Strong relaxation limit and uniform time asymptotics of the Jin-Xin model in the L^p framework

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Abstract

We investigate the diffusive relaxation limit and the time-asymptotic stability of the Jin-Xin model toward viscous conservation laws in $\mathbb{R}^d$ with $d \geq 1$. First, we establish uniform regularity estimates with respect to both the time and the relaxation parameter $\varepsilon > 0$, for initial data in hybrid Besov spaces based on general L^p-norms. This uniformity enables us to derive $\mathcal{O}(\varepsilon)$ bounds on the difference between solutions of the viscous conservation law and its associated Jin-Xin approximation, thus justifying the strong convergence of the Jin-Xin hyperbolic relaxation. Furthermore, under an additional condition on the initial data, for instance, that the low frequencies belong to $L^{p/2}(\mathbb{R}^d)$, we show that the $L^p(\mathbb{R}^d)$-norm of the solution to the Jin-Xin model decays at the optimal rate $(1+t)^{-d/2p}$ while the $L^p(\mathbb{R}^d)$-norm of its difference with the solution of the associated viscous conservation law decays at the enhanced rate $\varepsilon(1+t)^{-d/2p-1/2}$.

Keywords: Jin-Xin approximation; Hyperbolic relaxation; Diffusion limit; Asymptotic behavior; Partially dissipative systems; Littewood-Paley decomposition

1 Introduction

1.1 Presentation of the model

The return to equilibria of perturbed systems (relaxation phenomenon) occurs in a wide variety of physical situations, such as the blood flow with friction, non-equilibrium gas dynamics, kinetic theory, traffic flows, etc., see [34, 36, 37]. Liu [30] first studied the relaxation of 2×2 hyperbolic systems in one spatial dimension. Then, Chen, Levermore & Liu [8, 9] continued this investigation in the context of weak solutions. In 1995, Jin and Xin [28] introduced relaxation schemes for systems of conservation laws in arbitrary space dimensions that have been widely employed in numerical analysis and scientific computing, e.g., cf. [23, 24, 26]. We will be concentrating on their relaxation procedure here.

We investigate the following diffusely scaled version of the Jin-Xin system (cf. [27, 28]):

$$\begin{cases} \dfrac{\partial}{\partial t}u + \displaystyle\sum_{i=1}^{d} \dfrac{\partial}{\partial x_i} v_i = 0, \\ \varepsilon^2 \dfrac{\partial}{\partial t} v_i + A_i \dfrac{\partial}{\partial x_i} u = -\big(v_i - f_i(u)\big), \qquad i = 1, 2, ..., d, \end{cases} \tag{1.1}$$

where $t>0$ and $\boldsymbol{x}\in\mathbb{R}^d$ denote the time and space variables, $d\geq 1$ is the dimension and $\varepsilon>0$ stands for the relaxation parameter. The unknowns are $u=u(t,\boldsymbol{x})\in\mathbb{R}^n$ and $\boldsymbol{v}=(v_1,v_2,\cdots,v_d)$ with $v_i=v_i(t,\boldsymbol{x})\in\mathbb{R}^n$. The nonlinear term $f(u)=(f_1(u),f_2(u),\cdots,f_d(u))$ with $f_i(u):\mathbb{R}^n\to\mathbb{R}^n$ depends on u smoothly and satisfies $f_i(0)=\partial_{u_k}f_i(0)=0$ with $k=1,...,d$. The constant coefficient matrices A_i are taken as $A_i=a_iI_n$ with $a_i>0$ and I_n the unit matrix.

As $\varepsilon\to 0$, the dynamics of System (1.1) is formally governed by the viscous conservation law

$$\frac{\partial}{\partial t}u^*+\sum_{i=1}^{d}\frac{\partial}{\partial x_i}f_i(u^*)=\sum_{i=1}^{d}\frac{\partial}{\partial x_i}(A_i\frac{\partial}{\partial x_i}u^*),\tag{1.2}$$

and Darcy's law

$$v_i^*=-A_i\frac{\partial}{\partial x_i}u^*+f_i(u^*).\tag{1.3}$$

An explicit example of (1.2) is the two-dimensional Burgers equations

$$\frac{\partial}{\partial t}u+\frac{\partial}{\partial x_1}(u_1u)+\frac{\partial}{\partial x_2}(u_2u)-\Delta u=0,\tag{1.4}$$

where $u=(u_1,u_2)\in\mathbb{R}^2$. System (1.4) is similar to the pressureless incompressible Navier-Stokes equations used to model unsaturated flows [13]. The Jin-Xin approximation of (1.4) reads

$$\begin{cases}\frac{\partial}{\partial t}u+\frac{\partial}{\partial x_1}v_1+\frac{\partial}{\partial x_2}v_2=0,\\ \varepsilon^2\frac{\partial}{\partial t}v_1+\frac{\partial}{\partial x_1}u=-(v_1-u_1u),\\ \varepsilon^2\frac{\partial}{\partial t}v_2+\frac{\partial}{\partial x_2}u=-(v_2-u_2u).\end{cases}\tag{1.5}$$

The key point of the approximation (1.5) is that it modifies the nature of the system under study. Indeed, (1.4) is parabolic while (1.5) is purely hyperbolic. Therefore, if the approximation is valid in a sufficiently strong sense, this procedure justifies the use of hyperbolic approaches to study parabolic equations. For instance, the reader may refer to [28] for details concerning the relevance of the Jin-Xin approximation for numerical analysis. Here, our goal is to extend the validity of the diffusive Jin-Xin approximation in the context of strong solutions being perturbations of small initial data.

There has been a lot of studies devoted to the mathematical analysis for the Jin-Xin relaxation system in the one-dimensional setting. Chern [12] investigated the long-time effect of relaxation and proved that the corresponding solution tends to a diffusion wave asymptotically-in-time in terms of the Chapman-Enskog expansion. Natalini [33] and Bianchini [3] justified the relaxation convergence of the Jin-Xin approximation of hyperbolic conservation laws. Jin and Liu [27] studied the relaxation limit of Jin-Xin system under the diffusive scaling for initial data around traveling waves. Bouchut, Guarguaglini & Natalini [6] considered the diffusive relaxation of BGK type approximations for the Jin-Xin system. Mei and Rubino [32] got the time-convergence rates of solutions to the initial boundary value problem for the Jin-Xin system toward traveling waves on the half line. Orive and Zuazua [35] reformulated the system in a damped wave equation and derived algebraic time-decay rates of solutions on the real line. Huang, Pan & Wang [25] obtained the nonlinear stability of contact waves for the Jin-Xin model with the decay rate $(1+t)^{-\frac{1}{4}}$. Bianchini [4] derived the sharp time-decay estimates of solutions to the Jin-Xin system, in the case $f'(0)\neq 0$, which are uniform with respect to ε and provided the convergence to a nonlinear heat equation both asymptotically in time and in the relaxation limit.

For the high-dimensional case, there are fewer results. Crin-Barat and Shou [17] justified the uniform convergence of the multi-dimensional Jin-Xin system (1.1) toward viscous conservation laws (1.2) as $\varepsilon\to 0$ with an explicit rate in L^2-type Besov spaces.

To the best of our knowledge, the long-time asymptotics of the multi-dimensional Jin-Xin system (1.19) has not been established in previous references. Moreover, since the Cauchy problem of the limiting viscous conservation law (1.2) is globally well-posed in general Besov spaces of L^p-type, it is relevant to study the relaxation limit of the Jin-Xin system in a framework adapted to the limiting system. That is the two problem that we tackle in the present paper.

1.2 Main results

Our main results read as follows. First we state a global well-posedness result for viscous conservation laws in a L^p framework. For the definition of Besov spaces, see Section 2.

Theorem 1.1. *Let $d\geq 1$, $n\geq 1$ and $1\leq p\leq\infty$. There exists a generic constant $\eta_0>0$ such that if u_0^* satisfies $u_0^*\in \dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}$ and*

$$\|u_0^*\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}}\leq\eta_0, \tag{1.6}$$

then, System (1.2) has a unique global solution $u^*\in\mathcal{C}(\mathbb{R}^+;\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})$. Furthermore, it holds that

$$\|u^*\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})}+\|u^*\|_{L^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+1}_{p,1}\cap\dot{B}^{\frac{d}{p}+2}_{p,1})}+\|\boldsymbol{v}^*\|_{L^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}\leq C\|u_0^*\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}}, \tag{1.7}$$

where $\boldsymbol{v}^*$ is given by Darcy's law (1.3), and $C>0$ is a constant independent of time.

Next, we establish the global well-posedness of solutions for System (1.1) in a hybrid L^p-L^2 framework.

Theorem 1.2. *Assume $\varepsilon>0$ and $n\geq 1$. Let p satisfy*

$$\begin{cases} 1\leq p\leq 4, & \text{if}\quad d=1,2,\\ \dfrac{6}{5}\leq p\leq 4, & \text{if}\quad d=3,\\ \dfrac{2d}{d+2}\leq p\leq\dfrac{2d}{d-2}, & \text{if}\quad d\geq 4,\end{cases} \tag{1.8}$$

and set the threshold J_ε between low and high frequencies:

$$J_\varepsilon=-[\log\varepsilon]+k_0 \tag{1.9}$$

with some generic integer k_0. There exists a constant $\eta_1>0$ independent of ε such that for initial datum $(u_0,\boldsymbol{v}_0)$ satisfying $u_0^\ell\in\dot{B}^{\frac{d}{p}-1}_{p,1}$, $\boldsymbol{v}_0^\ell\in\dot{B}^{\frac{d}{p}}_{p,1}$, $(u_0^h,\boldsymbol{v}_0^h)\in\dot{B}^{\frac{d}{2}}_{2,1}$ and

$$\mathcal{X}_{p,0}\triangleq\|u_0\|^\ell_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}}+\varepsilon^2\|\boldsymbol{v}_0\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1}}+(1+\varepsilon)\|u_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}+\varepsilon(1+\varepsilon)\|\boldsymbol{v}_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\leq\eta_1, \tag{1.10}$$

then, System (1.1) *admits a unique global strong solution* $(u,\boldsymbol{v})$ *satisfying* $u^\ell\in\mathcal{C}(\mathbb{R}^+;\dot B^{\frac{d}{p}-1}_{p,1})$, $\boldsymbol{v}^\ell\in\mathcal{C}(\mathbb{R}^+;\dot B^{\frac{d}{p}}_{p,1})$, $(u^h,\boldsymbol{v}^h)\in\mathcal{C}(\mathbb{R}^+;\dot B^{\frac{d}{2}}_{2,1})$ *and*

$$
\begin{aligned}
&\|u\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1}\cap\dot B^{\frac{d}{p}}_{p,1})}+\|u\|^\ell_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1}\cap\dot B^{\frac{d}{p}+2}_{p,1})}+(1+\varepsilon)\|u\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}}_{2,1})}+(\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2})\|u\|^h_{L^1_t(\dot B^{\frac{d}{2}}_{2,1})}\\
&\quad+\varepsilon^2\|\boldsymbol{v}\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}}_{p,1}\cap\dot B^{\frac{d}{p}+1}_{p,1})}+\|\boldsymbol{v}\|^\ell_{L^1_t(\dot B^{\frac{d}{p}}_{p,1}\cap\dot B^{\frac{d}{p}+1}_{p,1})}+(\varepsilon+\varepsilon^2)\|\boldsymbol{v}\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{2}}_{2,1})}+(1+\frac{1}{\varepsilon})\|\boldsymbol{v}\|^h_{L^1_t(\dot B^{\frac{d}{2}}_{2,1})}\\
&\quad\le C\mathcal{X}_{p,0},
\end{aligned}
\tag{1.11}
$$

where $C>0$ *is a constant independent of time and* ε.

