\|(w_j, r_j, u_j)\|_{L^2}^2 \lesssim \mathcal{L}_j(t) \lesssim (1 + \eta_*)\|(w_j, r_j, u_j)\|_{L^2}^2,
$$

and

$$
\begin{aligned}
\mathcal{H}_j(t) &\gtrsim \frac{1}{\tau}\|u_j\|_{L^2}^2 + \frac{1}{\varepsilon}\|w_j\|_{L^2}^2 + \frac{\eta_*}{\tau}2^{-2j}\left(\|\nabla r_j\|_{L^2} - \|\nabla w_j\|_{L^2}^2 - \frac{1}{\tau^2}\|u_j\|_{L^2}\right) \\
&\gtrsim \frac{1}{\tau}(1 - \eta_*)\|u_j\|_{L^2}^2 + \frac{1}{\varepsilon}(1 - \eta_*)\|w_j\|_{L^2}^2 + \frac{\eta_*}{\tau}\|r_j\|_{L^2},
\end{aligned}
$$

where one has used the condition $\varepsilon \leq \tau$. Thus, we can choose a sufficiently small constant $\eta_* > 0$ independent of ε and τ so that

$$
\mathcal{L}_j(t) \sim \|(w_j, r_j, u_j)\|_{L^2}^2, \qquad \mathcal{H}_j(t) \gtrsim \frac{1}{\tau}\|(w_j, r_j, u_j)\|_{L^2}^2 \gtrsim \frac{1}{\tau}\mathcal{L}_j(t).
\tag{3.41}
$$

Dividing the two sides of (3.40) by $\sqrt{\mathcal{L}_j(t) + \eta}$ for any $\eta > 0$, we have

$$
\begin{aligned}
\frac{d}{dt}\sqrt{\mathcal{L}_j(t) + \eta} &+ \frac{1}{\tau}\sqrt{\mathcal{L}_j(t) + \eta} - \frac{\eta}{\tau\sqrt{\mathcal{L}_j(t) + \eta}} \\
&\lesssim \left(\|\operatorname{div} v\|_{L^\infty} + \|v\|_{L^\infty} + \sum_{i=1}^{5}\|\partial_t H_i\|_{L^\infty}\right)\|(r_j, w_j, u_j)\|_{L^2} \\
&+ \|\dot{\Delta}_j(S_1, S_2, S_3)\|_{L^2} + \|(T_j^1, T_j^2, T_j^3)\|_{L^2}.
\end{aligned}
$$

From (3.41) and the embedding $\dot{B}^{\frac{d}{2}} \hookrightarrow L^\infty$ we get after integrating the above inequality over $[0, t]$ and taking the limit as $\eta \to 0$ that

$$
\begin{aligned}
\tau\|(w, r, u)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})}^h &+ \|(w, r, u)\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})}^h \lesssim \tau\|(w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}+1}}^h \\
&+ \int_0^t \left(\|v(s)\|_{\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}} + \sum_{i=1}^{5}\|\partial_t H_i(s)\|_{\dot{B}^{\frac{d}{2}}}\right)\tau\|(w, r, u)(s)\|_{\dot{B}^{\frac{d}{2}+1}}^h ds \\
&+ \tau\sum_{j \geq J_\tau - 1} 2^{j(\frac{d}{2}+1)}\|(T_j^1, T_j^2, T_j^3)\|_{L_t^1(L^2)} + \tau\|(S_1, S_2, S_3)\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})}^h.
\end{aligned}
\tag{3.42}
$$

According to the commutator estimate (2.5), it follows that

$$
\begin{aligned}
\tau\sum_{j \geq J_\tau - 1} 2^{j(\frac{d}{2}+1)}\|(T_j^1, T_j^2, T_j^3)\|_{L_t^1(L^2)} &\lesssim \int_0^t \|v(s)\|_{\dot{B}^{\frac{d}{2}+1}}\|(w, r, u)(s)\|_{\dot{B}^{\frac{d}{2}+1}} ds \\
&+ \sum_{i=1}^{4}\|H_i\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})}\left(\frac{1}{\varepsilon}\|w\|_{L_t^1(\dot{B}^{\frac{d}{2}})} + \tau\|r\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})} + \|u\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})}\right) \\
&\lesssim \mathcal{Z}(t)\mathcal{X}(t).
\end{aligned}
$$

24

This with inequality (3.42) leads to

$$\tau \|(w,r,u)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})} + \|(w,r,u)\|^h_{L^1_t(\dot{B}^{\frac{d}{2}+1})}$$

$$\lesssim \tau \|(w_0,r_0,u_0)\|^h_{\dot{B}^{\frac{d}{2}+1}} + \tau \|(S_1,S_2,S_3)\|^h_{L^1_t(\dot{B}^{\frac{d}{2}+1})} \tag{3.43}$$

$$+ \mathcal{Z}(t)\mathcal{X}(t) + \int_0^t \mathcal{V}(s)\mathcal{X}(s)ds.$$

On the other hand, for any $\eta > 0$, we deduce from inequality (3.35) that

$$\frac{d}{dt}\sqrt{\|w_j\|^2_{L^2} + \eta} + \frac{1}{\varepsilon}\sqrt{\|w_j\|^2_{L^2} + \eta}$$

$$\lesssim 2^j\|u_j\|_{L^2} + \|(\partial_t H_1, \partial_t H_5)\|_{L^\infty}\|w_j\|_{L^2}$$

$$+ \|\mathrm{div}\, v\|_{L^\infty}\|w_j\|_{L^2} + \|v\|_{L^\infty}\|w_j\|_{L^2} + \|\dot{\Delta}_j S_1\|_{L^2} + \|T^1_j\|_{L^2},$$

which together with (3.43) implies

$$\frac{1}{\varepsilon}\|w\|^h_{L^1_t(\dot{B}^{\frac{d}{2}})} \lesssim \|w_0\|^h_{\dot{B}^{\frac{d}{2}}} + \|u\|^h_{L^1_t(\dot{B}^{\frac{d}{2}+1})}$$

$$+ \|S_1\|^h_{L^1_t(\dot{B}^{\frac{d}{2}})} + \tau \sum_{j \geq J_\tau - 1} 2^{j(\frac{d}{2}+1)}\|T^1_j\|_{L^1_t(L^2)}$$

$$+ \int_0^t \left(\|(\partial_t H_1, \partial_t H_5)(s)\|_{\dot{B}^{\frac{d}{2}}} + \|v(s)\|_{\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}}\right)\|w(s)\|_{\dot{B}^{\frac{d}{2}}}ds$$

$$\lesssim \tau\|(w_0,r_0,u_0)\|^h_{\dot{B}^{\frac{d}{2}+1}} + \tau\|(S_1,S_2,S_3)\|^h_{L^1_t(\dot{B}^{\frac{d}{2}+1})}$$

$$+ \mathcal{Z}(t)\mathcal{X}(t) + \int_0^t \mathcal{V}(s)\mathcal{X}(s)ds.$$

Thanks to inequality (2.2), one has

$$\|(w,r,u)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})} \lesssim \tau\|(w,r,u)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}})} \lesssim \tau\|(w,r,u)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})}. \tag{3.44}$$

Finally, the remain estimates in (3.31) can be achieved similarly to (3.44). We omit the details here and complete the proof of Lemma 3.2. $\qquad\square$

3.3   Recovering the $\dot{B}^{\frac{d}{2}+1}$-estimates

As explained in Remark 3.3, we need to establish the uniform $L^\infty_t(\dot{B}^{\frac{d}{2}+1})$-norm estimate of (w,r,u) which in fact leads to the uniform control of $\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})$-norms for $(\frac{1}{\sqrt{\varepsilon}}w, \frac{1}{\sqrt{\tau}}u)$ at both low and high frequencies as a byproduct.

Lemma 3.3. *Let $T > 0$, and the threshold J_τ be given by (2.1). Then for any $t \in (0,T)$, the solution (w,r,u) to the linear problem $(3.1)_2$-$(3.1)_4$ satisfies*

$$\|(w,r,u)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1})} + \frac{1}{\sqrt{\varepsilon}}\|w\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})} + \frac{1}{\sqrt{\tau}}\|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1})}$$

$$\lesssim \|(w_0,r_0,u_0)\|_{\dot{B}^{\frac{d}{2}+1}} + \|(S_1,S_2,S_3)\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})} \tag{3.45}$$

$$+ (\eta + \sqrt{\mathcal{Z}(t)})\mathcal{X}(t) + \frac{1}{\eta}\int_0^t \mathcal{V}(s)\mathcal{X}(s)ds,$$

where $\eta \in (0,1)$ is a constant to be chosen.

Proof. We perform a L^2-in-time type estimates and make use of the decay-in-τ of u for L^2-time type norms. In fact, for any $j \in \mathbb{Z}$, by combining inequalities (3.34)-(3.35) together, we get

$$
\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^d} &\frac{1}{2} \Big(\frac{h_5 + H_5}{h_1 + H_1} |w_j|^2 + \frac{h_4 + H_4}{h_3 + H_3} |r_j|^2 + |u_j|^2 \Big) dx \\
&+ \int_{\mathbb{R}^d} \Big(\frac{1}{\tau} |u_j|^2 + \frac{(h_2 + H_2)(h_5 + H_5)}{\varepsilon(h_1 + H_1)} |w_j|^2 \Big) dx \\
&\lesssim \|\operatorname{div} v\|_{L^\infty} \|(w_j, r_j, u_j)\|_{L^2}^2 + \Big(\sum_{i=1}^5 \|\partial_t H_i\|_{L^\infty} + \|v\|_{L^\infty} \Big) \|(w_j, r_j)\|_{L^2}^2 \\
&+ \|\nabla H_4\|_{L^\infty} \|(w_j, r_j)\|_{L^2} \|u_j\|_{L^2} + \|T_j^1\|_{L^2} \|w_j\|_{L^2} \\
&+ \|T_j^2\|_{L^2} \|r_j\|_{L^2} + \|T_j^3\|_{L^2} \|u_j\|_{L^2} + \|\dot{\Delta}_j(S_1, S_2, S_3)\|_{L^2} \|(w_j, r_j, u_j)\|_{L^2}.
\end{aligned}
\tag{3.46}
$$

Furthermore, from (3.46) we have

$$
\begin{aligned}
\|(w, r, u)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} &+ \frac{1}{\sqrt{\varepsilon}} \|w\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}+1})} + \frac{1}{\sqrt{\tau}} \|u\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}+1})} \\
&\lesssim \|(w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}+1}} + \|v\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})}^{\frac{1}{2}} \|(w, r, u)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} \\
&+ \Big(\int_0^t \Big(\sum_{i=1}^5 \|\partial_t H_i(s)\|_{\dot{B}^{\frac{d}{2}}} + \|v(s)\|_{\dot{B}^{\frac{d}{2}}} \Big) \|(w, r)(s)\|_{\dot{B}^{\frac{d}{2}+1}} ds \Big)^{\frac{1}{2}} \|(w, r)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})}^{\frac{1}{2}} \\
&+ \Big(\|H_4\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} \frac{1}{\sqrt{\tau}} \|u\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}+1})} \sqrt{\tau} \|(w, r)\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}+1})} \Big)^{\frac{1}{2}} \\
&+ \sum_{j \in \mathbb{Z}} 2^{j(\frac{d}{2}+1)} \Big(\int_0^t \big(\|T_j^1\|_{L^2} \|w_j\|_{L^2} + \|T_j^2\|_{L^2} \|r_j\|_{L^2} + \|T_j^3\|_{L^2} \|u_j\|_{L^2} \big) ds \Big)^{\frac{1}{2}} \\
&+ \Big(\|(S_1, S_2, S_3)\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})} \|(w, r, u)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} \Big)^{\frac{1}{2}}.
\end{aligned}
\tag{3.47}
$$

The right-hand side of inequality (3.47) can be estimated as follows. By the commutator estimate (2.5), we have

$$
\begin{aligned}
\sum_{j \in \mathbb{Z}} &2^{j(\frac{d}{2}+1)} \Big(\int_0^t \|T_j^1\|_{L^2} \|w_j\|_{L^2} ds \Big)^{\frac{1}{2}} \\
&\lesssim \Big(\|w\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} \int_0^t \|v(s)\|_{\dot{B}^{\frac{d}{2}+1}} \|w(s)\|_{\dot{B}^{\frac{d}{2}+1}} ds \\
&\quad + \|H_1\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} \|w\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} \|u\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})} + \|H_2\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} \frac{1}{\varepsilon} \|w\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}+1})}^2 \Big)^{\frac{1}{2}} \\
&\lesssim \sqrt{\mathcal{Z}(t)} \mathcal{X}(t) + \Big(\int_0^t \mathcal{V}(s) \mathcal{X}(s) ds \Big)^{\frac{1}{2}} \sqrt{\mathcal{X}(t)}.
\end{aligned}
$$

Similarly, it holds

$$
\begin{aligned}
\sum_{j \in \mathbb{Z}} &2^{j(\frac{d}{2}+1)} \Big(\int_0^t \|T_j^2\|_{L^2} \|r_j\|_{L^2} ds \Big)^{\frac{1}{2}} \\
&\lesssim \Big(\|r\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} \int_0^t \|v(s)\|_{\dot{B}^{\frac{d}{2}+1}} \|r(s)\|_{\dot{B}^{\frac{d}{2}+1}} ds + \|H_3\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} \|r\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} \|u\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})} \Big)^{\frac{1}{2}} \\
&\lesssim \sqrt{\mathcal{Z}(t)} \mathcal{X}(t) + \Big(\int_0^t \mathcal{V}(s) \mathcal{X}(s) ds \Big)^{\frac{1}{2}} \sqrt{\mathcal{X}(t)},
\end{aligned}
$$

26

and

$$\sum_{j \in \mathbb{Z}} 2^{j(\frac{d}{2}+1)} \Big(\int_0^t \|T_j^3\|_{L^2} \|u_j\|_{L^2} ds \Big)^{\frac{1}{2}}$$

$$\lesssim \Big(\|u\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} \int_0^t \|v(s)\|_{\dot{B}^{\frac{d}{2}+1}} \|u(s)\|_{\dot{B}^{\frac{d}{2}+1}} ds + \|(H_4, H_5)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} \|r\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} \|u\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})} \Big)^{\frac{1}{2}}$$

$$\lesssim \sqrt{\mathcal{Z}(t)}\mathcal{X}(t) + \Big(\int_0^t \mathcal{V}(s)\mathcal{X}(s)ds \Big)^{\frac{1}{2}} \sqrt{\mathcal{X}(t)}.$$

We conclude from the above estimates that

$$\|(w, r, u)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} + \frac{1}{\sqrt{\varepsilon}} \|w\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}+1})} + \frac{1}{\sqrt{\tau}} \|u\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}+1})}$$

$$\lesssim \|(w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}+1}} + \|(S_1, S_2, S_3)\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})}$$

$$+ \sqrt{\mathcal{Z}(t)}\mathcal{X}(t) + \Big(\int_0^t \mathcal{V}(s)\mathcal{X}(s)ds \Big)^{\frac{1}{2}} \sqrt{\mathcal{X}(t)}.$$

Applying Hölder's inequality to the above estimate leads to inequality (3.45). $\qquad\square$

4 Global well-posedness for the nonlinear problems

4.1 The Cauchy problem of System (BN)

In this section, we prove the uniform in ε and τ global existence and uniqueness of solutions to the Cauchy problem for (BN) subject to the initial data $(\alpha_{\pm,0}, \rho_{\pm,0}, u_0)$. i.e. Theorem 1.1. For simplicity, we omit the superscript concerning the parameters ε and τ in this section.