Then, thanks to the uniform bounds obtained in Theorem 1.2 we justify the strong relaxation limit of System (1.1) with an explicit converge rate in the L^p framework.

Theorem 1.3. *For* $d\ge1$ *and* $\varepsilon>0$, *let* $(u,\boldsymbol{v})$ *be the solution of System* (1.1) *associated to the initial datum* $(u_0,\boldsymbol{v}_0)$ *given by Theorem 1.2,* u^* *be the solution of System* (1.2) *associated to the initial datum* u_0^* *given by Theorem 1.1, and* $\boldsymbol{v}^*$ *be given by Darcy's law* (1.3)*. Assume further that*

$$
\begin{cases}
p=2, & \text{if}\quad d=1,\\
2\le p\le 4, & \text{if}\quad d=2,3,\\
2\le p\le \dfrac{2d}{d-2}, & \text{if}\quad d\ge4,
\end{cases}
\tag{1.12}
$$

and

$$
\varepsilon\|\boldsymbol{v}_0^\ell\|_{\dot B^{\frac{d}{p}}_{p,1}}\ \text{is uniformly bounded.}
\tag{1.13}
$$

Then, the following convergence estimates hold:

$$
\|u-u^*\|_{\widetilde L^\infty(\mathbb{R}^+;\dot B^{\frac{d}{p}-1}_{p,1})}+\|\boldsymbol{v}-\boldsymbol{v}^*\|_{L^1(\mathbb{R}^+;\dot B^{\frac{d}{p}}_{p,1})}\le C\|u_0-u_0^*\|_{\dot B^{\frac{d}{p}-1}_{p,1}}+C\varepsilon,
\tag{1.14}
$$

where $C>0$ *is a constant independent of time and* ε.

Finally, we exhibit sharp decay estimates for the solution of (1.1) and for its difference with the solution of (1.2) when u and u^* are associated to the same initial data.

Theorem 1.4. *For* $d\ge1$ *and* $0<\varepsilon<1$, *let* $(u,\boldsymbol{v})$ *be the global solution to* (1.1) *subject to the initial datum* $(u_0,\boldsymbol{v}_0)$ *given by Theorem 1.2. In addition to* (1.10)*, assume further that* p *satisfies* (1.12) *and*

$$
\|(u_0^\ell,\varepsilon\boldsymbol{v}_0^\ell)\|_{\dot B^{\sigma_1}_{p,\infty}}\ \text{is uniformly bounded with}\ -\frac{d}{p}\le\sigma_1\le\frac{d}{p}-1.
\tag{1.15}
$$

Then, for all $t>0$,

$$
\|(u,\varepsilon\boldsymbol{v})(t)\|_{\dot B^{\sigma}_{p,1}}\le C(1+t)^{-\frac12(\sigma-\sigma_1)},\qquad \sigma_1<\sigma\le\frac{d}{p},
\tag{1.16}
$$

where $C>0$ *is a constant independent of time and* ε.

Moreover, let u^* *be the global solution of System* (1.2) *supplemented with the initial datum* u_0 *given by Theorem 1.1. Then, for all* $t>0$, *the difference* $u-u^*$ *satisfies*

$$
\|(u-u^*)(t)\|_{\dot B^{\sigma}_{p,1}}\le C\varepsilon(1+t)^{-\frac12(\sigma-\sigma_1)-\frac12},\quad \sigma_1<\sigma\le\frac{d}{p}-1.
\tag{1.17}
$$

1.3 Comments on our main results

Some remarks on Theorems 1.2-1.4 are in order:

1. The low-frequency regularity properties (1.11) of u correspond to (1.7) verified by the solution u^* of the limiting system (1.2). As the relaxation parameter $\varepsilon \to 0$, the low-frequency region $|\xi| \leq 2^{J_\varepsilon} \sim \varepsilon^{-1}$ will cover the whole frequencies and the high-frequency regime disappears. This reveals, qualitatively, the diffusive relaxation process from the Jin-Xin system to viscous conservation laws.

2. In Theorem 1.2, we derive uniform regularity estimates for the solution to System (1.19) in L^p-L^2 hybrid Besov spaces i.e. the low frequencies in L^p-based spaces and the high-frequency ones in L^2-based spaces. Note that in (1.10), the norm of $\boldsymbol{v}_0$ and the high-frequency norm of u_0 can be arbitrarily large as long as ε is suitably small. Due to the dispersive structure in the high-frequency regime, the well-posedness of the hyperbolic system (1.19) cannot be entirely justified in L^p-based spaces for $p \neq 2$ (see Brenner [7]).

3. Compared with the work [17], Theorem 1.2 not only exhibits a more general L^p-type functional setting but also lower regularity assumptions on $\boldsymbol{v}$. Moreover, the restriction $0 < \varepsilon < 1$ required in [17] can be relaxed to the full range $\varepsilon > 0$ so as to describe the so-called overdamping phenomenon for the Jin-Xin system (see Figure 1).

4. Theorem 1.3 provides a rigorous justification of the strong relaxation limit from System (1.1) to System (1.2) for ill-prepared initial data, namely, $u_0^\varepsilon \to u_0^*$ as $\varepsilon \to 0$ without further smallness or decay-in-ε assumption on u_0^ε or $\boldsymbol{v}^\varepsilon$.

5. By virtue of Theorem 1.4, the solution of System (1.2) can be viewed as both the relaxation limit and the time-asymptotic profile of the solution of System (1.1). The time-decay rates in (1.16) are optimal in the sense at they are the same as those are obtainable for the heat equation. In the case $-d/p < \sigma_1 < 0$, due to the embedding $\dot{B}^0_{p,1} \hookrightarrow L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d) \hookrightarrow \dot{B}^{\sigma_1}_{p,\infty}$ with $q = \frac{dp}{d-p\sigma_1} \in [p/2, p)$, Theorem 1.4 implies that under the stronger condition $(u_0^\ell, \varepsilon \boldsymbol{v}_0^\ell) \in L^q(\mathbb{R}^d)$, the solution $(u, \boldsymbol{v})$ satisfies
$$\|(u, \varepsilon \boldsymbol{v})(t)\|_{L^p} \lesssim (1+t)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})},$$
Moreover, for $2 \leq p \leq d$ such that $\frac{d}{p} - 1 \geq 0$, the difference $u - u^*$ verifies
$$\|(u-u^*)(t)\|_{L^p} \lesssim \varepsilon (1+t)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}.$$
Additionally, in our computations (see (5.21)), the high frequencies of the solutions undergo faster time-decay rates than the rates obtained for the full solution in (1.17) due to the damping effect of the solution in this regime.

6. In Theorem 1.3-1.4, we require $2 \leq p \leq 2d$ due to $\dot{B}^{\frac{d}{2}}_{2,1} \hookrightarrow \dot{B}^{\frac{d}{p}}_{p,1}$ and the technical limitation when using product laws. In the case $p < 2$, by $\dot{B}^{\frac{d}{p}}_{p,1} \hookrightarrow \dot{B}^{\frac{d}{2}}_{2,1}$, one can establish similar estimates as in (1.13)-(1.17) for L^2-type norms.

7. Different from the Green function method used for instance in [4] concerning the 1-d case, our proof of Theorem 1.4 relies on a pure energy argument with explicit dependence of the parameter ε and may be of interest in the mathematical analysis of other relaxation problems.

1.4 Strategies of proofs

1.4.1 Spectral behavior of the solution

In order to understand the behavior of the solution to (1.1) with respect to ε, we analyse the eigenvalues of the associated linearized system of (1.1) as follows. Taking the Fourier transform of the linearisation of (1.19) with respect to $\boldsymbol{x}$, we obtain

$$\partial_t \begin{pmatrix} \hat{u} \\ \varepsilon\hat{v}_1 \\ \varepsilon\hat{v}_2 \\ \vdots \\ \varepsilon\hat{v}_d \end{pmatrix} = \widehat{\mathbb{A}}(\xi) \begin{pmatrix} \hat{u} \\ \varepsilon\hat{v}_1 \\ \varepsilon\hat{v}_2 \\ \vdots \\ \varepsilon\hat{v}_d \end{pmatrix}, \qquad \widehat{\mathbb{A}}(\xi) \triangleq \begin{pmatrix} 0 & -i\frac{1}{\varepsilon}\xi_1 & -i\frac{1}{\varepsilon}\xi_2 & \cdots & -i\frac{1}{\varepsilon}\xi_d \\ -i\frac{1}{\varepsilon}a_1\xi_1 & -\frac{1}{\varepsilon^2} & 0 & \cdots & 0 \\ -i\frac{1}{\varepsilon}a_2\xi_2 & 0 & -\frac{1}{\varepsilon^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -i\frac{1}{\varepsilon}a_d\xi_d & 0 & 0 & \cdots & -\frac{1}{\varepsilon^2} \end{pmatrix}.$$

Then, we compute the eigenvalues of the matrix $\widehat{\mathbb{A}}(\xi)$ from the determinant

$$\det(\widehat{\mathbb{A}}(\xi) - \lambda I_d) = (\lambda + \frac{1}{\varepsilon^2})^{d-1}(\lambda^2 + \frac{1}{\varepsilon^2}\lambda + \frac{1}{\varepsilon^2}\sum_{i=1}^{d} a_i|\xi_i|^2) = 0.$$

Solving this equation, we obtain

$$\begin{cases} \lambda_i = -\dfrac{1}{\varepsilon^2}, & i = 1, 2, \cdots, d-1, \\ \lambda_d = -\dfrac{1}{2\varepsilon^2} + \dfrac{1}{2\varepsilon}\sqrt{\dfrac{1}{\varepsilon^2} - 4\displaystyle\sum_{i=1}^{d} a_i|\xi_i|^2}, \\ \lambda_{d+1} = -\dfrac{1}{2\varepsilon^2} - \dfrac{1}{2\varepsilon}\sqrt{\dfrac{1}{\varepsilon^2} - 4\displaystyle\sum_{i=1}^{d} a_i|\xi_i|^2}. \end{cases}$$

The eigenvalues have following properties:

- In the low-frequency region $|\xi| \lesssim \varepsilon^{-1}$, all the eigenvalues are real, and we have $\lambda_d \sim -|\xi|^2$ and $\lambda_{d+1} \sim -\varepsilon^{-2}$. This implies that the parabolic effect and the damping effect coexist.
- In the high-frequency region $|\xi| \gtrsim \varepsilon^{-1}$, the complex conjugated eigenvalues λ_i $(i = d, d+1)$ have $-\frac{1}{2}\varepsilon^{-2}$ as real parts.