Proof of Theorem 1.1: Let $(\alpha_{\pm,0}, \rho_{\pm,0}, u_0)$ satisfy the smallness condition (1.7) and denote

$$\mathcal{X}_0 := \|(\alpha_{\pm,0} - \bar{\alpha}_\pm, \rho_{\pm,0} - \bar{\rho}_\pm, u_0)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}}.$$

Let (y_0, w_0, r_0) be the perturbation of $(\alpha_{\pm,0}, \rho_{\pm,0})$ given by (1.22).

- *Step 1: Construction of approximation sequence*

For any $n \geq 1$, we define the regularized perturbation

$$(y_0^n, w_0^n, r_0^n, u_0^n)(x) = (\dot{S}_n y_0, \dot{S}_n w_0, \dot{S}_n r_0, \dot{S}_n u_0)(x),$$

where $\dot{S}_n$ is the low-frequency cut-off operator (see Section 2). One can verify that $(y_0^n, w_0^n, r_0^n, u_0^n)$ is smooth and converges to (y_0, w_0, r_0, u_0) strongly in $\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}$ as $n \to \infty$. In addition, due to Lemmas 2.3 and 2.5, there exists a constant C_0^* independent of n, ε and τ such that

$$\|(y_0^n, w_0^n, r_0^n, u_0^n)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}} \leq C_0^* \mathcal{X}_0. \tag{4.1}$$

Set $(y^0, w^0, r^0, u^0) := (0,0,0,0)$. For any $n \geq 0$, we consider the approximate scheme for (1.23) as follows:

$$\begin{cases} \partial_t y^{n+1} + u^n \cdot \nabla y^{n+1} = 0, \\[2mm] \partial_t w^{n+1} + u^n \cdot \nabla w^{n+1} + (\bar{F}_1 + G_1^n)\mathrm{div}\, u^{n+1} + (\bar{F}_2 + G_2^n)\dfrac{w^{n+1}}{\varepsilon} = 0, \\[2mm] \partial_t r^{n+1} + u^n \cdot \nabla r^{n+1} + (\bar{F}_3 + G_3^n)\mathrm{div}\, u^{n+1} = F_4^n \dfrac{(w^n)^2}{\varepsilon}, \\[2mm] \partial_t u^{n+1} + u^n \cdot \nabla u^{n+1} + \dfrac{u^{n+1}}{\tau} + (\bar{F}_0 + G_0^n)(\nabla r^{n+1} + (\gamma_+ - \gamma_-)\nabla w^{n+1}) = 0, \\[2mm] (y^{n+1}, w^{n+1}, r^{n+1}, u^{n+1})(0,x) = (y_0^n, w_0^n, r_0^n, u_0^n)(x), \end{cases} \tag{4.2}$$

with $F_i^n = F_i^{\varepsilon,\tau}(y^n, w^n, r^n)$, $\bar{F}i$ and $G_i^n = G_i^{\varepsilon,\tau}(y^n, w^n, r^n)$ defined in (1.24), (1.25) and (1.26), respectively. We define the functional space $\mathbb{E}_t$ associated to the following norm:

$$\begin{aligned} \|(y,w,r,u)\|_{\mathbb{E}_t} :=\; & \|(y,w,r,u)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} + \|(\partial_t y, \partial_t w, \partial_t r, \partial_t u)\|_{L_t^1(\dot{B}^{\frac{d}{2}})} \\ & + \frac{1}{\varepsilon}\|w\|_{L_t^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})} + \frac{1}{\sqrt{\varepsilon}}\|w\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} \\ & + \tau\|r\|_{L_t^1(\dot{B}^{\frac{d}{2}+1} \cap \dot{B}^{\frac{d}{2}+2})}^{\ell} + \|r\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})}^{h} + \tau\|r\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})} + \sqrt{\tau}\|r\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} \\ & + \|u\|_{L_t^1(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} + \frac{1}{\sqrt{\tau}}\|u\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})}. \end{aligned}$$

For any fixed $n \geq 1$, we assume that (y^n, w^n, r^n, u^n) satisfies

$$\|(y^n, w^n, r^n, u^n)\|_{\mathbb{E}_t} + \frac{1}{\tau}\|u^n + \tau(\bar{F}_0 + G_0^{n-1})\nabla r^n\|_{L_t^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})} \leq 2C_0 C_0^* \mathcal{X}_0, \quad t > 0, \tag{4.3}$$

where the constants C_0 and C_0^* are given by (3.3) and (4.1), respectively. Since the initial data is smooth, by virtue of the classical theorems for the transport equation $(4.2)_1$ and the symmetric hyperbolic system $(4.2)_2$-$(4.2)_4$ (cf. [2, 4]), there exists a unique global solution $(y^{n+1}, w^{n+1}, r^{n+1}, u^{n+1}) \in \mathcal{C}(\mathbb{R}_+; H^s)$ with all $s > \frac{d}{2} + 1$.

- *Step 2: Uniform estimate*

Our goal is to show that $(y^{n+1}, w^{n+1}, r^{n+1}, u^{n+1})$ also satisfies the estimate (4.3) uniformly in n, ε, τ and time. To this end, we first let $\mathcal{X}_0 \leq 1$. It follows from (4.3) and the composition estimates in Lemma 2.5 that

$$\sum_{i=0}^{4} \|G_i^n\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} \leq C_2^* \|(y^n, r^n, w^n)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})}, \tag{4.4}$$

and similarly,

$$\sum_{i=0}^{4} \|\partial_t G_i^n\|_{L_t^1(\dot{B}^{\frac{d}{2}})} \leq C_3^* \|(\partial_t y^n, \partial_t w^n, \partial_t r^n)\|_{L_t^1(\dot{B}^{\frac{d}{2}})}, \tag{4.5}$$

for some universal constants C_2^* and C_3^*. According to (4.1) and (4.4), we can justify the condition (3.2) provided that

$$\mathcal{X}_0 \leq c_1^* := \frac{c}{2C_0 C_0^* C_2^*}.$$

28

Hence, we are able to employ the uniform a-priori estimate established in Proposition 3.1 to obtain

$$\|(y^{n+1}, w^{n+1}, r^{n+1}, u^{n+1})\|_{\mathbb{E}_t} + \frac{1}{\tau}\|u^{n+1} + \tau(\bar{F}_0 + G_0^n)\nabla r^{n+1}\|_{L_t^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})}$$

$$\leq C_0 \exp\Big(C_0 \int_0^t \big(\|u^n(s)\|_{\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1}} + \sum_{i=0}^4 \|\partial_t G_i^n(s)\|_{\dot{B}^{\frac{d}{2}}}\big) ds\Big) \tag{4.6}$$

$$\times \Big(\|(y_0^n, w_0^n, r_0^n, u_0^n)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}} + \|F_4^n \frac{(w^n)^2}{\varepsilon}\|_{L_t^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})}\Big).$$

Applying (4.3), Lemma 2.3 and 2.5 gives directly

$$\|F_4^n \frac{(w^n)^2}{\varepsilon}\|_{L_t^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} \leq \frac{C_4^*}{\varepsilon}\|w^n\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})}^2, \tag{4.7}$$

where $C_4^* > 0$ is a universal constant. Combining (4.1), (4.3), (4.5), (4.6) and (4.7) together, we have

$$\|(y^{n+1}, w^{n+1}, r^{n+1}, u^{n+1})\|_{\mathbb{E}_t} + \frac{1}{\tau}\|u^{n+1} + \tau(\bar{F}_0 + G_0^n)\nabla r^{n+1}\|_{L_t^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})}$$

$$\leq C_0 e^{2(1+C_3^*)C_0^2 C_0^* \mathcal{X}_0}\Big(C_0^* \mathcal{X}_0 + C_4^*(2C_0 C_0^* \mathcal{X}_0)^2\Big) \leq 2C_0 C_0^* \mathcal{X}_0,$$

as long as

$$\mathcal{X}_0 \leq c_2^* := \min\Big\{\frac{1}{2(1+C_3^*)C_0^2 C_0^* \log \frac{3}{2}}, \frac{2}{9(2C_0 C_0^*)^2 C_4^*}\Big\}$$

such that $e^{2(1+C_3^*)C_0^2 C_0^* \mathcal{X}_0} \leq \frac{3}{2}$ and $C_4^*(2C_0 C_0^* \mathcal{X}_0)^2 \leq \frac{1}{3}\mathcal{X}_0$. Thus, the uniform estimate (4.3) holds true for any $n \geq 0$.

- *Step 3: Strong convergence*

The uniform estimate (4.3) enables us to obtain the weak compactness of the approximate sequence. In order to pass the limit in every nonlinear term of (4.2) as $n \to \infty$, one needs to have robust strong compactness in a suitable sense. Classical compact embedding theorem merely gives the strong convergence locally in space-time and up to a subsequence, which is not enough for System (4.2). In what follows, we show that the strong convergence holds in $\mathbb{R}_+ \times \mathbb{R}^d$ for the whole sequence.

Lemma 4.1. *There exists a small constant $c_3^* \in (0, \min\{1, c_1^*, c_2^*\}]$ and a limit (y, w, r, u) such that if $\mathcal{X}_0 \leq c_3^*$, then as $n \to \infty$,*

$$(y^n, w^n, r^n, u^n) \to (y, w, r, u) \quad \text{strongly in} \quad L^\infty(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}}). \tag{4.8}$$

In particular, we have

$$(y^n, w^n, r^n, u^n) \to (y, w, r, u) \quad \text{strongly in} \quad L^\infty(\mathbb{R}_+; L^d \cap L^\infty). \tag{4.9}$$

Proof. In order to show (4.8), one needs to perform uniform energy estimates on the error unknown

$$(\widetilde{y}^n, \widetilde{w}^n, \widetilde{r}^n, \widetilde{u}^n) := (y^{n+1} - y^n, w^{n+1} - w^n, r^{n+1} - r^n, u^{n+1} - u^n).$$

Following the framework in Section 3, we aim to estimate the functional

$$\widetilde{\mathcal{X}}^n(t) = \|(\widetilde{y}^n, \widetilde{w}^n, \widetilde{r}^n, \widetilde{u}^n)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})} + \frac{1}{\varepsilon}\|\widetilde{w}^n\|_{L_t^1(\dot{B}^{\frac{d}{2}-1})} + \frac{1}{\sqrt{\varepsilon}}\|\widetilde{w}^n\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})}$$

$$+ \tau \|\widetilde{r}^n\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \|\widetilde{r}^n\|^h_{L^1_t(\dot{B}^{\frac{d}{2}})} + \sqrt{\tau}\|\widetilde{r}^n\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}$$

$$+ \|\widetilde{u}^n\|_{L^1_t(\dot{B}^{\frac{d}{2}})} + \frac{1}{\sqrt{\tau}}\|\widetilde{u}^n\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})} + \frac{1}{\tau}\|\widetilde{u}^n + \tau(\bar{F}_0 + G_0^{n-1})\nabla\widetilde{r}^n\|_{L^1_t(\dot{B}^{\frac{d}{2}-1})}.$$

To this matter, one can verify that for any $n \geq 1$, $(\widetilde{y}^n, \widetilde{w}^{n+1}, \widetilde{r}^{n+1}, \widetilde{u}^{n+1})$ solves

$$\begin{cases} \partial_t \widetilde{y}^{n+1} + u^n \cdot \nabla\widetilde{y}^{n+1} = \widetilde{S}_1^n, \\ \partial_t \widetilde{w}^{n+1} + u^n \cdot \nabla\widetilde{w}^{n+1} + (\bar{F}_1 + G_1^n)\mathrm{div}\,\widetilde{u}^{n+1} + (\bar{F}_2 + G_2^n)\dfrac{\widetilde{w}^{n+1}}{\varepsilon} = \widetilde{S}_2^n, \\ \partial_t \widetilde{r}^{n+1} + u^n \cdot \nabla\widetilde{r}^{n+1} + (\bar{F}_3 + G_3^n)\mathrm{div}\,\widetilde{u}^{n+1} = \widetilde{S}_3^n, \\ \partial_t \widetilde{u}^{n+1} + u^n \cdot \nabla\widetilde{u}^{n+1} + \dfrac{\widetilde{u}^{n+1}}{\tau} + (\bar{F}_0 + G_0^n)(\nabla\widetilde{r}^{n+1} + (\gamma_+ - \gamma_-)\nabla\widetilde{w}^{n+1}) = \widetilde{S}_4^n, \\ (\widetilde{y}^{n+1}, \widetilde{w}^{n+1}, \widetilde{r}^{n+1}, \widetilde{u}^{n+1})(0,x) = \dot{\Delta}_n(y_0, w_0, r_0, u_0)(x), \end{cases} \tag{4.10}$$

with

$$\begin{cases} \widetilde{S}_1^n := -\widetilde{u}^n \cdot \nabla y^n, \\ \widetilde{S}_2^n := -\widetilde{u}^n \cdot \nabla w^n - (G_1^n - G_1^{n-1})\mathrm{div}\,u^n - (G_2^n - G_2^{n-1})\dfrac{w^n}{\varepsilon}, \\ \widetilde{S}_3^n := -\widetilde{u}^n \cdot \nabla r^n - (G_3^n - G_3^{n-1})\mathrm{div}\,u^n + (F_4^n - F_4^{n-1})\dfrac{(w^n)^2}{\varepsilon} + F_4^{n-1}\dfrac{(w^n + w^{n-1})\widetilde{w}^n}{\varepsilon}, \\ \widetilde{S}_4^n := -\widetilde{u}^n \cdot \nabla u^n - (G_0^n - G_0^{n-1})(\nabla r^n + (\gamma_+ - \gamma_-)\nabla w^n). \end{cases}$$

First, employing Lemma 2.7 to $(4.10)_1$ yields

$$\|\widetilde{y}^{n+1}\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})} \lesssim \exp\Big(\|u^n\|_{L^1_t(\dot{B}^{\frac{d}{2}+1})}\Big)\Big(\|\dot{\Delta}_n y_0\|_{\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}}} + \|\widetilde{S}_1^n\|_{L^1_t(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})}\Big). \tag{4.11}$$

By similar computations on $(4.10)_2$-$(4.10)_4$ as in Lemma 3.1, we have the low-frequency estimate at the $\dot{B}^{\frac{d}{2}-1}$ regularity level:

$$\|(\widetilde{w}^{n+1}, \widetilde{r}^{n+1}, \widetilde{u}^{n+1})\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})} + \tau\|\widetilde{r}^{n+1}\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1})} + \sqrt{\tau}\|\widetilde{r}^{n+1}\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})}$$

$$+ \frac{1}{\varepsilon}\|\widetilde{w}^{n+1}\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1})} + \frac{1}{\sqrt{\varepsilon}}\|\widetilde{w}^{n+1}\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})} + \frac{1}{\tau}\|\widetilde{u}^{n+1} + \tau(\bar{F}_0 + G_0^n)\nabla\widetilde{r}^{n+1}\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1})} \tag{4.12}$$

$$\lesssim \|\dot{\Delta}_n(w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}-1}} + \|(\widetilde{S}_2^n, \widetilde{S}_3^n, \widetilde{S}_4^n)\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1})} + \mathcal{Z}^n(t)\widetilde{\mathcal{X}}^{n+1}(t) + \int_0^t \mathcal{V}^n(s)\widetilde{\mathcal{X}}^{n+1}(s)\,ds,$$

with

$$\begin{cases} \mathcal{Z}^n(t) = \displaystyle\sum_{i=0}^3 \|G_i^n\|_{\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}+1}}, \\ \mathcal{V}^n(t) = \|u^n\|_{\dot{B}^{\frac{d}{2}}\cap\dot{B}^{\frac{d}{2}+1}} + \displaystyle\sum_{i=0}^3 \|\partial_t G_i^n\|_{\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}+1}}. \end{cases}$$