The above spectral analysis reveals that it is suitable to split the frequencies into a low-frequency region $|\xi| \lesssim \varepsilon^{-1}$ and a high-frequency region $|\xi| \gtrsim \varepsilon^{-1}$ so as to study (1.1). On the other hand, since no dispersive effects are present in the low-frequency region $|\xi| \lesssim \varepsilon^{-1}$, as all the eigenvalues in low frequencies are purely real, a L^p-based functional framework is feasible in this regime.

Such choice of a threshold J_ε satisfying $2^{J_\varepsilon} \sim \varepsilon^{-1}$ allows us to tackle the so-called overdamping phenomena [40] that is usually a major obstacle to study the relaxation limit. The overdamping phenomena refers to the fact that *as the friction coefficient ε^{-1} gets larger, the decay rates do not necessarily increase and achieve* the maximum at a optimal threshold (cf. Figure 1 below). Here, decomposing the frequency-space with a suitably chosen threshold enables to capture perfectly the ε-dependency of solutions.

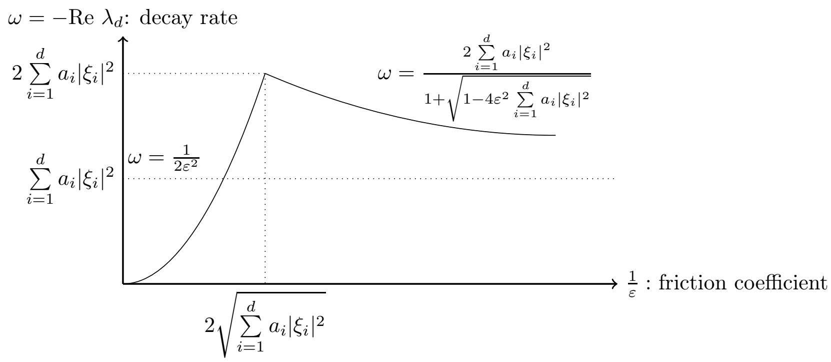

Figure 1: Overdamping phenomenon for System (1.1).

1.4.2 Effective unknowns and relaxation limit

To achieve our results, we first introduce an effective unknown

$$\boldsymbol{z} = (z_1, z_2, \cdots, z_d) \quad \text{with} \quad z_i \triangleq A_i \frac{\partial}{\partial x_i} u + v_i, \qquad i = 1, 2, ..., d, \tag{1.18}$$

which allows to capture the sharp dissipative effects observed in spectral behaviors for low frequencies. In fact, in order to decouple System (1.1), we rewrite (1.1) in terms of $(u, \boldsymbol{z})$ as

$$\begin{cases} \dfrac{\partial}{\partial t} u - \displaystyle\sum_{i=1}^d \frac{\partial}{\partial x_i}\Big(A_i \frac{\partial}{\partial x_i} u\Big) = -\sum_{i=1}^d \frac{\partial}{\partial x_i} z_i, \\ \dfrac{\partial}{\partial t} z_k + \dfrac{1}{\varepsilon^2} z_k = A_k \dfrac{\partial}{\partial x_k}\Big(\displaystyle\sum_{i=1}^d \frac{\partial}{\partial x_i}\Big(A_i \frac{\partial}{\partial x_i} u\Big)\Big) - A_k \frac{\partial}{\partial x_k} \sum_{i=1}^d \frac{\partial}{\partial x_i} z_i + \frac{1}{\varepsilon^2} f_k(u), \quad k = 1, 2, ..., d. \end{cases} \tag{1.19}$$

Note that the high-order linear terms in the right-hand-side of (1.19) can be absorbed if the threshold J_ε takes the form (1.9) for a suitably small k_0 independent of ε. Such unknown (1.18) is different from the previous work [17] and allows us to improve the regularity assumptions by avoiding difficult terms. Meanwhile, the high-frequency analysis is based on the construction of a Lyapunov functional as in [17] but with additional parameter weights. To overcome the major difficulty caused by the quadratic nonlinear term $f(u)$ in our functional setting, some new composition estimates are established in hybrid Besov spaces with explicit dependence on the threshold (refer to Lemmas 6.10 and 6.11).

In order to obtain the convergence estimate (1.14), we introduce another effective unknown

$$\boldsymbol{Z} = (Z_1, Z_2, \cdots, Z_d) \quad \text{with} \quad Z_i \triangleq A_i \frac{\partial}{\partial x_i} u + v_i - f_i(u). \tag{1.20}$$

Then, we are able to rewrite the equation $(1.1)_1$ of u as

$$\frac{\partial}{\partial t} u - \sum_{i=1}^d \frac{\partial}{\partial x_i}(A_i \frac{\partial}{\partial x_i} u) = -\sum_{i=1}^d \frac{\partial}{\partial x_i} Z_i + \sum_{i=1}^d \frac{\partial}{\partial x_i} f_i(u). \tag{1.21}$$

Note that (1.19) can be viewed as the structure of the viscous conservation law (1.2) coupled with the reminder $-\sum_{i=1}^d \frac{\partial}{\partial x_i} Z_i$. Indeed, from (1.2) and (1.21), the difference $\delta u \triangleq u - u^*$ solves

$$\partial_t \delta u - \sum_{i=1}^d \frac{\partial}{\partial x_i}(A_i \frac{\partial}{\partial x_i} \delta u) = -\sum_{i=1}^d \frac{\partial}{\partial x_i} Z_i - \sum_{i=1}^d \frac{\partial}{\partial x_i}\big(f_i(u) - f_i(u^*)\big). \tag{1.22}$$

To analyze (1.22) and derive $\mathcal{O}(\varepsilon)$ bounds in low frequencies, we establish the following key decay-in-ε estimate:

$$\frac{1}{\varepsilon}\|\boldsymbol{Z}^{\ell}\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{d}{p}}_{p,1})} \lesssim \varepsilon\|\boldsymbol{v}_0^{\ell}\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \mathcal{X}_{p,0}, \tag{1.23}$$

which comes essentially from the damped structure of $\boldsymbol{Z}$ (refer to Proposition 4.1). Meanwhile, the convergence rate for high frequencies follows directly from the bounds in (1.11) combined with Bernstein-type estimates.

1.4.3 Large-time asymptotics

Finally, we explain the main ideas concerning the time asymptotics obtained in Theorem 1.3. Since the solution of System (1.1) are purely damped in the high-frequency regime and that the component Z is also damped in the low-frequency region, the slow variable that will dictate the decay is u^{ℓ}. To that matter, multiplying (1.21) by t^{α} with any given $\alpha > 1$ and using maximal regularity estimates and real interpolation, we have

$$\begin{aligned}\|\tau^{\alpha}u^{\ell}\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \|\tau^{\alpha}u^{\ell}\|_{\widetilde{L}^{1}_t(\dot{B}^{\frac{d}{p}+2}_{p,1})} &\lesssim \int_0^t \tau^{\alpha-1}\|u^{\ell}\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\,d\tau \\ &\lesssim t^{\alpha-\frac{1}{2}(\frac{d}{p}-\sigma_1)}\|u^{\ell}\|_{\widetilde{L}^{\infty}_t(\dot{B}^{\sigma_1}_{p,\infty})} + o(1)\|\tau^{\alpha}u^{\ell}\|_{\widetilde{L}^{1}_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}.\end{aligned} \tag{1.24}$$

Under the assumption (1.15), the time-weighted estimate (1.24) allows us to establish the low-frequency evolution of the $\dot{B}^{\sigma_1}_{p,1}$-norm:

$$\|u^{\ell}\|_{\widetilde{L}^{\infty}(\dot{B}^{\sigma_1}_{p,\infty})} + \frac{1}{\varepsilon}\|\boldsymbol{Z}^{\ell}\|_{L^1(\mathbb{R}_+;\dot{B}^{\sigma_1}_{p,\infty})} \lesssim \|(u_0^{\ell},\varepsilon\boldsymbol{v}_0^{\ell})\|_{\dot{B}^{\sigma_1}_{p,\infty}} + \mathcal{X}_{p,0}. \tag{1.25}$$

This approach is based on the previous works [15,18,29,38] but requires more elaborate weighted estimates with respect to the relaxation parameter ε (see Proposition 5.1). Moreover, we observe that when $\delta u|_{t=0}$, the spatial derivative of Z_i in (1.22) and the $\dot{B}^{\sigma_1}_{p,1}$-norm convergence of $\boldsymbol{Z}$ in (1.25) implies the additional estimate of the difference δu:

$$\frac{1}{\varepsilon}\|\delta u^{\ell}\|_{\widetilde{L}^{\infty}(\dot{B}^{\sigma_1-1}_{p,\infty})} \lesssim \frac{1}{\varepsilon}\|\boldsymbol{Z}^{\ell}\|_{L^1(\mathbb{R}_+;\dot{B}^{\sigma_1}_{p,\infty})} \lesssim \|(u_0^{\ell},\varepsilon\boldsymbol{v}_0^{\ell})\|_{\dot{B}^{\sigma_1}_{p,\infty}} + \mathcal{X}_{p,0}. \tag{1.26}$$

Compared to the $\dot{B}^{\sigma_1}_{p,\infty}$-evolution of u^{ℓ} in (1.25), the key ingredient (1.26) not only provides lower-order regularity that leads to faster decay rates but also yields a $\mathcal{O}(\varepsilon)$ bound in time-decay estimates. Performing similar time-weighted energy estimates on the difference equation (1.22) and taking advantage of (1.26) enable to establish the enhanced decay estimates (1.17).

1.5 Outline of the paper

The rest of this paper is organized as follows. In Section 2, we briefly recall the Littlewood-Paley decomposition, Besov spaces and Chemin-Lerner spaces. In Section 3, we establish uniform a-priori estimates and prove Theorem 1.2. The rigorous justification of the relaxation limit from System (1.1) to System (1.2) is performed in Section 4. Section 5 is devoted to the proof of the sharp decay estimates in Theorem 1.3. Some technical lemmas and the proof of Theorem 1.1 are relegated to the appendix.

2 Preliminary

We list some notations that are used frequently throughout the paper.

Notations. For simplicity, C denotes some positive constant that is independent of ε and time. $A \lesssim B$ ($A \gtrsim B$) means that both $A \leq CB$ ($A \geq CB$), while $A \sim B$ means that both $A \lesssim B$ and $A \gtrsim B$. For a Banach space X, $p \in [1, \infty]$ and $T > 0$, the notation $L^p(0, T; X)$ or $L^p_T(X)$ designates the set of measurable functions $f : [0, T] \to X$ with $t \mapsto \|f(t)\|_X$ in $L^p(0, T)$, endowed with the norm $\|\cdot\|_{L^p_T(X)} \triangleq \|\|\cdot\|_X\|_{L^p(0,T)}$, and $\mathcal{C}([0, T]; X)$ denotes the set of continuous functions $f : [0, T] \to X$. Let $\mathcal{F}(f) = \widehat{f}$ and $\mathcal{F}^{-1}(f) = \check{f}$ be the Fourier transform of f and its inverse.

Then, we recall the Littlewood-Paley decomposition and the definitions of Besov spaces. The reader can refer to Chapters 2 and 3 in [2] for more details. Choose a smooth radial non-increasing function $\chi(\xi)$ compactly supported in $B(0, \frac{4}{3})$ and satisfying $\chi(\xi) = 1$ in $B(0, \frac{3}{4})$. Then, $\varphi(\xi) \triangleq \chi(\xi/2) - \chi(\xi)$ satisfies

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\cdot) = 1, \quad \text{Supp } \varphi \subset \Big\{\xi \in \mathbb{R}^d \mid \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\Big\}.$$

For any $j \in \mathbb{Z}$, define the homogeneous dyadic block

$$\dot{\Delta}_j u \triangleq \mathcal{F}^{-1}\big(\varphi(2^{-j}\cdot)\mathcal{F}(u)\big) = 2^{jd} h(2^j\cdot) \star u, \qquad h \triangleq \mathcal{F}^{-1}\varphi.$$

We also define the low-frequency cut-off operator

$$\dot{S}_j \triangleq \sum_{j' \leq j-1} \dot{\Delta}_j.$$

Let $\mathcal{S}'_h$ stand for the set of tempered distributions z on $\mathbb{R}^d$ such that $\dot{S}_j z \to 0$ uniformly as $j \to \infty$ (i.e., modulo polynomials). One has

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \quad \text{in } \mathcal{S}', \quad \forall u \in \mathcal{S}'_h, \qquad \dot{\Delta}_j \dot{\Delta}_l u = 0, \quad \text{if} \quad |j - l| \geq 2.$$

With the help of those dyadic blocks, we give the definitions of homogeneous Besov spaces and mixed space-time Besov spaces as follow. For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the homogeneous Besov space $\dot{B}^s_{p,r}$ is defined by

$$\dot{B}^s_{p,r} \triangleq \big\{u \in \mathcal{S}'_h \ : \ \|u\|_{\dot{B}^s_{p,r}} \triangleq \|\{2^{js}\|\dot{\Delta}_j u\|_{L^p}\}_{j\in\mathbb{Z}}\|_{l^r} < \infty\big\}.$$

For $T > 0$, $s \in \mathbb{R}$ and $1 \leq \varrho, r, q \leq \infty$, we recall a class of mixed space-time Besov spaces $\widetilde{L}^\varrho(0, T; \dot{B}^s_{p,r})$ introduced by Chemin-Lerner [10]:

$$\widetilde{L}^\varrho(0, T; \dot{B}^s_{p,r}) \triangleq \big\{u \in L^\varrho(0, T; \mathcal{S}'_h) \ : \ \|u\|_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})} \triangleq \|\{2^{js}\|\dot{\Delta}_j u\|_{L^\varrho_T(L^p)}\}_{j\in\mathbb{Z}}\|_{l^r} < \infty\big\}.$$

By Minkowski's inequality, it holds

$$\|u\|_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})} \leq (\geq) \|u\|_{L^\varrho_T(\dot{B}^s_{p,r})} \quad \text{if } r \geq (\leq)\rho,$$

where $\|\cdot\|_{L^\varrho_T(\dot{B}^s_{p,r})}$ is the usual Lebesgue-Besov norm.

In order to restrict Besov norms to the low frequency part and the high-frequency part, we write $\|\cdot\|^h_{\dot{B}^s_{q_1,1}}$ and $\|\cdot\|^\ell_{\dot{B}^s_{q_2,1}}$ to denote Besov semi-norms with respect to the threshold J_ε defining as (1.9), that is,

$$\|u\|^\ell_{\dot{B}^{s_2}_{q_2,r}} \triangleq \Big(\sum_{j\leq J_\varepsilon} \Big(2^{s_2 j}\|\dot{\Delta}_j u\|_{L^{q_2}}\Big)^r\Big)^{\frac{1}{r}} \quad \text{and} \quad \|u\|^h_{\dot{B}^{s_1}_{q_1,r}} \triangleq \Big(\sum_{j\geq J_\varepsilon-1} \Big(2^{s_1 j}\|\dot{\Delta}_j u\|_{L^{q_1}}\Big)^r\Big)^{\frac{1}{r}}. \tag{2.1}$$

One can deduce that for all $\sigma_0 > 0$,

$$\|u\|^{\ell}_{\dot{B}^{s_1}_{q_1,r}} \le 2^{\sigma_0 J_\varepsilon}\|u\|^{\ell}_{\dot{B}^{s_1-\sigma_0}_{q_1,r}} \quad \text{and} \quad \|u\|^{h}_{\dot{B}^{s_1}_{q_1,r}} \le 2^{-\sigma_0 J_\varepsilon+\sigma_0}\|u\|^{h}_{\dot{B}^{s_1+\sigma_0}_{q_1,r}}. \tag{2.2}$$

We also introduce the low-high-frequency decomposition $u = u^\ell + u^h$ with

$$u^\ell \triangleq \sum_{j\le J_\varepsilon-1}\dot{\Delta}_j u = \dot{S}_{J_\varepsilon}u \quad \text{and} \quad u^h \triangleq \sum_{j\ge J_\varepsilon}\dot{\Delta}_j u = (\mathrm{Id}-\dot{S}_{J_\varepsilon})u.$$

It is easy to check that

$$\|u\|^{\ell}_{\dot{B}^{s_1}_{q_1,r}} \le 2^{\sigma_0 J_\varepsilon}\|u\|^{\ell}_{\dot{B}^{s_1-\sigma_0}_{q_1,r}}, \quad \|u^\ell\|_{\dot{B}^{s}_{q_1,r}} \le \|u\|^{\ell}_{\dot{B}^{s}_{q_1,r}} \quad \text{and} \quad \|u^h\|_{\dot{B}^{s}_{q_1,r}} \le \|u\|^{h}_{\dot{B}^{s}_{q_1,r}}. \tag{2.3}$$

3 Uniform a priori estimates and global well-posedness

In this section, we give the key a-priori estimates leading to the global existence of solutions for (1.1).

Proposition 3.1. *Let p satisfy (1.8). For any $\varepsilon > 0$ and given time $T > 0$, let $(u, \boldsymbol{v})$ be a solution to System (1.1) satisfying, for $0 \le t < T$,*

$$\|u\|_{L^\infty_t(L^\infty)} \le 1.$$

Then, for all $0 \le t < T$, it holds that

$$\mathcal{X}_p(t) \le C\big(\mathcal{X}_{p,0} + \mathcal{X}_p(t)^2\big), \tag{3.1}$$

where $\mathcal{X}_p(t)$ is defined by

$$\begin{aligned}\mathcal{X}_p(t) \triangleq{}& \|u\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})} + \|u\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1}\cap\dot{B}^{\frac{d}{p}+2}_{p,1})} + (1+\varepsilon)\|u\|^{h}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \Big(\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}\Big)\|u\|^{h}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\\ &+\varepsilon^2\|\boldsymbol{v}\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})} + \|\boldsymbol{v}\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})} + (\varepsilon+\varepsilon^2)\|\boldsymbol{v}\|^{h}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \Big(1+\frac{1}{\varepsilon}\Big)\|\boldsymbol{v}\|^{h}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}.\end{aligned} \tag{3.2}$$

The proof of Proposition 3.1 is divided into two cases: the low and high frequencies.

3.1 Low-frequency analysis

Let $\boldsymbol{z}$ be the effective unknown defined by (1.18) and set $\boldsymbol{z}|_{t=0} = \boldsymbol{z}_0 = (z_{1,0},\cdots,z_{d,0})$ with $z_{0,i} \triangleq A_i\frac{\partial}{\partial x_i}u_0 + v_{0,i}$. Recall that $(u,\boldsymbol{z})$ satisfies (1.19). By virtue of Lemma 6.8 for $(1.19)_1$, there exists a generic constant $C_1 > 0$ such that

$$\|u\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})} + \|u\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1}\cap\dot{B}^{\frac{d}{p}+2}_{p,1})} \le C_1\Big(\|u_0\|^{\ell}_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}} + \|\boldsymbol{z}\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}\Big). \tag{3.3}$$

As for $\boldsymbol{z}$, according to (2.2) and Lemma 6.9 for $(1.19)_2$, one has

$$\begin{aligned}&\varepsilon^2\|\boldsymbol{z}\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})} + \|\boldsymbol{z}\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}\\ &\quad\le C_2\Big(\varepsilon^2\|\boldsymbol{z}_0\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1}} + \varepsilon^2 2^{2J_\varepsilon}\|u\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1}\cap\dot{B}^{\frac{d}{p}+2}_{p,1})} + \varepsilon^2 2^{2J_\varepsilon}\|\boldsymbol{z}\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}\Big)\\ &\qquad + C_2\|f(u)\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})},\end{aligned} \tag{3.4}$$

where C_2 is a universal constant. Then, substituting (3.3) into (3.4), we deduce

$$\begin{aligned}
&\varepsilon^2\|z\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}+\|z\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}\\
&\le C_2\varepsilon^2\|z_0\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1}}+C_1C_2\varepsilon^2 2^{2J_\varepsilon}\|u_0\|^\ell_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}}\\
&\quad+C_2(C_1+1)\varepsilon^2 2^{2J_\varepsilon}\|z\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}+C_2\|f(u)\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}.
\end{aligned}\tag{3.5}$$

Here the threshold J_ε takes the form (1.9) such that $\varepsilon 2^{J_\varepsilon}=2^{k_0}$. Therefore, one is able to choose the integer k_0 such that

$$2^{2k_0}<\frac{1}{2C_2(C_1+1)}.\tag{3.6}$$

Combining (3.3), (3.5) and (3.6) together, we obtain

$$\begin{aligned}
&\|u\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})}+\|u\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1}\cap\dot{B}^{\frac{d}{p}+2}_{p,1})}+\varepsilon^2\|z\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}+\|z\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}\\
&\lesssim\|u_0\|^\ell_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}}+\varepsilon^2\|z_0\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1}}+\|f(u)\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}.
\end{aligned}\tag{3.7}$$

Next, we bound the nonlinear term $f(u)$. Due to $p\ge\max\{1,\frac{2d}{d+2}\}$, it holds, by composition estimates in Lemma 6.9 with $(s,\sigma)=(d/p,d/2)$, that

$$\begin{aligned}
\|f(u)\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}&\lesssim\int_0^t\Big(\big(\|u^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}}+\|u\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big)\|u\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}}+\big(2^{J_\varepsilon}\|u^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|u\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big)\|u\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\Big)d\tau\\
&\lesssim\Big(\|u\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}\Big)^2+\Big(\|u\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}\Big)^2+\|u\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\frac{1}{\varepsilon}\|u\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}.
\end{aligned}\tag{3.8}$$