Moreover, as in Lemma 3.2, one can obtain the following estimate in the high-frequency region:

$$\|(\widetilde{w}^{n+1}, \widetilde{r}^{n+1}, \widetilde{u}^{n+1})\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1})\cap L^1_t(\dot{B}^{\frac{d}{2}})} + \frac{1}{\tau}\|\widetilde{u}^{n+1} + \tau(\bar{F}_0 + G_0^n)\nabla\widetilde{r}^{n+1}\|^h_{L^1_t(\dot{B}^{\frac{d}{2}-1})}$$

$$+ \frac{1}{\varepsilon}\|\widetilde{w}^{n+1}\|^h_{L^1_t(\dot{B}^{\frac{d}{2}-1})} + \frac{1}{\sqrt{\varepsilon}}\|\widetilde{w}^{n+1}\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}-1})} + \sqrt{\tau}\|\widetilde{r}^{n+1}\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}})} \tag{4.13}$$

$$\lesssim \|\dot{\Delta}_n(w_0, r_0, u_0)\|^h_{\dot{B}^{\frac{d}{2}}} + \|(\widetilde{S}_2^n, \widetilde{S}_3^n, \widetilde{S}_4^n)\|^h_{L^1_t(\dot{B}^{\frac{d}{2}})} + \mathcal{Z}^n(t)\widetilde{\mathcal{X}}^{n+1}(t) + \int_0^t \mathcal{V}^n(s)\widetilde{\mathcal{X}}^{n+1}(s)\,ds.$$

Finally, following the proof of Lemma 3.3, we also have

$$
\begin{aligned}
\|(\widetilde{w}^{n+1}, \widetilde{r}^{n+1}, \widetilde{u}^{n+1})\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})} &+ \frac{1}{\sqrt{\varepsilon}}\|\widetilde{w}^{n+1}\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}})} + \frac{1}{\sqrt{\tau}}\|\widetilde{u}^{n+1}\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}})} \\
&\lesssim \|\dot{\Delta}_n(w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}}} + \|(\widetilde{S}_2^n, \widetilde{S}_3^n, \widetilde{S}_4^n)\|_{L_t^1(\dot{B}^{\frac{d}{2}})} \\
&\quad + (\widetilde{\eta} + \sqrt{\mathcal{Z}^n(t)})\widetilde{\mathcal{X}}^{n+1}(t) + \frac{1}{\widetilde{\eta}}\int_0^t \mathcal{V}^n(s)\widetilde{\mathcal{X}}^{n+1}(s)ds,
\end{aligned}
\tag{4.14}
$$

where $\widetilde{\eta} > 0$ is a constant to be chosen. Combining (4.11)-(4.14) together, we arrive at

$$
\begin{aligned}
\widetilde{\mathcal{X}}^{n+1}(t) &\lesssim \|\dot{\Delta}_n(y_0, w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}}} + \|(\widetilde{S}_2^n, \widetilde{S}_3^n, \widetilde{S}_4^n)\|_{L_t^1(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})} \\
&\quad + (\widetilde{\eta} + \sqrt{\mathcal{Z}^n(t)} + \mathcal{Z}^n(t))\widetilde{\mathcal{X}}^{n+1}(t) + (1 + \frac{1}{\widetilde{\eta}})\int_0^t \mathcal{V}^n(s)\widetilde{\mathcal{X}}^{n+1}(s)ds.
\end{aligned}
\tag{4.15}
$$

Now we estimate the right-hand side of (4.15) as follow. First, due to $\Delta_{n'}\dot{\Delta}_n = 0$ with $|n - n'| \geq 2$, one has

$$
\|\dot{\Delta}_n(y_0, w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}}} \lesssim \sum_{|n'-n|\leq 1}(2^{(\frac{d}{2}-1)n'} + 2^{\frac{d}{2}n'})\|\dot{\Delta}_{n'}(y_0, w_0, r_0, u_0)\|_{L^2}.
$$

Thence, applying uniform estimate (4.3) leads to

$$
\begin{cases}
\mathcal{Z}^n(t) \lesssim \|(y^n, w^n, r^n)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}+1})} \lesssim \mathcal{X}_0, \\
\displaystyle\int_0^t \mathcal{V}^n(s)ds \lesssim \|(\partial_t y^n, \partial_t w^n, \partial_t r^n)\|_{L_t^1(\dot{B}^{\frac{d}{2}})} \lesssim \mathcal{X}_0.
\end{cases}
$$

Regarding the nonlinear terms on the right-hand side of (4.15), one deduces from (2.4), (2.6)-(2.8) and (4.3) that

$$
\begin{aligned}
\|(\widetilde{S}_1^n, &\widetilde{S}_2^n, \widetilde{S}_3^n, \widetilde{S}_4^n)\|_{L_t^1(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})} \\
&\lesssim \|(y^n, w^n, r^n)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}}\cap\dot{B}^{\frac{d}{2}+1})}\|\widetilde{u}^n\|_{L_t^1(\dot{B}^{\frac{d}{2}})} + \|(\widetilde{y}^n, \widetilde{w}^n, \widetilde{r}^n, \widetilde{u}^n)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})}\|u^n\|_{L_t^1(\dot{B}^{\frac{d}{2}}\cap\dot{B}^{\frac{d}{2}+1})} \\
&\quad + \frac{1}{\sqrt{\varepsilon}}\|(w^n, w^{n-1})\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}})}\frac{1}{\sqrt{\varepsilon}}\|\widetilde{w}^n\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})} \lesssim \mathcal{X}_0\widetilde{\mathcal{X}}^n(t).
\end{aligned}
$$

Gathering the above estimates into (4.15) and (4.11) and letting both $\widetilde{\eta}$ and $\mathcal{X}_0$ be sufficiently small, we obtain

$$
\widetilde{\mathcal{X}}^{n+1}(t) \lesssim \sum_{|n'-n|\leq 1}(2^{(\frac{d}{2}-1)n'} + 2^{\frac{d}{2}n'})\|\dot{\Delta}_{n'}(y_0, w_0, r_0, u_0)\|_{L^2} + \mathcal{X}_0\widetilde{\mathcal{X}}^n(t).
$$

Summing this over $n \geq 1$ leads to

$$
\sum_{n=1}^\infty \widetilde{\mathcal{X}}^{n+1}(t) \lesssim \|(y_0, w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}}} + \mathcal{X}_0\sum_{n=1}^\infty \widetilde{\mathcal{X}}^n(t).
$$

Given $(y^0, w^0, r^0, u^0) = (0, 0, 0, 0)$ and $(y^1, w^1, r^1, u^1) = (y_0, w_0, r_0, u_0)$, we take sufficiently small $\mathcal{X}_0$ to have

$$
\sum_{n=0}^\infty \widetilde{\mathcal{X}}^n(t) \lesssim \|(y_0, w_0, r_0, u_0)\|_{\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}}}.
\tag{4.16}
$$

31

Now we define $(y, w, r, u) := \sum_{n'=0}^{\infty} (\widetilde{y}^{n'}, \widetilde{w}^{n'}, \widetilde{r}^{n'}, \widetilde{u}^{n'})$. Thanks to (4.16) and

$$(y^n, w^n, r^n, u^n) = \sum_{n'=0}^{n} (\widetilde{y}^{n'}, \widetilde{w}^{n'}, \widetilde{r}^{n'}, \widetilde{u}^{n'}),$$

it follows that

$$\|(y^n, w^n, r^n, u^n) - (y, w, r, u)\|_{L_t^\infty (\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})} \leq \sum_{n' \geq n+1}^{\infty} \widetilde{\mathcal{X}}^{n'}(t) \underset{n \to \infty}{\to} 0.$$

Gathering the embeddings in Lemma 2.2, we get (4.9), which finishes the proof of Lemma 4.1.

$\square$

- *Step 4: Global existence*

Let $\mathcal{X}_0 \leq c_3^*$. By virtue of the strong convergence properties (4.8)-(4.9), one can pass to the limit as $n \to \infty$ in (4.2) and justify that the limit (y, w, r, u), obtained in Lemma 4.1, is indeed a global strong solution to the Cauchy problem (1.23). In addition, taking advantage of Fatou's property in Lemma 2.2, for all $t > 0$, we have

$$\|(y, w, r, u)\|_{\mathbb{E}_t} + \frac{1}{\tau} \|u + \tau(\bar{F}_0 + G_0)\nabla r\|_{L_t^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})}$$
$$\lesssim \liminf_{n \to \infty} \left(\|(y^n, w^n, r^n, u^n)\|_{\mathbb{E}_t} + \frac{1}{\tau} \|u^n + \tau(\bar{F}_0 + G_0^{n-1})\nabla r^n\|_{L_t^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})} \right) \lesssim \mathcal{X}_0. \tag{4.17}$$

To prove the time continuity property in (1.8)-(1.9), our proof relies on the uniform bound (4.17) and employs a reasoning analogous to that found in [22]. Since $\|(\partial_t y, \partial_t w, \partial_t r, \partial_t u)\|$ lies in $L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}})$, one has $(y, w, r, u) \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}})$. To recover $(y, w, r, u) \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})$, we shall investigate each equations separately. Recall that the solution (y, w, r, u) satisfies

$$\begin{cases} \partial_t y = -u \cdot \nabla y, \\ \partial_t w = -u \cdot \nabla w - (\bar{F}_1 + G_1)\operatorname{div} u - (\bar{F}_2 + G_2)\dfrac{w}{\varepsilon}, \\ \partial_t r = -u \cdot \nabla r - (\bar{F}_3 + G_3)\operatorname{div} u - F_4 \dfrac{w^2}{\varepsilon}, \\ \partial_t u = -u \cdot \nabla u - \dfrac{u}{\tau} - (\bar{F}_0 + G_0)\nabla r - (\gamma_+ - \gamma_-)(\bar{F}_0 + G_0)\nabla w. \end{cases}$$

As the right-hand side terms of the components y, r and w belong to $L^1(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1})$, we directly get $(y, r, w) \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1})$. Concerning the equation of u, its right-hand side lies in $L^2(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1})$ thus we can also recover that u belongs to $\mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1})$. We are left with recovering the time continuity of (y, w, r, u) in $\dot{B}^{\frac{d}{2}+1}$. First, for a fixed $j \in \mathbb{Z}$, each (y_j, w_j, r_j, u_j) is continuous in time with values in L^2 due to Bernstein's inequality. Now, thanks to $(y, w, r, u) \in L^\infty(\mathbb{R}_+; \dot{B}^{\frac{d}{2}+1})$, for any $\eta > 0$, there exists an large integer J_* such that, for all $t > 0$

$$\sum_{|j| \geq J_*} 2^{j(\frac{d}{2}+1)} \|(y_j, w_j, r_j, u_j)\|_{L_t^\infty(L^2)} < \frac{\eta}{2}.$$

Thus, for any time $t, t' \in \mathbb{R}_+$, we have

$$\|(y, w, r, u)(t) - (y, w, r, u)(t')\|_{\dot{B}^{\frac{d}{2}+1}} \leq \sum_{|j| < J_*} 2^{j(\frac{d}{2}+1)} \|\dot{\Delta}_j((y, w, r, u)(t) - (y, w, r, u)(t'))\|_{L^2}$$

32

$$+ \sum_{|j| \geq J_*} 2^{j(\frac{d}{2}+1)} \|\dot{\Delta}_j((y, w, r, u)(t) - (y, w, r, u)(t'))\|_{L^2}$$

$$\leq 2^{J_*} \|(y, w, r, u)(t) - (y, w, r, u)(t')\|_{\dot{B}^{\frac{d}{2}}} + \eta \underset{t \to t'}{\to} \eta.$$

Since η is an arbitrary constant, we get $(y, w, r, u) \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}+1})$. Finally, applying the inverse function theorem, we can see that once $\alpha_\pm$ and $\rho_\pm$ are determined by (1.21), then $(\alpha_\pm, \rho_\pm, u) \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})$ is the unique global strong solution to the original Cauchy problem of System (BN). Using (4.17), product laws and composite estimates, we are able to verify that $(\alpha_\pm, \rho_\pm, u)$ satisfies the properties (1.8)-(1.9). To complete the proof, we prove the uniqueness in our regularity framework below.

- *Step 5: Uniqueness*

The final step is to show the uniqueness of solutions to (BN) belonging to the regularity class (1.8). We emphasize that the proof does not require the smallness of regularity for initial data. It suffices to consider the reformulated system (1.23). Without loss of generality, as the parameters do not affect our argument for proving uniqueness, we set $\varepsilon = \tau = 1$. Let (y_1, w_1, r_1, u_1) and (y_2, w_2, r_2, u_2) be two solutions to (1.23) subject to the same initial data (y_0, w_0, r_0, u_0), satisfying (1.8) and $\bar{F}_i + G_i(y_j, r_j, w_j) > 0$ for $i = 0, 1, 2, 3$ and $j = 1, 2$. The difference

$$(\widetilde{y}, \widetilde{w}, \widetilde{r}, \widetilde{u}) := (y_1 - y_2, w_1 - w_2, r_1 - r_2, u_1 - u_2)$$

solves

$$\begin{cases} \partial_t \widetilde{y} + u_1 \cdot \nabla \widetilde{y} = \widetilde{S}_1, \\ \partial_t \widetilde{w} + u_1 \cdot \nabla \widetilde{w} + (\bar{F}_1 + G_1^1) \mathrm{div}\, \widetilde{u} + \bar{F}_2 \widetilde{w} = \widetilde{S}_2, \\ \partial_t \widetilde{r} + u_1 \cdot \nabla \widetilde{r} + (\bar{F}_3 + G_3^1) \mathrm{div}\, \widetilde{u} = \widetilde{S}_3, \\ \partial_t \widetilde{u} + u_1 \cdot \nabla \widetilde{u} + \widetilde{u} + (\bar{F}_0 + G_0^1)(\nabla \widetilde{r} + (\gamma_+ - \gamma_-)\nabla \widetilde{w}) = \widetilde{S}_4, \\ (\widetilde{y}, \widetilde{w}, \widetilde{r}, \widetilde{u})(0, x) = (0, 0, 0, 0), \end{cases} \tag{4.18}$$

where we denoted

$$\begin{cases} \widetilde{S}_1 := -\widetilde{u} \cdot \nabla y_2, \\ \widetilde{S}_2 := -\widetilde{u} \cdot \nabla w_2 - (G_1^1 - G_1^2)\mathrm{div}\, u_2 - (G_2^1 - G_2^2)w_2 - G_2^1 \widetilde{w}, \\ \widetilde{S}_3 := -\widetilde{u} \cdot \nabla r_2 - (G_3^1 - G_3^2)\mathrm{div}\, u_2 + (F_4^1 - F_4^2)(w_1)^2 - F_4^2(w_1 + w_2)\widetilde{w}, \\ \widetilde{S}^4 := -\widetilde{u} \cdot \nabla u_2 - (G_0^1 - G_0^2)(\nabla r_2 + (\gamma_+ - \gamma_-)\nabla w_2). \end{cases}$$

Here $G_i^l := G_i(y_l, w_l, r_l)$ and $F_4^l := F_4(y_l, w_l, r_l)$ with $i = 0, 1, 2, 3$, and $l = 1, 2$. Applying Lemma 2.7 to (4.18) implies that, for all $t > 0$,

$$\|\widetilde{y}(t)\|_{\dot{B}^{\frac{d}{2}}} \lesssim \exp\left(\|u_1\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})}\right) \int_0^t \|\widetilde{S}_1^n\|_{\dot{B}^{\frac{d}{2}}}\, ds. \tag{4.19}$$

Through the application of the weighted Lyapunov functional method, as outlined in (3.32)-(3.36), we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} \left(W_1 |\widetilde{w}_j|^2 + W_2 |\widetilde{r}_j|^2 + |\widetilde{u}_j|^2\right) dx + \int_{\mathbb{R}^d} \left(W_3 |\widetilde{w}_j|^2 + |\widetilde{u}_j|^2\right) dx$$

$$\lesssim \left(\|(\partial_t y_1, \partial_t w_1, \partial_t r_1, \nabla y_1, \nabla w_1, \nabla r_1)\|_{L^\infty} + \|u_1\|_{W^{1,\infty}}\right) \|(\widetilde{w}_j, \widetilde{r}_j, \widetilde{u}_j)\|_{L^2}^2 \tag{4.20}$$

$$+ \left(\|(\widetilde{T}_{1,j}, \widetilde{T}_{2,j}, \widetilde{T}_{3,j})\|_{L^2} + \|\dot{\Delta}_j(\widetilde{S}_2, \widetilde{S}_3, \widetilde{S}_4)\|_{L^2}\right) \|(\widetilde{w}_j, \widetilde{r}_j, \widetilde{u}_j)\|_{L^2},$$

where $W_i = W_i(y_1, w_1, r_1) > 0$, $i = 1, 2, 3$ are some smooth weight functions depending on $\bar{F}_i + G_i^1 > 0$, and the commutator terms are given by

$$\begin{cases} \widetilde{T}_{1,j} := [u_1, \dot{\Delta}_j]\nabla\widetilde{w} + [G_1^1, \dot{\Delta}_j]\mathrm{div}\,\widetilde{u}, \\ \widetilde{T}_{2,j} := [u_1, \dot{\Delta}_j]\nabla\widetilde{r} + [G_3^1, \dot{\Delta}_j]\mathrm{div}\,\widetilde{u}, \\ \widetilde{T}_{3,j} := [u_1, \dot{\Delta}_j]\nabla\widetilde{u} + [G_0^1, \dot{\Delta}_j]\nabla\widetilde{r} + [G_0^1, \dot{\Delta}_j]\nabla\widetilde{w}. \end{cases}$$

Integrating (4.20) over $[0, t]$, taking the square root of both sides and summing the resulting estimate over $j \in \mathbb{Z}$ with the weight $2^{\frac{d}{2}j}$, we get

$$\|(\widetilde{w}, \widetilde{r}, \widetilde{u})\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})} + \|\widetilde{u}\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}})} + \|\widetilde{w}\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}})}$$
$$\lesssim \bigg(\int_0^t \Big((\|(\partial_t y_1, \partial_t w_1, \partial_t r_1, \nabla y_1, \nabla w_1, \nabla r_1)\|_{L^\infty} + \|u_1\|_{W^{1,\infty}})\|(\widetilde{w}, \widetilde{r}, \widetilde{u})\|_{\dot{B}^{\frac{d}{2}}}$$
$$+ \sum_{j\in\mathbb{Z}} 2^{j\frac{d}{2}}\|(\widetilde{T}_{1,j}, \widetilde{T}_{2,j}, \widetilde{T}_{3,j})\|_{L^2} + \|(\widetilde{S}_2, \widetilde{S}_3, \widetilde{S}_4)\|_{\dot{B}^{\frac{d}{2}}} \Big)ds \bigg)^{\frac{1}{2}} \|(\widetilde{w}, \widetilde{r}, \widetilde{u})\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})}^{\frac{1}{2}},$$

which together with Young's inequality implies

$$\|(\widetilde{w}, \widetilde{r}, \widetilde{u})(t)\|_{\dot{B}^{\frac{d}{2}}} \lesssim \int_0^t \Big((\|(\partial_t y_1, \partial_t w_1, \partial_t r_1, \nabla y_1, \nabla w_1, \nabla r_1)\|_{L^\infty} + \|u_1\|_{W^{1,\infty}})\|(\widetilde{w}, \widetilde{r}, \widetilde{u})\|_{\dot{B}^{\frac{d}{2}}}$$
$$+ \sum_{j\in\mathbb{Z}} 2^{j\frac{d}{2}}\|(\widetilde{T}_{1,j}, \widetilde{T}_{2,j}, \widetilde{T}_{3,j})\|_{L^2} + \|(\widetilde{S}_2, \widetilde{S}_3, \widetilde{S}_4)\|_{\dot{B}^{\frac{d}{2}}} \Big)ds, \qquad t > 0. \tag{4.21}$$

Using the classical commutator estimate in Lemma 2.4 implies

$$\sum_{j\in\mathbb{Z}} 2^{j\frac{d}{2}}\|(\widetilde{T}_{1,j}, \widetilde{T}_{2,j}, \widetilde{T}_{3,j})\|_{L^2} \lesssim \|(y_1, r_1, w_1, u_1)\|_{\dot{B}^{\frac{d}{2}+1}} \|(\widetilde{w}, \widetilde{r}, \widetilde{u})\|_{\dot{B}^{\frac{d}{2}}}.$$

In addition, according to standard product laws and composite estimates in Lemmas 2.3 and 2.5, the nonlinear terms $(\widetilde{S}_1, \widetilde{S}_2, \widetilde{S}_3, \widetilde{S}_4)$ can be handled as

$$\|(\widetilde{S}_1, \widetilde{S}_2, \widetilde{S}_3, \widetilde{S}_4)\|_{\dot{B}^{\frac{d}{2}}} \lesssim \Big(\|(y_2, w_2, r_2, u_2)\|_{\dot{B}^{\frac{d}{2}+1}} + \|(w_1, w_2)\|_{\dot{B}^{\frac{d}{2}}} \Big)\|(\widetilde{y}, \widetilde{w}, \widetilde{r}, \widetilde{u})\|_{\dot{B}^{\frac{d}{2}}}.$$

Substituting the above two estimates into (4.19) and (4.21) and then employing Grönwall's inequality, we end up with $\|(\widetilde{y}, \widetilde{w}, \widetilde{r}, \widetilde{u})(t)\|_{\dot{B}^{\frac{d}{2}}} = 0$ for all $t > 0$, which concludes the proof of Theorem 1.1.

4.2 The Cauchy problem of Systems (K) and (PM)

We provide a brief explanation of the proof of the global existence and uniqueness for System (PM). The proof of the result for System (K) (Theorem 1.2) follows a very similar procedure, so we omit the details here for brevity. The uniformity of the estimate (1.9) for System (K) allows us to construct solutions for System (PM) by taking the limit as the relaxation parameter $\tau \to 0$.

The following lemma states the uniform estimate verified by the solutions of System (K$_\tau$), which are rescaled from estimate (1.12) obtained in Theorem 1.2 for System (K).

Lemma 4.2. *Let $(\alpha_\pm^\tau, \rho_\pm^\tau, u^\tau)$ be the global solution to the Cauchy problem of System* (K) *subject to the initial data $(\alpha_{\pm,0}^\tau, \rho_{\pm,0}^\tau, u_0^\tau)$ given by Theorem 1.2 and $(\beta_\pm^\tau, \varrho_\pm^\tau, v^\tau)$ be defined by the diffusive scaling* (1.2), *then it holds that*

$$
\begin{aligned}
\|(\beta_\pm^\tau - \bar\alpha_\pm, \varrho_\pm^\tau - \bar\rho_\pm)\|_{\widetilde{L}^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} & \\
+ \|(\Pi^\tau - \bar{P}, \varrho_\pm^\tau - \bar\rho_\pm)\|_{L^1(\dot{B}^{\frac{d}{2}+1})} &+ \|(\Pi^\tau - \bar{P}, \varrho_\pm^\tau - \bar\rho)\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} \\
+ \|v^\tau\|_{L^1(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} &+ \|v^\tau\|_{\widetilde{L}^2(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1})} + \frac{1}{\tau}\|z^\tau\|_{L^1(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})} \\
&\leq C\|(\alpha_{\pm,0}^\tau - \bar\alpha_\pm, \rho_{\pm,0}^\tau - \bar\rho_\pm, u_0^\tau)\|_{\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}+1}},
\end{aligned}
\tag{4.22}
$$

with $z^\tau := v^\tau + \frac{1}{\varrho^\tau}\nabla\Pi^\tau$, and $C > 0$ a universal constant.

   Proof of Theorem 1.3: Assume that the initial data $(\beta_{\pm,0}, \varrho_{\pm,0})$ satisfies (1.13). For any $\tau \in (0,1)$, we define the regularized data as

$$(\alpha_{\pm,0}^\tau, \rho_{\pm,0}^\tau)(x) := \dot{S}_{[\frac{1}{\tau}]}(\beta_{\pm,0}, \varrho_{\pm,0})(x) \quad \text{and} \quad u_0^\tau(x) := 0.$$

Hence, by employing Theorem 1.2 we can obtain a sequence $(\alpha_\pm^\tau, \rho_\pm^\tau, u^\tau)$, which is the global solution to System (K) subject to the initial data $(\alpha_{\pm,0}^\tau, \rho_{\pm,0}^\tau, u_0^\tau)$. Taking the diffusive scaling (1.2), one has that $(\beta_\pm^\tau, \varrho_\pm^\tau, v^\tau)$ is the global solution to System (K_τ) subject to the initial data $(\alpha_{\pm,0}^\varepsilon, \rho_{\pm,0}^\varepsilon, u_0^\tau/\tau)$. In view of the uniform estimate (4.22) established in Lemma 4.2, the Aubin-Lions lemma and the cantor diagonal process, there exists a limit $(\beta_\pm, \varrho_\pm)$ such that as $\tau \to 0$, up to a subsequence, $\chi(\beta_\pm^\tau, \varrho_\pm^\tau)$ converges to $\chi(\beta_\pm, \varrho_\pm)$ in $C([0,T]; \dot{B}^s)$ $(s < \frac{d}{2}+1)$ strongly for any given time $T > 0$ and $\chi \in C_c^\infty(\mathbb{R}^d \times [0,T])$. Thus, we can check that $(\beta_\pm, \varrho_\pm)$ solves System (PM) in the sense of distributions. Furthermore, taking advantage of the Fatou property and the optimal regularity estimate in Lemma 2.6 for the equation of Π, we can conclude (1.14). Finally, the uniqueness can be obtained in a simple fashion. The interested reader may also refer to [19, 20] for more details.

5   Relaxation limits with convergence rates

5.1   Pressure-relaxation limit: System (BN) to System (K)

   In this section, we prove Theorem 1.4 related to the convergence rate of the relaxation process between System (BN) and System (K). Let $(\alpha_\pm^{\varepsilon,\tau}, \rho_\pm^{\varepsilon,\tau}, u^{\varepsilon,\tau})$ and $(\alpha_\pm^\tau, \rho_\pm^\tau, u^\tau)$ be the global solutions to System (BN) with the initial data $(\alpha_{\pm,0}^{\varepsilon,\tau}, \rho_{\pm,0}^{\varepsilon,\tau}, u_0^{\varepsilon,\tau})$ and System (K) with the initial data $(\alpha_{\pm,0}^{\varepsilon,\tau}, \rho_{\pm,0}^{\varepsilon,\tau}, u_0^{\varepsilon,\tau})$ given by Theorems 1.1 and 1.2, respectively. Denote the error variables

$$
\begin{aligned}
(\delta\alpha_\pm, \delta\rho_\pm, \delta u) &:= (\alpha_\pm^{\varepsilon,\tau} - \alpha_\pm^\tau, u^{\varepsilon,\tau} - u^\tau), \\
(\delta\rho, \delta P_\pm, \delta P) &:= (\rho_\pm^{\varepsilon,\tau} - \rho_\pm^\tau, \rho^{\varepsilon,\tau} - \rho^\tau, P_\pm(\rho_\pm^{\varepsilon,\tau}) - P_\pm(\rho_\pm^\tau), P^{\varepsilon,\tau} - P^\tau),
\end{aligned}
$$

and the initial data of δP

$$\delta P|_{t=0} = P_0^{\varepsilon,\tau} - P_0^\tau, \quad P_0^\varepsilon := \alpha_{+,0}^{\varepsilon,\tau} P_+(\rho_{+,0}^{\varepsilon,\tau}) + \alpha_{-,0}^{\varepsilon,\tau} P_-(\rho_{-,0}^{\varepsilon,\tau}), \quad P_0^\tau := P_+(\rho_{+,0}^\tau). \tag{5.1}$$

First, to avoid dealing with difficult nonlinearities in the equation of $\delta\alpha_\pm$, we work with the following purely transported variable instead of $\delta\alpha_\pm$:

$$\delta Y := \frac{\alpha_+^{\varepsilon,\tau}\rho_+^{\varepsilon,\tau}}{\rho^{\varepsilon,\tau}} - \frac{\alpha_+^\tau\rho_+^\tau}{\rho^\tau}. \tag{5.2}$$

with the initial data

$$\delta Y|_{t=0} = Y_0^{\varepsilon,\tau} - Y_0^\tau, \quad Y_0^{\varepsilon,\tau} := \frac{\alpha_{+,0}^{\varepsilon,\tau}\rho_{+,0}^{\varepsilon,\tau}}{\alpha_{+,0}^{\varepsilon,\tau}\rho_{+,0}^{\varepsilon,\tau} + \alpha_{-,0}^{\varepsilon,\tau}\rho_{-,0}^{\varepsilon,\tau}}, \quad Y_0^\tau := \frac{\alpha_{+,0}^\tau\rho_{+,0}^\tau}{\alpha_{+,0}^\tau\rho_{+,0}^\tau + \alpha_{-,0}^\tau\rho_{-,0}^\tau}. \tag{5.3}$$

Lemma 5.1. *For $d \geq 3$, under the assumption (1.17), δY satisfies the following estimate:*

$$\|\delta Y\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})} \lesssim \sqrt{\varepsilon\tau} + o(1)\|\delta u\|_{L_t^1(\dot{B}^{\frac{d}{2}-1})}. \tag{5.4}$$

Proof. Since the equation of δY reads

$$\partial_t\delta Y + u^{\varepsilon,\tau}\cdot\nabla\delta Y = -\delta u\cdot\nabla\frac{\alpha_+^\tau\rho_+^\tau}{\rho^\tau},$$