Similarly, by Lemma 6.9 with $(s,\sigma)=(d/p+1,d/2)$ and (3.10) one has

$$\begin{aligned}
\|f(u)\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}&\lesssim\int_0^t\Big(\big(\|u\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}}+\|u\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big)\|u\|^\ell_{\dot{B}^{\frac{d}{p}+1}_{p,1}}+2^{J_\varepsilon}\big(2^{J_\varepsilon}\|u\|^\ell_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|u\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big)\|u\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\Big)d\tau\\
&\lesssim\Big(\|u\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}\Big)^2+\|u\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\|u\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}\\
&\quad+\|u\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\frac{1}{\varepsilon^2}\|u\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\frac{1}{\varepsilon}\Big(\|u\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}\Big)^2.
\end{aligned}\tag{3.9}$$

From the interpolation inequalities in Lemma 6.3 there holds that

$$\begin{cases}
\|u\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}\lesssim\Big(\|u\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1}\cap B^{\frac{d}{p}}_{p,1})}\Big)^{\frac12}\Big(\|u\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1}\cap B^{\frac{d}{p}+2}_{p,1})}\Big)^{\frac12}\lesssim\mathcal{X}_p(t),\\
(1+\frac{1}{\varepsilon})\|u\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}\lesssim\Big((1+\varepsilon)\|u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\Big)^{\frac12}\Big((\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2})\|u\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\Big)^{\frac12}\lesssim\mathcal{X}_p(t).
\end{cases}\tag{3.10}$$

Combining (3.7), (3.8), (3.9) and (3.10) together, we have

$$\begin{aligned}
&\|u\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})}+\|u\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1}\cap\dot{B}^{\frac{d}{p}+2}_{p,1})}+\varepsilon^2\|z\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}+\|z\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}\\
&\lesssim\|u_0\|^\ell_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}}+\varepsilon^2\|z_0\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1}}+\mathcal{X}_p(t)^2.
\end{aligned}\tag{3.11}$$

To recover the information on $\boldsymbol{v}$, one deduces from (1.18) and (2.2) that

$$\|\boldsymbol{v}\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})} \lesssim \|u\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1}\cap\dot{B}^{\frac{d}{p}+2}_{p,1})} + \|\boldsymbol{z}\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}, \tag{3.12}$$

$$\begin{aligned}\varepsilon^2\|\boldsymbol{v}\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})} &\lesssim \varepsilon^2 2^{2J_\varepsilon}\|u\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})} + \varepsilon^2\|\boldsymbol{z}\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}\\ &\lesssim \|u\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})} + \varepsilon^2\|\boldsymbol{z}\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}.\end{aligned} \tag{3.13}$$

From (3.11), (3.12) and (3.13) we obtain

$$\begin{aligned}&\|u\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})} + \|u\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1}\cap\dot{B}^{\frac{d}{p}+2}_{p,1})} + \varepsilon^2\|\boldsymbol{v}\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})} + \|\boldsymbol{v}\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}\\ &\quad+\varepsilon^2\|\boldsymbol{z}\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})} + \|\boldsymbol{z}\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})} \lesssim \mathcal{X}_{p,0} + \mathcal{X}_p(t)^2.\end{aligned} \tag{3.14}$$

3.2 High-frequency analysis

For any $j \geq J_\varepsilon - 1$, we localize System (1.1) as follows

$$\begin{cases}\dfrac{\partial}{\partial t}\dot{\Delta}_j u + \displaystyle\sum_{i=1}^d \dfrac{\partial}{\partial x_i}\dot{\Delta}_j v_i = 0,\\[2mm] \varepsilon^2\dfrac{\partial}{\partial t}\dot{\Delta}_j v_i + A_i\dfrac{\partial}{\partial x_i}\dot{\Delta}_j u + \dot{\Delta}_j v_i = \dot{\Delta}_j f_i(u), \quad i = 1, 2, ..., d.\end{cases} \tag{3.15}$$

Multiplying $(3.15)_1$ by $\dot{\Delta}_j u$ and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j u\|^2_{L^2} - \sum_{i=1}^d \int_{\mathbb{R}^d}\dot{\Delta}_j v_i\cdot\frac{\partial}{\partial x_i}\dot{\Delta}_j u\ d\boldsymbol{x} = 0. \tag{3.16}$$

Meanwhile, taking the inner product of $(3.15)_2$ with $\dfrac{1}{a_i}\dot{\Delta}_j v_i$ and summing over $1 \leq i \leq d$, we get

$$\begin{aligned}&\sum_{i=1}^d \frac{\varepsilon^2}{2a_i}\frac{d}{dt}\|\dot{\Delta}_j v_i\|^2_{L^2} + \sum_{i=1}^d\int_{\mathbb{R}^d}\dot{\Delta}_j v_i\cdot\frac{\partial}{\partial x_i}\dot{\Delta}_j u\ d\boldsymbol{x} + \sum_{i=1}^d\frac{1}{a_i}\|\dot{\Delta}_j v_i\|^2_{L^2}\\ &\quad= \sum_{i=1}^d\frac{1}{a_i}\int_{\mathbb{R}^d}\dot{\Delta}_j f_i(u)\cdot\dot{\Delta}_j v_i\ d\boldsymbol{x}.\end{aligned} \tag{3.17}$$

Adding (3.16) and (3.17) together yields

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^d\Big(\|\dot{\Delta}_j u\|^2_{L^2} + \frac{\varepsilon^2}{a_i}\|\dot{\Delta}_j v_i\|^2_{L^2}\Big) + \sum_{i=1}^d\frac{1}{a_i}\|\dot{\Delta}_j v_i\|^2_{L^2} = \sum_{i=1}^d\frac{1}{a_i}\int_{\mathbb{R}^d}\dot{\Delta}_j f_i(u)\cdot\dot{\Delta}_j v_i\ d\boldsymbol{x}. \tag{3.18}$$

To derive the dissipation for u, we perform the following cross estimate

$$\begin{aligned}&\sum_{i=1}^d\frac{1}{a_i}\frac{d}{dt}\int_{\mathbb{R}^d}\dot{\Delta}_j v_i\cdot\frac{\partial}{\partial x_i}\dot{\Delta}_j u\ d\boldsymbol{x} + \frac{1}{\varepsilon^2}\sum_{i=1}^d\|\frac{\partial}{\partial x_i}\dot{\Delta}_j u\|^2_{L^2}\\ &\quad-\sum_{i=1}^d\frac{1}{a_i}\int_{\mathbb{R}^d}\frac{\partial}{\partial x_i}\dot{\Delta}_j v_i\cdot\sum_{i=1}^k\frac{\partial}{\partial x_k}\dot{\Delta}_j v_k\ d\boldsymbol{x} + \frac{1}{\varepsilon^2}\sum_{i=1}^d\frac{1}{a_i}\int_{\mathbb{R}^d}\dot{\Delta}_j v_i\cdot\frac{\partial}{\partial x_i}\dot{\Delta}_j u\ d\boldsymbol{x}\\ &= \frac{1}{\varepsilon^2}\sum_{i=1}^d\frac{1}{a_i}\int_{\mathbb{R}^d}\dot{\Delta}_j f_i(u)\cdot\frac{\partial}{\partial x_i}\dot{\Delta}_j u\ d\boldsymbol{x}.\end{aligned} \tag{3.19}$$

For any $j\geq J_\varepsilon-1$, the linear combination of (3.18) and (3.19) leads to

$$\frac{d}{dt}\mathcal{L}_j^2(t)+\frac{1}{\varepsilon^2}\mathcal{H}_j^2(t)\leq\sum_{i=1}^d\frac{1}{a_i}\|\dot\Delta_jf_i(u)\|_{L^2}\big(\frac{1}{\varepsilon}\|\dot\Delta_j\boldsymbol{v}\|_{L^2}+\frac{2^{-2j}\zeta}{\varepsilon^2}\|\frac{\partial}{\partial x_i}\dot\Delta_ju\|_{L^2}\big),\tag{3.20}$$

where the Lyapunov functional $\mathcal{L}_j^2(t)$ and its dissipation rate $\mathcal{H}_j^2(t)$ are defined by

$$\begin{cases}\mathcal{L}_j^2(t)\triangleq\frac12\|\dot\Delta_ju\|_{L^2}^2+\sum_{i=1}^d\frac{1}{2a_i}\|\varepsilon\dot\Delta_jv_i\|_{L^2}^2+2^{-2j}\zeta\sum_{i=1}^d\frac{1}{a_i}\int_{\mathbb{R}^d}\dot\Delta_jv_i\cdot\frac{\partial}{\partial x_i}\dot\Delta_ju\,d\boldsymbol{x},\\ \mathcal{H}_j^2(t)\triangleq\sum_{i=1}^d\frac{1}{a_i}\|\dot\Delta_jv_i\|_{L^2}^2+\frac{2^{-2j}\zeta}{\varepsilon^2}\sum_{i=1}^d\Big(\|\frac{\partial}{\partial x_i}\dot\Delta_ju\|_{L^2}^2\\ \qquad-\frac{\varepsilon^2}{a_i}\int_{\mathbb{R}^d}\frac{\partial}{\partial x_i}\dot\Delta_jv_i\cdot\sum_{i=1}^k\frac{\partial}{\partial x_k}\dot\Delta_jv_k\,d\boldsymbol{x}+\frac{1}{a_i}\int_{\mathbb{R}^d}\dot\Delta_jv_i\cdot\frac{\partial}{\partial x_i}\dot\Delta_ju\,d\boldsymbol{x}\Big)\end{cases}$$

for $\zeta>0$ suitably small. By Bernstein's inequality and Young's inequality, we deduce

$$\mathcal{L}_j^2(t)\sim\|\dot\Delta_j(u,\varepsilon\boldsymbol{v})\|_{L^2}^2,\qquad\mathcal{H}_j^2(t)\gtrsim\frac{1}{\varepsilon^2}\|\dot\Delta_j(u,\varepsilon\boldsymbol{v})\|_{L^2}^2.\tag{3.21}$$

Thus, it follows from (3.20) and (3.21) that

$$\frac{d}{dt}\mathcal{L}_j^2(t)+\frac{1}{\varepsilon^2}\mathcal{L}_j^2(t)\lesssim\frac{1}{\varepsilon}\sum_{i=1}^d\|\dot\Delta_jf_i(u)\|_{L^2}\mathcal{L}_j(t).\tag{3.22}$$

Dividing (3.22) by $\sqrt{\mathcal{L}_j(t)+\nu_0}$, integrating the resulting inequality over $[0,t]$ and then, taking the limit as $\nu_0\to0$, we reach

$$\|\dot\Delta_j(u,\varepsilon\boldsymbol{v})\|_{L^2}+\frac{1}{\varepsilon^2}\|\dot\Delta_j(u,\varepsilon\boldsymbol{v})\|_{L^1_t(L^2)}\lesssim\|\dot\Delta_j(u_0,\varepsilon\boldsymbol{v}_0)\|_{L^2}+\frac{1}{\varepsilon}\sum_{i=1}^d\|\dot\Delta_jf_i(u)\|_{L^1_t(L^2)},$$

which yields

$$\begin{aligned}&(1+\varepsilon)\|(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde L^\infty_t(\dot B^{\frac d2}_{2,1})}+(\frac1\varepsilon+\frac{1}{\varepsilon^2})\|(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde L^1_t(\dot B^{\frac d2}_{2,1})}\\&\lesssim(1+\varepsilon)\|(u_0,\varepsilon\boldsymbol{v}_0)\|^h_{\dot B^{\frac d2}_{2,1}}+(1+\frac1\varepsilon)\|f(u)\|^h_{\widetilde L^1_t(\dot B^{\frac d2}_{2,1})}.\end{aligned}\tag{3.23}$$

The nonlinear term $f(u)$ is analyzed as follows. Owing to $2^{J_\varepsilon}\sim\varepsilon^{-1}$, Lemma 6.11 with $(s,\sigma)=(d/2,d/p+1)$ and (3.10), we have

$$\begin{aligned}\|f(u)\|^h_{\widetilde L^1_t(B^{\frac d2}_{2,1})}&\lesssim\int_0^t\Big(\big(\|u\|^\ell_{\dot B^{\frac dp}_{p,1}}+\|u\|^h_{\dot B^{\frac d2}_{2,1}}\big)\|u\|^h_{\dot B^{\frac d2}_{2,1}}+2^{-J_\varepsilon}\big(2^{J_\varepsilon}\|u\|^\ell_{\dot B^{\frac dp-1}_{p,1}}+\|u\|^h_{\dot B^{\frac d2}_{2,1}}\big)\|u\|^\ell_{\dot B^{\frac dp+1}_{p,1}}\Big)d\tau\\&\lesssim\Big(\|u\|^\ell_{\widetilde L^2_t(\dot B^{\frac dp}_{p,1})}\Big)^2+\Big(\|u\|^h_{\widetilde L^2_t(\dot B^{\frac d2}_{2,1})}\Big)^2+\big(\|u\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac dp-1}_{p,1})}+\frac1\varepsilon\|u\|^h_{\widetilde L^\infty_t(\dot B^{\frac d2}_{2,1})}\big)\|u\|^\ell_{\widetilde L^1_t(\dot B^{\frac dp+1}_{p,1})}\\&\lesssim\mathcal{X}_p(t)^2.\end{aligned}\tag{3.24}$$

Similarly, one gets

$$\begin{aligned}\frac1\varepsilon\|f(u)\|^h_{\widetilde L^1_t(B^{\frac d2}_{2,1})}&\lesssim\frac1\varepsilon\int_0^t\Big(\big(\|u\|^\ell_{\dot B^{\frac dp}_{p,1}}+\|u\|^h_{\dot B^{\frac d2}_{2,1}}\big)\|u\|^h_{\dot B^{\frac d2}_{2,1}}+2^{-2J_\varepsilon}\big(2^{J_\varepsilon}\|u\|^\ell_{\dot B^{\frac dp-1}_{p,1}}+\|u\|^h_{\dot B^{\frac d2}_{2,1}}\big)\|u\|^\ell_{\dot B^{\frac dp+2}_{p,1}}\Big)d\tau\\&\lesssim\big(\|u\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac dp}_{p,1})}+\|u\|^h_{\widetilde L^\infty_t(\dot B^{\frac d2}_{2,1})}\big)\frac1\varepsilon\|u\|^h_{\widetilde L^1_t(\dot B^{\frac d2}_{2,1})}+\big(\|u\|^\ell_{\widetilde L^\infty_t(\dot B^{\frac dp-1}_{p,1})}+\varepsilon\|u\|^h_{\widetilde L^\infty_t(\dot B^{\frac d2}_{2,1})}\big)\|u\|^\ell_{\widetilde L^1_t(\dot B^{\frac dp+2}_{p,1})}\\&\lesssim\mathcal{X}_p(t)^2.\end{aligned}\tag{3.25}$$

It thus holds by (3.23), (3.24) and (3.25) that

$$(1+\varepsilon)\|(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}+(\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2})\|(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\lesssim\mathcal{X}_{p,0}+\mathcal{X}_p(t)^2. \tag{3.26}$$

The combination of estimates (3.14) and (3.26) gives rise to (3.1), and concludes the proof of Proposition 3.1.

3.3 Proof of global existence and uniqueness

In order to prove Theorem 1.2, we first need to justify the existence and uniqueness of local-in-time solutions for System (1.1). Define the space

$$E(T)\triangleq\{(u,\boldsymbol{v})\ :\ u^\ell\in\mathcal{C}([0,T_*];\dot{B}^{\frac{d}{p}-1}_{p,1}),\ \boldsymbol{v}^\ell\in\mathcal{C}([0,T_*];\dot{B}^{\frac{d}{p}}_{p,1}),\ (u^h,\boldsymbol{v}^h)\in\mathcal{C}([0,T_*];\dot{B}^{\frac{d}{2}}_{2,1})\}$$

and its associated norm

$$\|(u,\boldsymbol{v})\|_{E(T)}\triangleq\|u\|^\ell_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})}+(1+\varepsilon)\|u\|^h_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\varepsilon^2\|\boldsymbol{v}\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}+\varepsilon(1+\varepsilon)\|\boldsymbol{v}\|^h_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}.$$

We also denote the space of initial data by

$$E_0\triangleq\{(u_0,\boldsymbol{v}_0)\ :\ u_0^\ell\in\dot{B}^{\frac{d}{p}-1}_{p,1},\ \boldsymbol{v}_0^\ell\in\dot{B}^{\frac{d}{p}}_{p,1},\ (u_0^h,\boldsymbol{v}_0^h)\in\dot{B}^{\frac{d}{2}}_{2,1}\},$$

equipped with the norm $\|(u_0,\boldsymbol{v}_0)\|_{E_0}\triangleq\mathcal{X}_{p,0}$ with $\mathcal{X}_{p,0}$ defined in (1.10).

Theorem 3.1. (*Local well-posedness*) *Let p satisfy (1.8), and the threshold J_ε be given by (1.9). Assume $(u_0,\boldsymbol{v}_0)\in E_0$. Then, there exists a time $T_*>0$ such that System (1.1) associated to the initial datum $(u_0,\boldsymbol{v}_0)$ admits a unique strong solution $(u,\boldsymbol{v})\in E(T_*)$.*

With Theorem 3.1 in hand, we can conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. Under the assumption (1.10), Theorem 3.1 implies the existence and uniqueness of the solution $(u,\boldsymbol{v})$ to System (1.1) on $[0,T_{\max})$ with a maximal time $T_{\max}>0$. In light of the uniform a-priori estimate (3.1) obtained in Proposition 3.1 and a standard bootstrap argument, one can justify $T_{\max}=\infty$ and verify that the global solution $(u,\boldsymbol{v})$ fulfills the property (1.11). □

Proof of Theorem 3.1. The proof of Theorem 3.1 is divided into the following four steps.

- Step 1: Construction of the approximate sequence.

For $k=1,2,..$, define the regularized initial data

$$(u_0^k,\boldsymbol{v}_0^k)\triangleq\begin{cases}\chi(L_kx)\sum\limits_{|j|\le k}\dot{\Delta}_j(u_0,\boldsymbol{v}_0), & \text{if } p\ge 2,\\ \sum\limits_{|j|\le k}\dot{\Delta}_j(u_0,\boldsymbol{v}_0), & \text{if } p<2,\end{cases}$$

where $\chi(x)\in\mathcal{S}(\mathbb{R}^d)$ satisfies $\chi(0)=1$ and Supp $\mathcal{F}(\chi)(\xi)\subset B(0,1)$, and $L_k>0$ is a constant which satisfies $\lim\limits_{k\to\infty}L_k=0$ and will be chosen later.

We claim that $(u_0^k,\boldsymbol{v}_0^k)$ belongs to $H^{s_0}(\mathbb{R}^d)$ for $s_0\ge[\frac{d}{2}]+1$ and fixed $k\ge0$. Indeed, in the case $p\ge2$, one deduces from Bernstein's inequality and the embedding $\dot{B}^{\frac{d}{p}}_{p,1}\hookrightarrow L^\infty$ that

$$\|(u_0^k,\boldsymbol{v}_0^k)\|_{H^{s_0}(\mathbb{R}^d)}\lesssim2^{ks_0}\|(u_0^k,\boldsymbol{v}_0^k)\|_{L^2}\lesssim2^{ks_0}\|\chi(L_k\cdot)\|_{L^2}\|(u_0,v_0)\|_{L^\infty}<\infty.$$

As for the case $p<2$, due to Bernstein's inequality and Sobolev embeddings, one also has $(u_0^k,\boldsymbol{v}_0^k)\in W^{s_0+\frac{d}{p}-\frac{d}{2},p}(\mathbb{R}^d)\hookrightarrow H^{s_0}(\mathbb{R}^d)$.

Then, we explain that there exists a suitable large integer k_0 such that for all $k\geq k_0$, $(u_0^k,\boldsymbol{v}_0^k)$ has the following uniform bound

$$\|(u_0^k,\boldsymbol{v}_0^k)\|_{E_0}\lesssim\|(u_0,\boldsymbol{v}_0)\|_{E_0}. \tag{3.27}$$

We only justify (3.27) when $p\geq 2$ as the case $p<2$ is clear. Recalling the invariance of the norm $\dot{B}^{\frac{d}{p}}_{p,1}$ by spatial dilation (see, e.g., [2, Proposition 2.18]), we have $\|\chi(\frac{\cdot}{k})\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\sim\|\chi\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\lesssim 1$. Thus, for $p\geq 2$, the classical product law (6.2) and Bernstein's inequality ensure that

$$\|u_0^k\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}}\lesssim\|\chi\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|u_0\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\lesssim\|u_0\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}}+\|u_0\|^{h}_{\dot{B}^{\frac{d}{2}}_{2,1}}.$$

Employing the hybrid product law (6.4), we also have

$$\|u_0^k\|^{h}_{\dot{B}^{\frac{d}{2}}_{2,1}}\lesssim(\|\chi\|_{\dot{B}^{\frac{d}{p}}_{p,1}}+\|\chi\|_{\dot{B}^{\frac{d}{2}}_{2,1}})(\|u_0\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}}+\|u_0\|^{h}_{\dot{B}^{\frac{d}{2}}_{2,1}})\lesssim\|u_0\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}}+\|u_0\|^{h}_{\dot{B}^{\frac{d}{2}}_{2,1}}.$$

Similarly, one can obtain the desired bounds for $\boldsymbol{v}_0^k$ in (3.27). Then, it suffices to show

$$\lim_{k\to\infty}\|u_0^k-u_0\|^{\ell}_{\dot{B}^{\frac{d}{p}-1}_{p,1}}=0. \tag{3.28}$$

In fact, inspired by [1][Lemma 4.2] we decompose

$$u_0^k-u_0=(\lambda(L_kx)-1)\sum_{|j|\leq k}\dot{\Delta}_ju_0+\sum_{|j|\geq k+1}\dot{\Delta}_ju_0.$$