Lemma 2.7 and the product law (2.4) for $d \geq 3$ gives

$$\|\delta Y\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})} \lesssim \exp\Big(\|u^{\varepsilon,\tau}\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})}\Big)\Big(\sqrt{\varepsilon\tau} + \|\delta u\|_{L_t^1(\dot{B}^{\frac{d}{2}-1})}\|\nabla\frac{\alpha_+^\tau\rho_+^\tau}{\rho^\tau}\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})}\Big).$$

This together with the uniform estimate (1.9) leads to (5.4). $\qquad\square$

We are now ready to estimate $(\delta\alpha_\pm, \delta\rho_\pm, \delta P_\pm, \delta P)$. It is easy to verify that $P^{\varepsilon,\tau}$ satisfies

$$\partial_t P^{\varepsilon,\tau} + u^{\varepsilon,\tau}\cdot\nabla P^{\varepsilon,\tau} = -\big(\gamma_+\alpha_+^{\varepsilon,\tau}P_+(\rho_+^{\varepsilon,\tau}) + \gamma_-\alpha_-^{\varepsilon,\tau}P_-(\rho_-^{\varepsilon,\tau})\big)\text{div}\,u^{\varepsilon,\tau}$$
$$- \alpha_+^{\varepsilon,\tau}\alpha_-^{\varepsilon,\tau}\big((\gamma_+ - 1)P_+(\rho_+^{\varepsilon,\tau}) - (\gamma_- - 1)P_-(\rho_-^{\varepsilon,\tau})\big)\frac{P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})}{\varepsilon}. \tag{5.5}$$

And the equation of P^τ reads

$$\partial_t P^\tau + u^\tau\cdot\nabla P^\tau + \frac{\gamma_+\gamma_-P^\tau}{\gamma_+\alpha_-^\tau + \gamma_-\alpha_+^\tau}\text{div}\,u^\tau = 0. \tag{5.6}$$

However it is not suitable to estimate δP directly from (5.5)-(5.6) as the decay rate of $P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})$ can not be faster than ε in view of (1.9). To overcome this difficulty, we introduce an auxiliary unknown $Q^{\varepsilon,\tau} := P^{\varepsilon,\tau} - \Gamma_1^{\varepsilon,\tau}\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big)$ which verifies

$$\partial_t Q^{\varepsilon,\tau} + u^{\varepsilon,\tau}\cdot\nabla Q^{\varepsilon,\tau} + \Gamma_2^{\varepsilon,\tau}\text{div}\,u^{\varepsilon,\tau} = -\Gamma_3^{\varepsilon,\tau}\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big)\text{div}\,u^{\varepsilon,\tau}$$
$$+ (\partial_t\Gamma_1^{\varepsilon,\tau} + u^{\varepsilon,\tau}\cdot\nabla\Gamma_1^{\varepsilon,\tau})\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big), \tag{5.7}$$

with

$$\begin{cases}
\Gamma_1^{\varepsilon,\tau} := \dfrac{\alpha_+^{\varepsilon,\tau}\alpha_-^{\varepsilon,\tau}\big((\gamma_+ - 1)P_+(\rho_+^{\varepsilon,\tau}) - (\gamma_- - 1)P_-(\rho_-^{\varepsilon,\tau})\big)}{\gamma_+\alpha_-^{\varepsilon,\tau}P_+(\rho_+^{\varepsilon,\tau}) + \gamma_-\alpha_+^{\varepsilon,\tau}P_-(\rho_-^{\varepsilon,\tau})}, \\[3mm]
\Gamma_2^{\varepsilon,\tau} := \dfrac{\gamma_+\gamma_-P_+(\rho_+^{\varepsilon,\tau})P_-(\rho_-^{\varepsilon,\tau})}{\gamma_+\alpha_-P_+(\rho_+^{\varepsilon,\tau}) + \gamma_-\alpha_+^{\varepsilon,\tau}P_-(\rho_-^{\varepsilon,\tau})}, \\[3mm]
\Gamma_3^{\varepsilon,\tau} := \dfrac{\alpha_+^{\varepsilon,\tau}\alpha_-^{\varepsilon,\tau}\big(\gamma_+P_+(\rho_+^{\varepsilon,\tau}) - \gamma_-P_-(\rho_-^{\varepsilon,\tau})\big)}{\gamma_+\alpha_-^{\varepsilon,\tau}P_+(\rho_+^{\varepsilon,\tau}) + \gamma_-\alpha_+^{\varepsilon,\tau}P_-(\rho_-^{\varepsilon,\tau})}.
\end{cases}$$

With this formulation, it will be possible to derive the $\mathcal{O}(\varepsilon)$ bounds for the last term on the right-hand side of (5.7). Define

$$\delta Q := P^{\varepsilon,\tau} - P^\tau - \Gamma_1^{\varepsilon,\tau}\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big). \tag{5.8}$$

The next lemma implies that to estimate $(\delta\alpha_\pm, \delta\rho_\pm, \delta P_\pm, \delta P)$, it is sufficient to control $(\delta Y, \delta Q, P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau}))$.

Lemma 5.2. *For $d \geq 3$, under the assumption (1.17), the following estimates follow:*

$$\begin{cases} \|(\delta\alpha_\pm, \delta\rho_\pm, \delta\rho)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})} \lesssim \|(\delta Y, \delta Q, P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau}))\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})}, \\ \|\delta\rho_\pm\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1})} \lesssim \|(\delta Q, P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau}))\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1})}. \end{cases} \tag{5.9}$$

Proof. Due to (5.2) and

$$\delta\rho = (\rho_+^{\varepsilon,\tau} - \rho_-^{\varepsilon,\tau})\delta\alpha_+ + \alpha_+^\tau\delta\rho_+ + \alpha_-^\tau\delta\rho_-, \tag{5.10}$$

it holds that

$$\begin{aligned} \delta Y &= \frac{1}{\rho^{\varepsilon,\tau}\rho^\tau}\big(\rho_+^{\varepsilon,\tau}\rho^\tau\delta\alpha_+^{\varepsilon,\tau} + \alpha_+^\tau\rho^\tau\delta\rho_+ - \alpha_+^\tau\rho_+^\tau\delta\rho\big) \\ &= \frac{1}{\rho^{\varepsilon,\tau}\rho^\tau}\big((\alpha_-^\tau\rho_+^{\varepsilon,\tau}\rho_-^\tau + \alpha_+^\tau\rho_+^\tau\rho_-^{\varepsilon,\tau})\delta\alpha_+ + \alpha_+^\tau\alpha_-^\tau\rho_-^\tau\delta\rho_+ - \alpha_+^\tau\alpha_-^\tau\rho_+^\tau\delta\rho_-\big). \end{aligned}$$

This implies

$$\delta\alpha_+ = \frac{1}{\alpha_-^\tau\rho_+^{\varepsilon,\tau}\rho_-^\tau + \alpha_+^\tau\rho_+^\tau\rho_-^{\varepsilon,\tau}}\big(\rho^{\varepsilon,\tau}\rho^\tau\delta Y - \alpha_+^\tau\alpha_-^\tau\rho_-^\tau\delta\rho_+ + \alpha_+^\tau\alpha_-^\tau\rho_+^\tau\delta\rho_-\big). \tag{5.11}$$

Inserting (5.11) into (5.10), we have

$$\begin{aligned} \delta\rho &= \frac{\rho_+^{\varepsilon,\tau} - \rho_-^{\varepsilon,\tau}}{\alpha_-^\tau\rho_+^{\varepsilon,\tau}\rho_-^\tau + \alpha_+^\tau\rho_+^\tau\rho_-^{\varepsilon,\tau}}\big(\rho^{\varepsilon,\tau}\rho^\tau\delta Y - \alpha_+^\tau\alpha_-^\tau\rho_-^\tau\delta\rho_+ + \alpha_+^\tau\alpha_-^\tau\rho_+^\tau\delta\rho_-\big) \\ &\quad + \alpha_+^\tau\delta\rho_+ + \alpha_-^\tau\delta\rho_-. \end{aligned} \tag{5.12}$$

Moreover, we have

$$\delta P_\pm = \delta\rho_\pm \int_0^1 P'_\pm(\theta\rho_\pm^{\varepsilon,\tau} + (1-\theta)\rho_\pm^\tau)d\theta \qquad \text{and} \qquad \delta P = \alpha_+^{\varepsilon,\tau}(P_+^{\varepsilon,\tau} - P_-^{\varepsilon,\tau}) + \delta P_-. \tag{5.13}$$

Using the previous uniform estimates (1.9) and (4.22), the product laws (2.3)-(2.4) and the composition estimates (2.6)-(2.7), for some constant states $\bar{\Gamma}_i > 0$ $(i = 1, 2, 3)$, we have

$$\sum_{i=1}^3(\|\Gamma_i - \bar{\Gamma}_i\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}+1})} + \|\partial_t\Gamma_i\|_{L_t^1(\dot{B}^{\frac{d}{2}})}) = o(1). \tag{5.14}$$

Therefore, (5.9) follows from (5.8), (5.11)-(5.14), the product laws (2.3)-(2.4) and the fact $\delta\alpha_+ = -\delta\alpha_+$. $\qquad\square$

The next lemma pertains to $\mathcal{O}(\sqrt{\varepsilon\tau})$ bounds for $P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})$, which leads to the convergence rate $\sqrt{\varepsilon\tau}$.

Lemma 5.3. *For $d \geq 3$, under the assumption (1.17), the following estimate is valid:*

$$
\begin{aligned}
&\|P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})} \\
&\quad + \frac{1}{\sqrt{\varepsilon}}\|P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}-1})} \lesssim \sqrt{\varepsilon\tau}.
\end{aligned}
\tag{5.15}
$$

Proof. It is easy to verify from (BN) that $P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})$ satisfies the damped equation

$$
\begin{aligned}
&\partial_t\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big) + u^{\varepsilon,\tau}\cdot\nabla\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big) \\
&\quad + \frac{c_*}{\varepsilon}\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big) \\
&= ((\gamma_+\alpha_-^{\varepsilon,\tau}P_+(\rho_+^{\varepsilon,\tau}) + \gamma_-\alpha_+^{\varepsilon,\tau}P_-(\rho_-^{\varepsilon,\tau})) - c_*)\frac{1}{\varepsilon}\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big) \\
&\quad - \big(\gamma_+P_+(\rho_+^{\varepsilon,\tau}) - \gamma_-P_-(\rho_-^{\varepsilon,\tau})\big)\operatorname{div}u^{\varepsilon,\tau} := W_1 + W_2,
\end{aligned}
\tag{5.16}
$$

with $c_* := (\gamma_+\bar{\alpha}_- + \gamma_-\bar{\alpha}_+)\bar{P}$. Thence the L^2-in-time type estimate in Lemma 2.7 for the damped transport equation (5.16) leads to

$$
\begin{aligned}
&\|P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})} + \frac{1}{\sqrt{\varepsilon}}\|P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})} \\
&\lesssim \exp\Big(\|u^{\varepsilon,\tau}\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})}\Big)\Big(\sqrt{\varepsilon\tau} + \sqrt{\varepsilon}\|(W_1,W_2)\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})}\Big).
\end{aligned}
$$

By (1.9) and (2.4), there holds

$$
\begin{aligned}
\sqrt{\varepsilon}\|W_1\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})} &\lesssim \|(\alpha_\pm^{\varepsilon,\tau}-\bar{\alpha}_\pm, \rho_\pm^{\varepsilon,\tau}-\bar{\rho}_\pm)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})}\frac{1}{\sqrt{\varepsilon}}\|P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})} \\
&\lesssim o(1)\frac{1}{\sqrt{\varepsilon}}\|P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})},
\end{aligned}
$$

and

$$
\sqrt{\varepsilon}\|W_2\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-2}\cap\dot{B}^{\frac{d}{2}-1})} \lesssim \sqrt{\varepsilon}\|u^{\varepsilon,\tau}\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})} \lesssim \sqrt{\varepsilon\tau}.
$$

Therefore, we gain (5.15). $\qquad\square$

We are going to estimate $(\delta Q, \delta u)$. By virtue of (BN), (K) and (5.6)-(5.7), $(\delta Q, \delta u)$ satisfies the following equations of damped Euler type with rough coefficients:

$$
\begin{cases}
\partial_t\delta Q + u^{\varepsilon,\tau}\cdot\nabla\delta Q + \Gamma_2^{\varepsilon,\tau}\operatorname{div}\delta u = \delta F_1, \\
\partial_t\delta u + u^{\varepsilon,\tau}\cdot\nabla\delta u + \frac{1}{\bar{\rho}}\nabla\delta Q + \big(\frac{1}{\rho^{\varepsilon,\tau}} - \frac{1}{\rho^\tau}\big)\nabla P^\tau + \frac{\delta u}{\tau} = \delta F_2,
\end{cases}
\tag{5.17}
$$

with the nonlinear terms

$$
\begin{cases}
\delta F_1 = -\delta u\cdot\nabla P^\tau - \Big(\Gamma_2^{\varepsilon,\tau} - \dfrac{\gamma_+\gamma_-P^\tau}{\gamma_+\alpha_-^\tau + \gamma_-\alpha_+^\tau}\Big)\operatorname{div}u^\tau \\
\qquad - \Gamma_3^{\varepsilon,\tau}\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big)\operatorname{div}u^{\varepsilon,\tau} + (\partial_t\Gamma_1^{\varepsilon,\tau} + u^{\varepsilon,\tau}\cdot\nabla\Gamma_1^{\varepsilon,\tau})\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big), \\
\delta F_2 := -\delta u\cdot\nabla u^\tau - \dfrac{1}{\rho^{\varepsilon,\tau}}\nabla\Big(\Gamma_1^{\varepsilon,\tau}\big(P_+(\rho_+^{\varepsilon,\tau}) - P_-(\rho_-^{\varepsilon,\tau})\big)\Big).
\end{cases}
$$

In order to establish the uniform-in-τ convergence estimates, we follow the ideas in Section 3 to overcome the issue caused by the overdamping phenomenon.