For any $\eta>0$, one can find a suitable large integer $k_0^*=k_0^*(\eta)$ such that for all $k\geq k_0^*$,

$$\|\sum_{|j|\geq k+1}\dot{\Delta}_ju_0\|^{\ell}_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\leq C\sum_{j\leq -k+1,\ j\leq J_\varepsilon}2^{(\frac{d}{p}-1)j}\|\dot{\Delta}_ju_0\|_{L^p}<\eta.$$

On the other hand, Since $\operatorname{Supp}\mathcal{F}(\chi(L_k\cdot))\subset B(0,L^k)$ and

$$\operatorname{Supp}\mathcal{F}(\sum_{j\leq k}\dot{\Delta}_ju_0)\subset\{\xi\in\mathbb{R}^d:\frac{3}{4}2^{-k}\leq|\xi|\leq\frac{8}{3}2^k\},$$

we have $\mathcal{F}(u_0^k)\subset\{\xi\in\mathbb{R}^d:\frac{3}{8}2^{-k}\leq|\xi|\leq\frac{11}{3}2^k\}$ for $L_k\leq\frac{3}{8}2^{-k}$, so $\dot{\Delta}_{j'}u_0^k=0$ if $|j'|\geq k+3$. Hence, from the definition of χ it follows that

$$\begin{aligned}\|(\lambda(L_kx)-1)\sum_{|j|\leq k}\dot{\Delta}_ju_0\|^{\ell}_{\dot{B}^{\frac{d}{p}-1}_{p,1}}&\lesssim 2^k\|(\lambda(L_k\cdot)-1)\sum_{|j|\leq k}\dot{\Delta}_ju_0\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}}\\&\lesssim 2^k\|\lambda(L_k\cdot)-1\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|u_0\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\to 0\quad\text{as}\quad k\to\infty,\end{aligned}$$

as long as we choose a suitable constant L_k. Therefore, we have (3.28) and complete the proof of (3.27).

According to the classical local well-posedness theorem for hyperbolic systems (see, e.g., [19,31]), for fixed $k\geq k_0$ and any $s_0\geq[\frac{d}{2}]+1$, there exists a time T_k such that the Cauchy problem of System (1.19) associated with the initial datum $(u_0^k,\boldsymbol{v}_0^k)$ has a unique solution $(u^k,\boldsymbol{v}^k)\in\mathcal{C}([0,T_k];H^{s_0}(\mathbb{R}^d))$.

- Step 2: Uniform estimates.

Performing similar computations as in Section 3 and using(3.27), for all $k \geq k_0$ and $0 < t < T_k$, we have

$$\|(u^k, \boldsymbol{v}^k)\|_{E(T)} \lesssim \|(u_0, v_0)\|_{E_0} + \|f(u^k)\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1} \cap \dot{B}^{\frac{d}{p}+1}_{p,1})} + (1 + \frac{1}{\varepsilon})\|f(u^k)\|^{h}_{\widetilde{L}^1_t(B^{\frac{d}{2}}_{2,1})}. \tag{3.29}$$

Then, Lemmas 6.9 and 6.11 guarantee that

$$\begin{aligned}
\|f(u^k)\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1} \cap \dot{B}^{\frac{d}{p}+1}_{p,1})} &\lesssim T(1 + \frac{1}{\varepsilon})\|f(u^k)\|^{\ell}_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} \\
&\lesssim T(1 + \frac{1}{\varepsilon})\Big(\|u^k\|^{\ell}_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\Big)^2 + T(1 + \frac{1}{\varepsilon})\Big(\|u^k\|^{h}_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\Big)^2 \\
&\quad + \frac{T}{\varepsilon}(1 + \frac{1}{\varepsilon})\|u^k\|^{\ell}_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|u^k\|^{h}_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})},
\end{aligned}$$

and

$$\begin{aligned}
\|f(u^k)\|^{h}_{\widetilde{L}^1_t(B^{\frac{d}{2}}_{2,1})} &\lesssim T\Big(\|u^k\|^{\ell}_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\Big)^2 + T\Big(\|u^k\|^{h}_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\Big)^2 \\
&\quad + \frac{T}{\varepsilon}(\|u^k\|^{\ell}_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \frac{1}{\varepsilon}\|u^k\|^{h}_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})})\|u^k\|^{\ell}_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}.
\end{aligned}$$

Consequently, one can find a generic constant $C_* > 0$ such that

$$\|(u^k, \boldsymbol{v}^k)\|_{E(T)} \leq C_*\|(u_0, \boldsymbol{v}_0)\|_{E_0} + C_* T(1 + \frac{1}{\varepsilon})^2\|(u, \boldsymbol{v})\|^2_{E(T)}, \qquad k \geq k_0. \tag{3.30}$$

We define the time

$$T_* \triangleq \frac{1}{2C_*^2(1 + \frac{1}{\varepsilon})^2 E_0} \tag{3.31}$$

and the time set

$$I^k := \Big\{t \in [0, T_*] \ : \ \|(u^k, \boldsymbol{v}^k)\|_{E(t)} \leq 2C_*\|(u_0, \boldsymbol{v}_0)\|_{E_0}\Big\}. \tag{3.32}$$

It is clear that $T_* < T_k$. Due to the time continuity of $(u^k, \boldsymbol{v}^k)$ on $[0, T_k)$, I^k is a nonempty closed subset of $[0, T]$ for every $k \geq k_0$. By (3.30) and the definition (3.32) we have

$$\|(u^k, \boldsymbol{v}^k)\|_{E(t)} < 2C_*\|(u_0, \boldsymbol{v}_0)\|_{E_0}, \qquad t \in I^k, \quad k \geq k_0.$$

Again, using the time continuity of $(u^k, \boldsymbol{v}^k)$, one can show that there exists a ball $B(t, \eta_*)$ for a suitably small constant $\eta_* > 0$ such that $[0, T] \cap B(t, \eta_*) \subset I^k$, which implies that I^k is also an open subset of $[0, T]$. Hence, we have $I^k = [0, T]$, and the approximate sequence $(u^k, \boldsymbol{v}^k)$ satisfies the estimate $\|(u^k, \boldsymbol{v}^k)\|_{E(T_*)} \leq 2C_*\|(u_0, \boldsymbol{v}_0)\|_{E_0}$ which is uniform with respect to $k \geq k_0$.

- Step 3: Convergence of the approximate sequence

The uniform estimate established in Step 2 implies that there exists $(u, \boldsymbol{v})$ such that as $k \to \infty$, it holds up to a subsequence that

$$(u^k, \boldsymbol{v}^k) \overset{*}{\rightharpoonup} (u, \boldsymbol{v}) \qquad \text{in} \quad L^\infty(0, T_*; L^\infty(\mathbb{R}^d)).$$

In order to justify the convergence of $f(u^k)$ in $(1.19)_2$, one needs to show the strong compactness of $\{u^k\}_{k\geq k_0}$ in a suitable sense. From $(1.19)_1$ and the uniform estimate of $\boldsymbol{v}^k$ we have

$$\|\partial_t u^k\|_{L^\infty(0,T;\dot{B}^{\frac{d}{\max\{2,p\}}-1}_{\max\{2,p\},1})} \lesssim \|v^k\|^\ell_{L^\infty(0,T;\dot{B}^{\frac{d}{p}}_{p,1})} + \|v^k\|^h_{L^\infty(0,T;\dot{B}^{\frac{d}{2}}_{2,1})} \lesssim \|(u_0,\boldsymbol{v}_0)\|_{E_0}.$$

Gathering this and the compact embedding $\dot{B}^{\frac{d}{\max\{2,p\}}}_{\max\{2,p\},1} \hookrightarrow L^1_{loc}(\mathbb{R}^d)$, one infers from the Aubin-Lions lemma and the Cantor diagonal argument that, as $k\to\infty$, for any bounded set $K\subset\mathbb{R}^d$,

$$u^k \to u \qquad \text{in} \quad L^1(0,T_*;L^1(K)),$$

which implies that

$$\begin{aligned}
&\|f(u^k)-f(u)\|_{L^1(0,T_*;L^1(K))}\\
&\quad\leq \sup_{\tau\in[0,1]}\|f'(u+\tau(u^k-u))\|_{L^\infty(0,T_*;L^\infty(\mathbb{R}^d))}\|u^k-u\||_{L^1(0,T_*;L^1(K))}\to 0.
\end{aligned}$$

Therefore, $(u,\boldsymbol{v})$ indeed solves System (1.19) in the sense of distributions. Finally, we justify the time continuity of the solution. Taking advantage of $(1.19)_1$, for any $0\leq t_1,t_2\leq T_*$, we have

$$\|u^\ell(t_1)-u^\ell(t_2)\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \lesssim \|v\|^\ell_{L^\infty(0,T_*;\dot{B}^{\frac{d}{p}}_{p,1})}|t_1-t_2|.$$

This implies that $u^\ell\in\mathcal{C}([0,T_*];\dot{B}^{\frac{d}{p}-1}_{p,1})$. A similar argument leads to $\boldsymbol{v}^\ell\in\mathcal{C}([0,T_*];\dot{B}^{\frac{d}{p}}_{p,1})$. To deal with the high-frequency part, we consider the decomposition $(u,\boldsymbol{v})^h=S_{N_0}(u,\boldsymbol{v})^h+(\mathrm{Id}-S_{N_0})(u,\boldsymbol{v})^h$ for some integer N_0. From (1.19) and the given bounds on $(u,\boldsymbol{v})$, one can show that the high-frequency part satisfies $(u,\boldsymbol{v})^h\in\mathcal{C}([0,T_*];\dot{B}^{\frac{d}{2}-1}_{2,1})$. It thus follows that $S_{N_0}(u,\boldsymbol{v})^h\in\mathcal{C}([0,T_*];\dot{B}^{\frac{d}{2}}_{2,1})$ due to Bernstein's inequality. On the other hand, following similar arguments as in Subsection 3.2, we have $(u,\boldsymbol{v})^h\in\widetilde{L}^\infty(0,T_*;\dot{B}^{\frac{d}{2}}_{2,1})$, and therefore

$$\|(\mathrm{Id}-S_{N_0})(u,\boldsymbol{v})\|^h_{L^\infty_{T_*}(\dot{B}^{\frac{d}{2}}_{2,1})} \lesssim \sum_{j\geq\max\{J_\varepsilon,N_0\}-1} 2^{\frac{d}{2}j}\sup_{t\in[0,T_*]}\|\dot{\Delta}_j(u,\boldsymbol{v})\|_{L^2}$$

can be arbitrarily small as long as N_0 is chosen to be large enough. Consequently, we have $(u,\boldsymbol{v})^h\in\mathcal{C}([0,T_*];\dot{B}^{\frac{d}{2}}_{2,1})$.