Lemma 5.4. *Let $d \geq 3$, $0 < \varepsilon \leq \tau \leq 1$, and the threshold J_τ be given by* (2.1). *Then under the assumption* (1.17), *there holds*

$$\|(\delta Q, \delta u)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-2} \cap \dot{B}^{\frac{d}{2}-1})} + \tau\|\delta Q\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell + \|\delta Q\|_{L_t^1(\dot{B}^{\frac{d}{2}-1})}^h$$
$$+ \sqrt{\tau}\|\delta Q\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1})} + \frac{1}{\sqrt{\tau}}\|\delta u\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-2} \cap \dot{B}^{\frac{d}{2}-1})} + \|\delta u\|_{L_t^1(\dot{B}^{\frac{d}{2}-1})} \tag{5.18}$$
$$\lesssim \sqrt{\varepsilon\tau} + o(1)\|\delta Y\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-2} \cap \dot{B}^{\frac{d}{2}-1})}.$$

Proof. As in Section 3, we split the proof into three parts:

- **Step 1: $\dot{B}^{\frac{d}{2}-2}$-estimates in low frequencies**

We introduce the new damped mode (effective flux)

$$\delta z := \delta u + \frac{\tau}{\rho^{\varepsilon,\tau}}\nabla\delta Q + \tau\Big(\frac{1}{\rho^{\varepsilon,\tau}} - \frac{1}{\rho^\tau}\Big)\nabla P^\tau,$$

so that (5.17) is rewritten as

$$\begin{cases} \partial_t\delta Q - \dfrac{\bar{\Gamma}_2\tau}{\bar{\rho}}\Delta\delta Q = -\bar{\Gamma}_2\mathrm{div}\,z + \delta F_3, \\[2mm] \partial_t\delta z + \dfrac{\delta z}{\tau} = \dfrac{\tau}{\bar{\rho}}\nabla\Big(\dfrac{\bar{\Gamma}_2\tau}{\bar{\rho}}\Delta\delta Q - \bar{\Gamma}_2\mathrm{div}\,z\Big) + \delta F_4, \end{cases} \tag{5.19}$$

where $\bar{\Gamma}_2 > 0$ is the constant state of $\Gamma_2^{\varepsilon,\tau}$, and δF_i ($i = 3, 4$) is defined by

$$\begin{cases} \delta F_3 := -u^{\varepsilon,\tau} \cdot \nabla\delta Q - (\Gamma_2^{\varepsilon,\tau} - \bar{\Gamma}_2)\mathrm{div}\,\delta u \\[2mm] \qquad\quad + \bar{\Gamma}_2\tau\mathrm{div}\Big(\Big(\dfrac{1}{\rho^{\varepsilon,\tau}} - \dfrac{1}{\bar{\rho}}\Big)\nabla\delta Q + \Big(\dfrac{1}{\rho^{\varepsilon,\tau}} - \dfrac{1}{\rho^\tau}\Big)\nabla P^\tau\Big) + \delta F_1, \\[3mm] \delta F_4 := -u^{\varepsilon,\tau} \cdot \nabla\delta u + \dfrac{\tau}{\bar{\rho}}\nabla\delta F_3 \\[2mm] \qquad\quad + \tau\Big(\dfrac{1}{\rho^{\varepsilon,\tau}} - \dfrac{1}{\bar{\rho}}\Big)\nabla\partial_t\delta Q - \tau\partial_t\Big(\dfrac{1}{\rho^{\varepsilon,\tau}}\Big)\nabla\delta Q + \tau\partial_t\Big(\Big(\dfrac{1}{\rho^{\varepsilon,\tau}} - \dfrac{1}{\rho^\tau}\Big)\nabla P^\tau\Big) + \delta F_2. \end{cases}$$

Then by similar arguments used to get (3.13)-(3.16), we deduce from (5.19) and the choice (2.1) of the threshold J_τ that

$$\|(\delta Q, \delta z)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-2})}^\ell + \tau\|\delta Q\|_{L_t^1(\dot{B}^{\frac{d}{2}})}^\ell + \tau\|\partial_t\delta Q\|_{L_t^1(\dot{B}^{\frac{d}{2}-2})}^\ell$$
$$+ \sqrt{\tau}\|\delta Q\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1})}^\ell + \frac{1}{\tau}\|\delta z\|_{L_t^1(\dot{B}^{\frac{d}{2}-2})}^\ell \tag{5.20}$$
$$\lesssim \sqrt{\varepsilon\tau} + \|(\delta F_3, \delta F_4)\|_{L_t^1(\dot{B}^{\frac{d}{2}-2})}^\ell.$$

We first estimate δF_3. From (1.9), (5.14) and the product map $\dot{B}^{\frac{d}{2}-2} \times \dot{B}^{\frac{d}{2}} \to \dot{B}^{\frac{d}{2}-2}$ for $d \geq 3$, one obtains

$$\|u^{\varepsilon,\tau} \cdot \nabla\delta Q + (\Gamma_2^{\varepsilon,\tau} - \bar{\Gamma}_2)\mathrm{div}\,\delta u\|_{L_t^1(\dot{B}^{\frac{d}{2}-2})}$$
$$\lesssim \frac{1}{\sqrt{\tau}}\|u^{\varepsilon,\tau}\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}})}\sqrt{\tau}\|\delta Q\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1})} + \|\Gamma_2^{\varepsilon,\tau} - \bar{\Gamma}_2\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})}\|\delta u\|_{L_t^1(\dot{B}^{\frac{d}{2}-1})} \tag{5.21}$$
$$\lesssim o(1)\Big(\sqrt{\tau}\|\delta Q\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1})} + \|\delta u\|_{L_t^1(\dot{B}^{\frac{d}{2}-1})}\Big).$$

By virtue of (1.9), (2.2), (5.9), (5.15) and (2.4), we also have

$$\tau\Big\|\mathrm{div}\,\big((\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau\big)\Big\|^\ell_{L^1_t(\dot B^{\frac{d}{2}-2})}$$

$$\lesssim \|\delta\rho\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}-1})}\tau\|P^\tau-\bar P\|_{L^1_t(\dot B^{\frac{d}{2}+1})}$$

$$\lesssim o(1)\|(\delta Y,\delta Q)\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}-1})}+\sqrt{\varepsilon\tau}.$$

As in the previous analysis (3.19), the tricky nonlinear term can be estimated as

$$\tau\Big\|\mathrm{div}\,\big((\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\bar\rho})\nabla\delta Q\big)\Big\|^\ell_{L^1_t(\dot B^{\frac{d}{2}-2})}$$

$$\lesssim \tau\Big\|(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\bar\rho})\nabla\delta Q^\ell\Big\|^\ell_{L^1_t(\dot B^{\frac{d}{2}-1})}+\Big\|(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\bar\rho})\nabla\delta Q^h\Big\|^\ell_{L^1_t(\dot B^{\frac{d}{2}-2})}$$

$$\lesssim o(1)\Big(\tau\|\delta Q\|^\ell_{L^1_t(\dot B^{\frac{d}{2}})}+\|\delta Q\|^h_{L^1_t(\dot B^{\frac{d}{2}-1})}\Big).$$

Similarly, one can show

$$\|\delta F_1\|_{L^1_t(\dot B^{\frac{d}{2}-2})}\lesssim \frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde L^2_t(\dot B^{\frac{d}{2}-2})}\sqrt\tau\|Q^\tau\|_{\widetilde L^2_t(\dot B^{\frac{d}{2}})}$$

$$+\Big\|\Big(\Gamma_2^{\varepsilon,\tau}-\frac{\gamma_+\gamma_-P^\tau}{\gamma_+\alpha_-^\tau+\gamma_-\alpha_+^\tau}\Big)\Big\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}-2})}\|u^\tau\|_{L^1_t(\dot B^{\frac{d}{2}+1})}$$

$$+\|P_+(\rho_+^{\varepsilon,\tau})-P_-(\rho_-^{\varepsilon,\tau})\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}-2})}\|u^{\varepsilon,\tau}\|_{L^1_t(\dot B^{\frac{d}{2}+1})}$$

$$+(\|\partial_t\Gamma_1^{\varepsilon,\tau}\|_{L^1_t(\dot B^{\frac{d}{2}})}+\|u^{\varepsilon,\tau}\|_{L^1_t(\dot B^{\frac{d}{2}})}\|\nabla\Gamma_1^{\varepsilon,\tau}\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}})})\|P_+(\rho_+^{\varepsilon,\tau})-P_-(\rho_-^{\varepsilon,\tau})\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}-2})},$$

which implies

$$\|\delta F_1\|_{L^1_t(\dot B^{\frac{d}{2}-2})}\lesssim o(1)\Big(\|(\delta Y,\delta Q)\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}-2})}+\frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde L^2_t(\dot B^{\frac{d}{2}-2})}\Big)+\sqrt{\varepsilon\tau}. \qquad (5.22)$$

Therefore, we have

$$\|\delta F_3\|^\ell_{L^1_t(\dot B^{\frac{d}{2}-2})}\lesssim o(1)\Big(\|(\delta Y,\delta Q)\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}-2})}+\|\delta Q\|_{L^1_t(\dot B^{\frac{d}{2}})}+\sqrt\tau\|\delta Q\|_{\widetilde L^2_t(\dot B^{\frac{d}{2}-1})}$$

$$+\frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde L^2_t(\dot B^{\frac{d}{2}-2})}+\|\delta u\|_{L^1_t(\dot B^{\frac{d}{2}-1})}\Big)+\sqrt{\varepsilon\tau}. \qquad (5.23)$$

We turn to the estimate of δF_4. Similar calculations give

$$\Big\|-u^{\varepsilon,\tau}\cdot\nabla\delta u+\frac{\tau}{\rho^{\varepsilon,\tau}}\nabla\delta F_3\Big\|_{L^1_t(\dot B^{\frac{d}{2}-2})}\lesssim o(1)\|\delta u\|_{L^1_t(\dot B^{\frac{d}{2}-1})}+\|\delta F_3\|_{L^1_t(\dot B^{\frac{d}{2}-2})},$$

and

$$\Big\|\tau\partial_t(\frac{1}{\rho^{\varepsilon,\tau}})\nabla\delta Q\Big\|_{L^1_t(\dot B^{\frac{d}{2}-2})}\lesssim \|\partial_t\rho^{\varepsilon,\tau}\|_{L^1_t(\dot B^{\frac{d}{2}})}\|\delta Q\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}-1})}\lesssim o(1)\|\delta Q\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}-1})}.$$

For the third difficult term in δF_4, we apply (2.2), the product law (2.4) for $d\geq 3$ and the fact

$$\tau(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\bar\rho})\nabla\partial_t\delta Q=\tau\nabla\big((\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\bar\rho})\partial_t\delta Q\big)-\tau\nabla(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\bar\rho})\partial_t\delta Q$$

to have

$$\left\|\tau(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\bar\rho})\nabla\partial_t\delta Q\right\|_{L_t^1(\dot B^{\frac{d}{2}-2})}^{\ell} \lesssim \tau\left\|\nabla((\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\bar\rho})\partial_t\delta Q)\right\|_{L_t^1(\dot B^{\frac{d}{2}-2})}^{\ell} + \tau\left\|\nabla(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\bar\rho})\partial_t\delta Q\right\|_{L_t^1(\dot B^{\frac{d}{2}-2})}^{\ell}$$

$$\lesssim \left\|(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\bar\rho})\partial_t\delta Q\right\|_{L_t^1(\dot B^{\frac{d}{2}-2})}^{\ell} + \left\|\nabla(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\bar\rho})\partial_t\delta Q\right\|_{L_t^1(\dot B^{\frac{d}{2}-2})}^{\ell}$$

$$\lesssim \|\rho^{\varepsilon,\tau}-\bar\rho\|_{\widetilde L_t^\infty(\dot B^{\frac{d}{2}}\cap\dot B^{\frac{d}{2}+1})}\|\partial_t\delta Q\|_{L_t^1(\dot B^{\frac{d}{2}-2})}$$

$$\lesssim o(1)\|\partial_t\delta Q\|_{L_t^1(\dot B^{\frac{d}{2}-2})}.$$

Similarly, the term δF_2 can be easily estimated as follows:

$$\|\delta F_2\|_{L_t^1(\dot B^{\frac{d}{2}-2})} \lesssim \|\delta u\|_{L_t^1(\dot B^{\frac{d}{2}-1})}\|u^\tau\|_{\widetilde L_t^\infty(\dot B^{\frac{d}{2}})} + \|P_+(\rho_+^{\varepsilon,\tau})-P_-(\rho_-^{\varepsilon,\tau})\|_{L_t^1(\dot B^{\frac{d}{2}-1})}$$

$$\lesssim o(1)\|\delta u\|_{L_t^1(\dot B^{\frac{d}{2}-1})} + \sqrt{\varepsilon\tau} + \varepsilon$$

$$\lesssim o(1)\|\delta u\|_{L_t^1(\dot B^{\frac{d}{2}-1})} + \sqrt{\varepsilon\tau}.$$

To bound the term $\tau\partial_t\big((\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau\big)$, noticing that

$$\partial_t\delta\rho = -\mathrm{div}\big(\delta\rho u^{\varepsilon,\tau} + \rho^\tau\delta u\big),$$

we use (1.9), (5.9), (5.15) and (2.6)-(2.7) that

$$\left\|\tau\partial_t\big((\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau\big)\right\|_{L_t^1(\dot B^{\frac{d}{2}-2})}$$

$$\lesssim \|\partial_t\delta\rho\|_{L_t^1(\dot B^{\frac{d}{2}-2})}\|\nabla P^\tau\|_{\widetilde L_t^\infty(\dot B^{\frac{d}{2}})} + \|\delta\rho\|_{\widetilde L_t^\infty(\dot B^{\frac{d}{2}-2})}\tau\|\nabla P^\tau\|_{L_t^1(\dot B^{\frac{d}{2}})}$$

$$\lesssim o(1)\Big(\|(\delta Y,\delta Q)\|_{\widetilde L_t^\infty(\dot B^{\frac{d}{2}-2}\cap\dot B^{\frac{d}{2}-1})} + \|\delta u\|_{L_t^1(\dot B^{\frac{d}{2}-1})}\Big) + \sqrt{\varepsilon\tau}.$$

We thence get

$$\|\delta F_3\|_{L_t^1(\dot B^{\frac{d}{2}-2})}^{\ell} \lesssim o(1)\Big(\|(\delta Y,\delta Q)\|_{\widetilde L_t^\infty(\dot B^{\frac{d}{2}-2})} + \|\delta Q\|_{L_t^1(\dot B^{\frac{d}{2}})} + \sqrt{\tau}\|\delta Q\|_{\widetilde L_t^2(\dot B^{\frac{d}{2}-1})}$$

$$+ \frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde L_t^2(\dot B^{\frac{d}{2}-2})} + \|\delta u\|_{L_t^1(\dot B^{\frac{d}{2}-1})}\Big) + \sqrt{\varepsilon\tau}. \tag{5.24}$$

Substituting the above estimates (5.23)-(5.24) into (5.20) and taking advantage of $\delta u = \delta z - \frac{\tau}{\rho^{\varepsilon,\tau}}\nabla\delta Q - \tau(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau$, we obtain

$$\|(\delta Q,\delta z)\|_{\widetilde L_t^\infty(\dot B^{\frac{d}{2}-2})}^{\ell} + \tau\|\delta Q\|_{L_t^1(\dot B^{\frac{d}{2}})}^{\ell} + \sqrt\tau\|\delta Q\|_{\widetilde L_t^2(\dot B^{\frac{d}{2}-1})}^{\ell} + \frac{1}{\tau}\|\delta z\|_{L_t^1(\dot B^{\frac{d}{2}-2})}^{\ell}$$

$$+ \|\delta u\|_{\widetilde L_t^\infty(\dot B^{\frac{d}{2}-2})}^{\ell} + \frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde L_t^2(\dot B^{\frac{d}{2}-2})}^{\ell} + \|\delta u\|_{L_t^1(\dot B^{\frac{d}{2}-1})}^{\ell}$$

$$\lesssim o(1)\Big(\|(\delta Y,\delta Q)\|_{\widetilde L_t^\infty(\dot B^{\frac{d}{2}-2}\cap\dot B^{\frac{d}{2}-1})} + \tau\|\delta Q\|_{L_t^1(\dot B^{\frac{d}{2}})}^{\ell} \tag{5.25}$$

$$+ \|\delta Q\|_{L_t^1(\dot B^{\frac{d}{2}-1})}^{h} + \sqrt\tau\|\delta Q\|_{\widetilde L_t^2(\dot B^{\frac{d}{2}-1})}$$

$$+ \frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde L_t^2(\dot B^{\frac{d}{2}-2})} + \|\delta u\|_{L_t^1(\dot B^{\frac{d}{2}-1})} + \|\partial_t\delta Q\|_{L_t^1(\dot B^{\frac{d}{2}-2})}\Big) + \sqrt{\varepsilon\tau}.$$

- **Step 2: $\dot B^{\frac{d}{2}-2}$-estimates of $(\delta Q,\delta u)$ in high frequencies**

Applying Δ_j to (5.17), one gets

$$
\begin{cases}
\partial_t \dot{\Delta}_j \delta Q + u^{\varepsilon,\tau} \cdot \nabla \dot{\Delta}_j \delta Q + \Gamma_2^{\varepsilon,\tau} \operatorname{div} \dot{\Delta}_j \delta u = \dot{\Delta}_j \delta F_1 + \delta R_{1,j}, \\[2mm]
\partial_t \dot{\Delta}_j \delta u + u^{\varepsilon,\tau} \cdot \nabla \dot{\Delta}_j \delta u + \dfrac{1}{\rho^{\varepsilon,\tau}} \nabla \dot{\Delta}_j \delta Q + \dfrac{\dot{\Delta}_j \delta u}{\tau} \\[2mm]
\qquad = -\dot{\Delta}_j\big((\frac{1}{\rho^{\varepsilon,\tau}} - \frac{1}{\rho^{\tau}}) \nabla P^{\tau} \big) + \dot{\Delta}_j \delta F_2 + \delta R_{2,j} + \delta R_{3,j},
\end{cases}
$$

with

$$
\begin{cases}
\delta R_{1,j} := [u^{\varepsilon,\tau}, \dot{\Delta}_j] \nabla \delta Q + [\Gamma_2^{\varepsilon,\tau}, \dot{\Delta}_j] \operatorname{div} \dot{\Delta}_j \delta u, \\[2mm]
\delta R_{2,j} := [u^{\varepsilon,\tau}, \dot{\Delta}_j] \nabla \delta u, \\[2mm]
\delta R_{3,j} := [\frac{1}{\rho^{\varepsilon,\tau}}, \dot{\Delta}_j] \nabla \delta Q.
\end{cases}
$$

Similarly to the high-frequency analysis in Section 3.2, one gains

$$
\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^d} \big(\frac{1}{\rho^{\varepsilon,\tau}} |\dot{\Delta}_j \delta Q|^2 + \Gamma_2^{\varepsilon,\tau} |\dot{\Delta}_j \delta u|^2 \big) dx + \frac{1}{\tau} \|\dot{\Delta}_j \delta u\|_{L^2}^2 \\
&\lesssim \Big\| \big(\operatorname{div} u^{\varepsilon,\tau}, \nabla \Gamma_2^{\varepsilon,\tau}, \nabla \frac{1}{\rho^{\varepsilon,\tau}}, \partial_t \frac{1}{\rho^{\varepsilon,\tau}}, \partial_t \Gamma_2^{\varepsilon,\tau} \big) \Big\|_{\widetilde{L}^\infty} \|\dot{\Delta}_j \delta Q\|_{L^2} \|\dot{\Delta}_j \delta u\|_{L^2} \\
&\quad + \Big\| \dot{\Delta}_j \big(\frac{1}{\rho^{\varepsilon,\tau}} - \frac{1}{\rho^{\tau}} \big) \nabla P^{\tau} \Big\|_{L^2} \|\dot{\Delta}_j \delta u\|_{L^2} + \|\dot{\Delta}_j (\delta F_1, \delta F_2)\|_{L^2} \|\dot{\Delta}_j (\delta Q, \delta u)\|_{L^2} \\
&\quad + \|\delta R_{1,j}\|_{L^2} \|\dot{\Delta}_j \delta Q\|_{L^2} + \|(\delta R_{2,j}, \delta R_{3,j})\|_{L^2} \|\dot{\Delta}_j \delta u\|_{L^2},
\end{aligned}
\tag{5.26}
$$

and the cross term

$$
\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^d} \dot{\Delta}_j \delta u \cdot \nabla \dot{\Delta}_j \nabla \delta P dx \\
&\quad + \int_{\mathbb{R}^d} \Big(\frac{1}{\rho^{\varepsilon,\tau}} |\nabla \dot{\Delta}_j \delta Q|^2 - \Gamma_2^{\varepsilon,\tau} |\operatorname{div} \dot{\Delta}_j \delta u_j|^2 + \frac{1}{\tau} \dot{\Delta}_j \delta u \cdot \nabla \dot{\Delta}_j \nabla \delta P \Big) dx \\
&\lesssim \Big(\|u^{\varepsilon,\tau}\|_{\widetilde{L}^\infty} \|\nabla \dot{\Delta}_j \delta u\|_{L^2} + \Big\| \dot{\Delta}_j \big((\frac{1}{\rho^{\varepsilon,\tau}} - \frac{1}{\rho^{\tau}}) \nabla P^{\tau} \big) \Big\|_{L^2} \\
&\quad + \|(\dot{\Delta}_j \delta F_2, \delta R_{2,j}, \delta R_{3,j})\|_{L^2} \Big) \|\nabla \dot{\Delta}_j \nabla \delta P\|_{L^2} \\
&\quad + (\|u^{\varepsilon,\tau}\|_{\widetilde{L}^\infty} \|\nabla \dot{\Delta}_j \delta Q\|_{L^2} + \|\dot{\Delta}_j \delta F_1\|_{L^2} + \|\delta R_{1,j}\|_{L^2}) \|\delta u\|_{L^2}.
\end{aligned}
\tag{5.27}
$$

For all $j \geq J_\tau$, multiplying (5.27) by a suitable small constant and adding the resulting inequality and (5.26) together, we can derive the Lyapunov inequality similar to (3.34)-(3.41) and then show the following L^1-in-time type estimate:

$$
\begin{aligned}
&\tau \|(\delta Q, \delta u)\|^h_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1})} + \|(\delta Q, \delta u)\|^h_{L_t^1(\dot{B}^{\frac{d}{2}-1})} \\
&\lesssim \sqrt{\tau}\varepsilon + (\|u^{\varepsilon,\tau}\|_{L_t^1(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} + \|(\partial_t \rho^{\varepsilon,\tau}, \partial_t \Gamma_2^{\varepsilon,\tau})\|_{L_t^1(\dot{B}^{\frac{d}{2}})}) \tau \|(\delta Q, \delta u)\|^h_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1})} \\
&\quad + \|(\rho^{\varepsilon,\tau} - \bar{\rho}, \Gamma_2^{\varepsilon,\tau} - \bar{\Gamma}_2)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}+1})} \|(\delta Q, \delta u)\|^h_{L_t^1(\dot{B}^{\frac{d}{2}-1})} + \tau \|(\delta F_1, \delta F_2)\|^h_{L_t^1(\dot{B}^{\frac{d}{2}-1})} \\
&\quad + \tau \Big\| (\frac{1}{\rho^{\varepsilon,\tau}} - \frac{1}{\rho^{\tau}}) \nabla P^{\tau} \Big\|^h_{L_t^1(\dot{B}^{\frac{d}{2}-1})} \\
&\quad + \tau \sum_{j \geq J_\tau - 1} 2^{(\frac{d}{2}-1)j} \|(\delta R_{1,j}, \delta R_{2,j}, \delta R_{3,j})\|_{L^2}.
\end{aligned}
$$

By (1.9), (5.9), (5.15), the product laws (2.4) and the commutator estimate (2.5), it is easy to show

$$\tau\Big\|\big(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^{\tau}}\big)\nabla P^{\tau}\Big\|^{h}_{L^{1}_{t}(\dot{B}^{\frac{d}{2}-1})}$$
$$\lesssim\|\delta\rho\|_{\widetilde{L}^{\infty}_{t}(\dot{B}^{\frac{d}{2}-1})}\tau\|P^{\tau}-\bar{P}\|_{L^{1}_{t}(\dot{B}^{\frac{d}{2}+1})}\lesssim o(1)\|(\delta Y,\delta Q)\|_{\widetilde{L}^{\infty}_{t}(\dot{B}^{\frac{d}{2}-1})}+\sqrt{\varepsilon\tau},$$

and

$$\tau\sum_{j\geq J_{\tau}-1}2^{(\frac{d}{2}-1)j}\|(\delta R_{1,j},\delta R_{2,j})\|_{L^{2}}$$
$$\lesssim\|\nabla u^{\varepsilon,\tau}\|_{L^{1}_{t}(\dot{B}^{\frac{d}{2}})}\|(\delta Q,\delta u)\|_{\widetilde{L}^{\infty}_{t}(\dot{B}^{\frac{d}{2}-1})}\lesssim o(1)\|(\delta Q,\delta u)\|_{\widetilde{L}^{\infty}_{t}(\dot{B}^{\frac{d}{2}-1})}.$$

For the tricky commutator term $R_{3,j}$, we have

$$\tau\sum_{j\geq J_{\tau}-1}2^{(\frac{d}{2}-1)j}\|R_{3,j}\|_{L^{1}_{t}(L^{2})}$$
$$\lesssim\tau\sum_{j\geq J_{\tau}-1}2^{\frac{d}{2}j}\Big\|[\frac{1}{\rho^{\varepsilon,\tau}},\dot{\Delta}_{j}]\nabla\delta Q^{\ell}\Big\|_{L^{1}_{t}(L^{2})}+\sum_{j\geq J_{\tau}-1}2^{(\frac{d}{2}-1)j}\Big\|[\frac{1}{\rho^{\varepsilon,\tau}},\dot{\Delta}_{j}]\nabla\delta Q^{h}\Big\|_{L^{1}_{t}(L^{2})}$$
$$\lesssim\|\nabla\rho^{\varepsilon,\tau}\|_{\widetilde{L}^{\infty}_{t}(\dot{B}^{\frac{d}{2}})}(\tau\|\delta Q^{\ell}\|_{L^{1}_{t}(\dot{B}^{\frac{d}{2}})}+\|\delta Q^{h}\|_{L^{1}_{t}(\dot{B}^{\frac{d}{2}-1})})$$
$$\lesssim o(1)\Big(\tau\|\delta Q\|^{\ell}_{L^{1}_{t}(\dot{B}^{\frac{d}{2}})}+\|\delta Q\|^{h}_{L^{1}_{t}(\dot{B}^{\frac{d}{2}-1})}\Big).$$

For δF_{1} and δF_{2}, similar computations give rise to

$$\|(\delta F_{1},\delta F_{2})\|_{L^{1}_{t}(\dot{B}^{\frac{d}{2}-1})}\lesssim\|\delta u\|_{L^{1}_{t}(\dot{B}^{\frac{d}{2}-1})}\|(P^{\tau}-\bar{P},u^{\tau})\|_{\widetilde{L}^{\infty}_{t}(\dot{B}^{\frac{d}{2}+1})}$$
$$+\Big\|\Gamma^{\varepsilon,\tau}_{2}-\frac{\gamma_{+}\gamma_{-}P^{\tau}}{\gamma_{+}\alpha^{\tau}_{-}+\gamma_{-}\alpha^{\tau}_{+}}\Big\|_{\widetilde{L}^{\infty}_{t}(\dot{B}^{\frac{d}{2}-1})}\|u^{\tau}\|_{L^{1}_{t}(\dot{B}^{\frac{d}{2}+1})}$$
$$+(\|u^{\varepsilon,\tau}\|_{L^{1}_{t}(\dot{B}^{\frac{d}{2}+1})}+\|\partial_{t}\Gamma^{\varepsilon,\tau}_{1}+u^{\varepsilon,\tau}\cdot\nabla\Gamma^{\varepsilon,\tau}_{1}\|_{L^{1}_{t}(\dot{B}^{\frac{d}{2}})})\|P_{+}(\rho^{\varepsilon,\tau}_{+})-P_{-}(\rho^{\varepsilon,\tau}_{-})\|_{\widetilde{L}^{\infty}_{t}(\dot{B}^{\frac{d}{2}-1})} \tag{5.28}$$
$$+\|P_{+}(\rho^{\varepsilon,\tau}_{+})-P_{-}(\rho^{\varepsilon,\tau}_{-})\|_{L^{1}_{t}(\dot{B}^{\frac{d}{2}-1}\cap\dot{B}^{\frac{d}{2}})}$$
$$\lesssim o(1)\Big(\|(\delta Y,\delta Q)\|_{\widetilde{L}^{\infty}_{t}(\dot{B}^{\frac{d}{2}-1})}+\|\delta u\|_{L^{1}_{t}(\dot{B}^{\frac{d}{2}-1})}\Big)+\sqrt{\varepsilon\tau}+\varepsilon.$$

We thus get

$$\|(\delta Q,\delta u)\|^{h}_{\widetilde{L}^{\infty}_{t}(\dot{B}^{\frac{d}{2}-2})}+\tau\|(\delta Q,\delta u)\|^{h}_{\widetilde{L}^{\infty}_{t}(\dot{B}^{\frac{d}{2}-1})}$$
$$+\|(\delta Q,\delta u)\|^{h}_{L^{1}_{t}(\dot{B}^{\frac{d}{2}-1})}+\sqrt{\tau}\|\delta Q\|^{h}_{\widetilde{L}^{2}_{t}(\dot{B}^{\frac{d}{2}-1})}+\frac{1}{\sqrt{\tau}}\|u\|^{h}_{\widetilde{L}^{2}_{t}(\dot{B}^{\frac{d}{2}-2})} \tag{5.29}$$
$$\lesssim o(1)\Big(\|(\delta Q,\delta u)\|_{\widetilde{L}^{\infty}_{t}(\dot{B}^{\frac{d}{2}-1})}+\tau\|\delta Q\|^{\ell}_{L^{1}_{t}(\dot{B}^{\frac{d}{2}})}$$
$$+\|\delta Q\|^{h}_{L^{1}_{t}(\dot{B}^{\frac{d}{2}-1})}+\tau\|\delta u\|_{L^{1}_{t}(\dot{B}^{\frac{d}{2}-1})}\Big)+\sqrt{\varepsilon\tau}.$$

- **Step 3: $\dot{B}^{\frac{d}{2}-1}$-estimates of $(\delta Q,\delta u)$ in all frequencies**

We need to further establish the uniform $\dot{B}^{\frac{d}{2}-1}$-bounds. To this end, owing to (5.26), we obtain the

L^2-in-time estimate

$$\|(\delta Q, \delta u)\|_{\widetilde{L}_t^\infty(\dot B^{\frac d2-1})} + \frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}$$

$$\lesssim \sqrt{\varepsilon\tau} + \Big(\|u^{\varepsilon,\tau}\|_{L_t^1(\dot B^{\frac d2})} + \|(\partial_t\rho^{\varepsilon,\tau},\partial_t\Gamma_2^{\varepsilon,\tau})\|_{L_t^1(\dot B^{\frac d2})}\Big)^{\frac12}\|(\delta Q,\delta u)\|_{\widetilde{L}_t^\infty(\dot B^{\frac d2-1})}^h$$

$$+ \|(\rho^{\varepsilon,\tau}-\bar\rho,\Gamma_2^{\varepsilon,\tau}-\bar\Gamma_2)\|_{\widetilde{L}_t^\infty(\dot B^{\frac d2+1})}^{\frac12}\Big(\sqrt\tau\|\delta Q\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}^h\Big)^{\frac12}\Big(\frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}^h\Big)^{\frac12}$$

$$+ \|(\delta F_1,\delta F_2)\|_{L_t^1(\dot B^{\frac d2-1})}^{\frac12}\|(\delta Q,\delta u)\|_{\widetilde{L}_t^\infty(\dot B^{\frac d2-1})}^{\frac12} + \Big\|(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau\Big\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}^{\frac12}\|\delta u\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}^{\frac12}$$

$$+ \sum_{j\in\mathbb Z}2^{(\frac d2-1)j}\Big(\int_0^t\big(\|\delta R_{1,j}\|_{L^2}\|\dot\Delta_j\delta Q\|_{L^2} + \|(\delta R_{2,j},\delta R_{3,j})\|_{L^2}\|\dot\Delta_j u\|_{L^2}\big)ds\Big)^{\frac12}.$$

One has

$$\Big\|(\frac{1}{\rho^{\varepsilon,\tau}}-\frac{1}{\rho^\tau})\nabla P^\tau\Big\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}^{\frac12}\|\delta u\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}^{\frac12}$$

$$\lesssim \|\delta\rho\|_{\widetilde{L}_t^\infty(\dot B^{\frac d2-1})}\Big(\sqrt\tau\|P^\tau-\bar P\|_{\widetilde{L}_t^2(\dot B^{\frac d2+1})}\Big)^{\frac12}\Big(\frac{1}{\sqrt\tau}\|\|\delta u\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}\Big)^{\frac12}$$

$$\lesssim o(1)\Big(\|(\delta Y,\delta Q)\|_{\widetilde{L}_t^\infty(\dot B^{\frac d2-1})} + \frac{1}{\sqrt\tau}\|\|\delta u\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}\Big) + \sqrt{\varepsilon\tau}.$$

Concerning the commutator terms, we have

$$\sum_{j\in\mathbb Z}2^{(\frac d2-1)j}\Big(\int_0^t\big(\|\delta R_{1,j}\|_{L^2}\|\dot\Delta_j\delta Q\|_{L^2} + \|\delta R_{2,j}\|_{L^2}\|\dot\Delta_j u\|_{L^2}\big)ds\Big)^{\frac12}$$

$$\lesssim \Big(\frac{1}{\sqrt\tau}\|\nabla u^{\varepsilon,\tau}\|_{\widetilde{L}_t^2(\dot B^{\frac d2})}\|\delta Q\|_{\widetilde{L}_t^\infty(\dot B^{\frac d2-1})}\sqrt\tau\|\delta Q\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}\Big)^{\frac12}$$

$$+ \Big(\frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}\|\nabla\Gamma_2^{\varepsilon,\tau}\|_{\widetilde{L}_t^\infty(\dot B^{\frac d2+1})}\sqrt\tau\|\delta Q\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}\Big)^{\frac12} + \Big(\|\nabla u^{\varepsilon,\tau}\|_{L_t^1(\dot B^{\frac d2})}\|\delta u\|_{\widetilde{L}_t^\infty(\dot B^{\frac d2-1})}^2\Big)^{\frac12}$$

$$+ \Big(\|\nabla\rho^{\varepsilon,\tau}\|_{\widetilde{L}_t^\infty(\dot B^{\frac d2})}\sqrt\tau\|\delta Q\|_{\widetilde{L}_t^\infty(\dot B^{\frac d2-1})}\frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}\Big)^{\frac12}$$

$$\lesssim o(1)\Big(\|(\delta Q,\delta u)\|_{\widetilde{L}_t^\infty(\dot B^{\frac d2-1})} + \sqrt\tau\|\delta Q\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})} + \frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}\Big).$$

Gathering (5.28) and the above three estimates, we have

$$\|(\delta Q,\delta u)\|_{\widetilde{L}_t^\infty(\dot B^{\frac d2-1})} + \frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}$$
$$\lesssim o(1)\Big(\|(\delta Y,\delta Q,\delta u)\|_{\widetilde{L}_t^\infty(\dot B^{\frac d2-1})} + \|\delta Q\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})} + \frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde{L}_t^2(\dot B^{\frac d2-1})}\Big) + \sqrt{\varepsilon\tau}. \tag{5.30}$$

- **Step 4: Proof of convergence rate**

Finally, as required in (5.25), one needs to estimate $\partial_t\delta Q$. We make use of the equation $(5.17)_1$, (5.21) and (5.22) to get

$$\|\partial_t\delta Q\|_{L_t^1(\dot B^{\frac d2-2})} \lesssim \|u^{\varepsilon,\tau}\cdot\nabla\delta Q + (\Gamma_2^{\varepsilon,\tau}-\bar\Gamma_2)\mathrm{div}\,\delta u\|_{L_t^1(\dot B^{\frac d2-2})}$$
$$+ \|\mathrm{div}\,\delta u\|_{L_t^1(\dot B^{\frac d2-2})} + \|\delta F_1\|_{L_t^1(\dot B^{\frac d2-2})}$$
$$\lesssim o(1)\Big(\|(\delta Y,\delta Q)\|_{\widetilde{L}_t^\infty(\dot B^{\frac d2-2})} + \frac{1}{\sqrt\tau}\|\delta u\|_{\widetilde{L}_t^2(\dot B^{\frac d2-2})}\Big)$$
$$+ \|\delta u\|_{L_t^1(\dot B^{\frac d2-1})} + \sqrt{\varepsilon\tau}. \tag{5.31}$$

Combining (5.4), (5.25), (5.29), (5.30) and (5.31) together, we end up with (5.18) which completes the proof of Lemma 5.4. □

5.2 Time-relaxation limit: System (K_τ) to System (PM)

This section is devoted to the proof of (1.20) in Theorem 1.4. Define the error variables

$$(\delta\beta_\pm, \delta\varrho_\pm, \delta\varrho, \delta\Pi, \delta v) := (\beta_\pm^\tau - \beta_\pm, \varrho_\pm^\tau - \varrho_\pm, \varrho^\tau - \varrho, \Pi^\tau - \Pi, v^\tau - v).$$

First, similarly to (5.1), instead of $\delta\beta$, we need to estimate the variable

$$\delta Z := \frac{\beta_+^\tau \varrho_+^\tau}{\varrho^\tau} - \frac{\beta_+ \varrho_+}{\varrho},$$

where the initial data of δz is

$$\delta Z|_{t=0} = Z_0^\tau - Z_0, \quad Z_0^\tau := \frac{\alpha_{+,0}^\tau \rho_{+,0}^\tau}{\alpha_{+,0}^\tau \rho_{+,0}^\tau + \alpha_{-,0}^\tau \rho_{-,0}^\tau}, \quad Z_0 := \frac{\beta_{+,0} \varrho_{+,0}}{\beta_{+,0} \varrho_{+,0} + \beta_{-,0} \varrho_{-,0}}. \tag{5.32}$$

Indeed, arguing similarly as in Lemma 5.2, we obtain from (K_τ) and (PM) that

$$\begin{cases}
\delta\beta_+ = \dfrac{1}{\beta_-^\tau \varrho_+^\tau \varrho_- + \beta_+ \varrho_+ \varrho_-^\tau} \left(\varrho^\tau \varrho \delta Z - \beta_+ \beta_- \varrho_- \delta\varrho_+ + \beta_+^\tau \beta_- \varrho_+ \delta\varrho_- \right), \\[2mm]
\delta\varrho^\tau = \dfrac{\varrho_+^\tau - \varrho_-^\tau}{\beta_- \varrho_+^\tau \varrho_- + \beta_+ \varrho_+ \varrho_-^\tau} \left(\varrho^\tau \varrho \delta Z - \beta_+ \beta_- \varrho_- \delta\varrho_+ + \beta_+ \beta_- \varrho_+ \delta\varrho_- \right) + \beta_+ \delta\varrho_+ + \beta_- \delta\varrho_-^\tau, \\[2mm]
\delta\Pi = \delta\varrho_+ \displaystyle\int_0^1 P_+'(\theta\varrho_+^\tau + (1-\theta)\varrho_+) d\theta = \delta\varrho_- \int_0^1 P_-'(\theta\varrho_-^\tau + (1-\theta)\varrho_-) d\theta,
\end{cases} \tag{5.33}$$

which leads to

$$\begin{cases}
\|\delta\varrho_\pm\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1})} + \|\delta\varrho_\pm\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})} \sim \|\delta\Pi\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1})} + \|\delta\Pi\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})}, \\[2mm]
\|\delta\beta_\pm\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1})} \lesssim \|(\delta Z, \delta\Pi)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1})}.
\end{cases} \tag{5.34}$$

It is therefore sufficient to estimate $(\delta\Pi, \delta v, \delta Z)$ to recover the information on all the error unknowns.

Next, note that δZ satisfies the transport equation

$$\partial_t \delta Z + v^\tau \cdot \nabla \delta Z = -\delta v \cdot \nabla \frac{\beta_+ \varrho_+}{\varrho}. \tag{5.35}$$

Using Lemma 2.7, (4.22) and the product law (2.4), we get

$$\begin{aligned}
\|\delta Z\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1} \cap \dot{B}^{\frac{d}{2}})} &\lesssim \exp\left(\|v^\tau\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})}\right) \|\delta v\|_{L_t^1(\dot{B}^{\frac{d}{2}})} \left\|\nabla \frac{\beta_+ \varrho_+}{\varrho}\right\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}} \cap \dot{B}^{\frac{d}{2}+1})} \\
&\lesssim o(1) \|\delta v\|_{L_t^1(\dot{B}^{\frac{d}{2}})}.
\end{aligned} \tag{5.36}$$

Then, we perform the key estimates of $\delta\Pi$. From (K_τ), it is easy to see

$$\partial_t \Pi^\tau + v^\tau \cdot \nabla \Pi^\tau = \frac{\gamma_+ \gamma_- \Pi^\tau}{\gamma_+ \beta_-^\tau + \gamma_- \beta_+^\tau} \text{div}\left(\frac{\nabla \Pi^\tau}{\varrho^\tau}\right) - \frac{\gamma_+ \gamma_- \Pi^\tau}{\gamma_+ \beta_-^\tau + \gamma_- \beta_+^\tau} \text{div } z^\tau, \qquad z^\tau := v^\tau + \frac{\nabla \Pi^\tau}{\varrho^\tau}.$$

Thence by the above equation and (PM), $\delta\Pi$ satisfies

$$
\begin{aligned}
\partial_t \delta\Pi - \bar{c}\Delta\delta\Pi = &-v^\tau \cdot \nabla\delta\Pi - \delta v \cdot \nabla\Pi \\
&+ \Big(\frac{\gamma_+\gamma_-\Pi^\tau}{\gamma_+\beta_+^\tau + \gamma_-\beta_-^\tau} - \frac{\gamma_+\gamma_-\Pi}{\gamma_+\beta_+ + \gamma_-\beta_-}\Big)\mathrm{div}\,\Big(\frac{\nabla\Pi^\tau}{\varrho^\tau}\Big) \\
&+ \frac{\gamma_+\gamma_-\Pi}{\gamma_+\beta_+ + \gamma_-\beta_-}\mathrm{div}\,\Big((\frac{1}{\varrho^\tau} - \frac{1}{\varrho})\nabla\Pi^\tau\Big) - \frac{\gamma_+\gamma_-\Pi^\tau}{\gamma_+\beta_-^\tau + \gamma_-\beta_+^\tau}\mathrm{div}\,z^\tau,
\end{aligned}
\tag{5.37}
$$

with the constant $\bar{c} := \frac{\gamma_+\gamma_- P_+(\bar{\rho}_+)}{(\gamma_+\bar{\alpha}_- + \gamma_-\bar{\alpha}_+)\bar{\rho}} > 0$. We mention that the convergence rate τ is $\mathcal{O}(\tau)$ bound comes from the uniform estimate (4.22) of the effective unknown z^τ. Indeed, by Lemma 2.6, the uniform estimate (4.22), the smallness of the initial data (1.7), the product laws (2.3) and the composition estimates (2.6)-(2.7), one obtains

$$
\begin{aligned}
\|\delta\Pi\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1})} + \|\delta\Pi\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})} &\lesssim \tau + \|v^\tau\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}})}\|\delta\Pi\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}-1})} + \|\delta v\|_{L_t^1(\dot{B}^{\frac{d}{2}})}\|\Pi - \bar{P}\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})} \\
&\quad + \|\delta\Pi\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}})}\|\Pi^\tau - \bar{P}\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}+1})} + \Big\|(\frac{1}{\varrho^\tau} - \frac{1}{\varrho})\nabla\Pi^\tau\Big\|_{L_t^1(\dot{B}^{\frac{d}{2}})} + \|z^\tau\|_{L_t^1(\dot{B}^{\frac{d}{2}})} \\
&\lesssim o(1)\Big(\|\delta Z\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})} + \|(\delta\varrho_\pm, \delta\Pi)\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1})} + \|(\delta\varrho_\pm, \delta\Pi)\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})}\Big) + \tau,
\end{aligned}
\tag{5.38}
$$

where we have used the key fact

$$
\begin{aligned}
\Big\|(\frac{1}{\varrho^\tau} - \frac{1}{\varrho})\nabla\Pi^\tau\Big\|_{L_t^1(\dot{B}^{\frac{d}{2}})} &\lesssim \|\delta Z\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})}\|\Pi^\tau - \bar{P}\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})} + \|\delta\varrho_\pm\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}})}\|\Pi^\tau - \bar{P}\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}+1})} \\
&\lesssim o(1)\Big(\|\delta Z\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})} + \|\delta\varrho_\pm\|_{\widetilde{L}_t^2(\dot{B}^{\frac{d}{2}})}\Big),
\end{aligned}
\tag{5.39}
$$

derived from (2.4), (4.22) and (5.33). Gathering (5.34) and (5.38) together, we get

$$
\|\delta\Pi\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1})} + \|\delta\Pi\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})} \lesssim o(1)\Big(\|\delta Z\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})} + \|\delta\Pi\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})}\Big) + \tau.
\tag{5.40}
$$

For the error unknown δv, in view of (4.22), (5.39) and $\delta v = (\frac{1}{\varrho^\tau} - \frac{1}{\varrho})\nabla\Pi^\tau - \frac{1}{\varrho}\nabla\delta\Pi + z^\tau$, it can be bounded by

$$
\begin{aligned}
\|\delta v\|_{L_t^1(\dot{B}^{\frac{d}{2}})} &\lesssim \|\delta\Pi\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})} + \Big\|(\frac{1}{\varrho^\tau} - \frac{1}{\varrho})\nabla\Pi^\tau\Big\|_{L_t^1(\dot{B}^{\frac{d}{2}})} + \|z^\tau\|_{L_t^1(\dot{B}^{\frac{d}{2}})} \\
&\lesssim \|\delta Z\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}})} + \|\delta\Pi\|_{\widetilde{L}_t^\infty(\dot{B}^{\frac{d}{2}-1})} + \|\delta\Pi\|_{L_t^1(\dot{B}^{\frac{d}{2}+1})} + \tau.
\end{aligned}
\tag{5.41}
$$

The combination of estimate (5.34) and inequalities (5.36)-(5.40) gives rise to estimate (1.20), which completes the proof of Theorem 1.4.

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