- Step 4: Proof of the uniqueness

For given time $T>0$, let $(u_1,\boldsymbol{v}_1)$ and $(u_2,\boldsymbol{v}_2)$ two solutions of System (1.1) in the space $E(T)$ with the same data $(u_0,\boldsymbol{v}_0)$. Then, $(U,\boldsymbol{V})=(u_1-u_2,\boldsymbol{v}_1-\boldsymbol{v}_2)$ satisfies

$$\begin{cases}
\dfrac{\partial}{\partial t}U+\displaystyle\sum_{i=1}^d\dfrac{\partial}{\partial x_i}V_i=0,\\[2mm]
\varepsilon^2\dfrac{\partial}{\partial t}V_i+A_i\dfrac{\partial}{\partial x_i}U+V_i=\big(f_i(u_1)-f_i(u_2)\big), \quad i=1,2,...,d.
\end{cases}$$

- Case 1: $p\geq 2$.

Arguing similarly as in Subsections 3.1-3.2, for $t\in(0,T]$, one can infer that

$$\begin{aligned}
&\|U\|^\ell_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|U\|^\ell_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}+\varepsilon\|(U,\varepsilon\boldsymbol{V})\|^h_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\frac{1}{\varepsilon}\|(U,\varepsilon\boldsymbol{V})\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\\
&\quad\lesssim\|f(u_1)-f(u_2)\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|f(u_1)-f(u_2)\|^h_{\widetilde{L}^1_t(B^{\frac{d}{2}}_{2,1})}.
\end{aligned} \tag{3.33}$$

We now bound the nonlinear part of (3.33). Since $p \geq 2$, we have $u_1, u_2 \in L^\infty(0,T;\dot{B}^{\frac{d}{p}}_{p,1})$ by virtue of Bernstein's inequality. It follows from Corollary 6.7 that

$$\begin{aligned}\|f(u_1)-f(u_2)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} &\lesssim \int_0^t \|U\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|(u_1,u_2)\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\, d\tau \\ &\lesssim \int_0^t \|(u_1,u_2)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \Big(\frac{1}{\varepsilon}\|U\|^{\ell}_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|U\|^{h}_{\dot{B}^{\frac{d}{2}}_{2,1}}\Big)\, d\tau.\end{aligned} \tag{3.34}$$

To bound the nonlinear term in the high-frequency region, we rewrite $f_i(u_1)-f_i(u_2) = \sum_{j=1}^d U_j b_{i,j}$ with $b_{i,j} \triangleq \int_0^1 \frac{\partial f}{\partial u_j}(u_2+\tau(u_1-u_2))\, d\tau$. Therefore, together with the product law in Lemma 6.5 and the composition estimates in Lemmas 6.10 and 6.11 it yields

$$\begin{aligned}&\|f(u_1)-f(u_2)\|^{h}_{\widetilde{L}^1_t(B^{\frac{d}{2}}_{2,1})} \\ &\lesssim \int_0^t \Big(\|U\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \sum_{i,j}^{d} \|b_{i,j}\|^{h}_{\dot{B}^{\frac{d}{2}}_{2,1}} + \sum_{i,j}^{d} \|b_{i,j}\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|U\|^{h}_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|U\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}}\Big)\, d\tau \\ &\lesssim (1+\frac{1}{\varepsilon}) \int_0^T \Big(\|(u_1,u_2)\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|(u_1,u_2)\|^{h}_{\dot{B}^{\frac{d}{2}}_{2,1}}\Big)\Big(\|U\|^{\ell}_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|U\|^{h}_{\dot{B}^{\frac{d}{2}}_{2,1}}\Big)\, d\tau.\end{aligned} \tag{3.35}$$

Inserting (3.34) and (3.35) into (3.33) and employing Grönwall's inequality yield $(u_1,\boldsymbol{v}_1) = (u_2,\boldsymbol{v}_2)$ for a.e. $(t,x) \in [0,T]\times\mathbb{R}^d$.

- Case 2: $p<2$.

As $p<2$, it is clear that $\dot{B}^{\frac{d}{p}}_{p,1} \hookrightarrow \dot{B}^{\frac{d}{2}}_{2,1}$. Therefore, we only need to prove the uniqueness in the L^2 framework. The L^2 energy method from Subsections 3.1-3.2 leads to

$$\begin{aligned}&\|U\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|U\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}+1}_{p,1})} + \varepsilon\|(U,\varepsilon\boldsymbol{V})\|^{h}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \frac{1}{\varepsilon}\|(U,\varepsilon\boldsymbol{V})\|^{h}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})} \\ &\lesssim \|f(u_1)-f(u_2)\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}.\end{aligned} \tag{3.36}$$

The composition estimate in Corollary 6.7 gives

$$\begin{aligned}\|f(u_1)-f(u_2)\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})} &\lesssim \int_0^t \|(u_1,u_2)\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|U\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\, d\tau \\ &\lesssim \int_0^t \|(u_1,u_2)\|_{\dot{B}^{\frac{d}{2}}_{2,1}} (\frac{1}{\varepsilon}\|U\|^{\ell}_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|U\|^{h}_{\dot{B}^{\frac{d}{2}}_{2,1}})\, d\tau.\end{aligned} \tag{3.37}$$

Hence, Grönwall's inequality implies the uniqueness in the case $p<2$. □

4 Strong relaxation limit

In this section, we prove Theorem 1.3. For clarity, we divide the proof into two steps. First, we establish additional regularity estimates of the effective unknown $\boldsymbol{Z}$.

Proposition 4.1. *Let $(u,\boldsymbol{v})$ be the solution to System (1.1) obtained in Theorem 1.2. If p is given by (1.12), then*

$$\|u\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})} + \|u\|_{\widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})} \lesssim \mathcal{X}_{p,0}, \tag{4.1}$$

where $\mathcal{X}_{p,0}$ is given by (1.10).

Moreover, under the additional assumption (1.13), we have

$$\varepsilon\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}+\frac{1}{\varepsilon}\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim\varepsilon\|\boldsymbol{v}_0^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}}+\mathcal{X}_{p,0},\tag{4.2}$$

where $\boldsymbol{Z}$ is defined by (1.20).

Proof. Thanks to (1.11), (2.2) and Bernstein's inequality, it holds that

$$\begin{aligned}&\|u\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim\|u\|^\ell_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})}+(1+\varepsilon)\|u\|^h_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{2}}_{2,1})}\lesssim\mathcal{X}_{p,0},\\&\|u\|_{\widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+1}_{p,1})}\lesssim\|u\|^\ell_{\widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+1}_{p,1})}+\|u\|_{\widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\frac{d}{2}}_{2,1})}\lesssim\mathcal{X}_{p,0},\end{aligned}$$

where we used that $p\geq 2$ and $2^{J_\varepsilon}\lesssim\varepsilon^{-1}$. Consequently, we get (4.1).

Next, to establish the low-frequency estimate (4.2) of $\boldsymbol{Z}$, we observe that $\boldsymbol{Z}$ has a damping effect in

$$\begin{aligned}\frac{\partial}{\partial t}Z_k+\frac{1}{\varepsilon^2}Z_k&=A_k\frac{\partial}{\partial x_k}\Big(\sum_{i=1}^d\frac{\partial}{\partial x_i}(A_i\frac{\partial}{\partial x_i}u)\Big)-A_k\frac{\partial}{\partial x_k}\sum_{i=1}^d\frac{\partial}{\partial x_i}Z_i\\&\quad+A_k\frac{\partial}{\partial x_k}\Big(\sum_{i=1}^d\frac{\partial}{\partial x_i}f_i(u)\Big)-\sum_{i,j=1}^d\frac{\partial}{\partial_{u_i}}f_k(u)\frac{\partial}{\partial x_i}v_j^i,\quad k=1,2,\cdots,d.\end{aligned}\tag{4.3}$$

Applying the low-frequency cut-off operator $\dot{S}_{J_\varepsilon}$ to (4.3), making use of Lemma 6.9 and the fact that $2^{J_\varepsilon}\lesssim\varepsilon^{-1}$, we obtain

$$\begin{aligned}\varepsilon\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}+\frac{1}{\varepsilon}\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}&\lesssim\varepsilon\|\boldsymbol{Z}_0^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}}+\varepsilon\|u^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+3}_{p,1})}+\varepsilon\|f(u)^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+1}_{p,1})}\\&\quad+\sum_{k,i,j=1}^d\varepsilon\|\Big(\frac{\partial}{\partial_{u_i}}f_k(u)\frac{\partial}{\partial x_i}v_j^i\Big)^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}.\end{aligned}\tag{4.4}$$

Here $\boldsymbol{Z}_0=(Z_{0,i},\cdots,Z_{0,d})$ with $Z_{0,k}\triangleq A_k\frac{\partial}{\partial x_k}u_0+v_{0,i}+f_k(u_0)$. Using the classical composition estimate in Lemma 6.6, the first term on the r.h.s of (4.4) is controlled by

$$\begin{aligned}\varepsilon\|\boldsymbol{Z}_0\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}}&\lesssim\varepsilon\|u_0\|^\ell_{\dot{B}^{\frac{d}{p}+1}_{p,1}}+\varepsilon\|\boldsymbol{v}_0\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}}+\varepsilon\|f(u_0)\|^\ell_{\dot{B}^{\frac{d}{p}+1}_{p,1}}\\&\lesssim\|u_0\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}}+\varepsilon\|\boldsymbol{v}_0\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}}+\|f(u_0)\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}}\\&\lesssim\|u_0\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}}+\|u_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}+\varepsilon\|\boldsymbol{v}_0\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}}.\end{aligned}\tag{4.5}$$

From (1.11), (2.2), (3.9) and $2^{J_\varepsilon}\lesssim\varepsilon^{-1}$, one gets

$$\begin{aligned}&\varepsilon\|u\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+3}_{p,1})}+\varepsilon\|f(u)^\ell\|_{\widetilde{L}^1(\mathbb{R}^+\dot{B}^{\frac{d}{p}+2}_{p,1})}\\&\quad\lesssim\|u\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+2}_{p,1})}+\|f(u)\|^\ell_{\widetilde{L}^1(\mathbb{R}^+\dot{B}^{\frac{d}{p}+1}_{p,1})}\lesssim\mathcal{X}^2_{p,0}.\end{aligned}\tag{4.6}$$

By virtue of (2.2), (4.1), Lemma 6.4, Lemma 6.6 and $-\frac{d}{p}\leq\frac{d}{p}-1$ due to $2\leq p\leq 2d$, one also has

$$
\begin{aligned}
\varepsilon\|\Big(\frac{\partial}{\partial_{u_i}}f_k(u)\frac{\partial}{\partial x_i}v_j^i\Big)^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}&\lesssim\|\frac{\partial}{\partial_{u_i}}f_k(u)\frac{\partial}{\partial x_i}v_j^i\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}-1}_{p,\infty})}\\
&\lesssim\|\frac{\partial}{\partial_{u_i}}f_k(u)\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}\|\frac{\partial}{\partial x_i}v_j^i\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}-1}_{p,\infty})}\\
&\lesssim\|u\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}\Big(\|v\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}+\|v\|^h_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{2}}_{2,1})}\Big)\lesssim\mathcal{X}^2_{p,0}.
\end{aligned}\tag{4.7}
$$

Inserting (4.5), (4.6) and (4.7) into (4.4), we obtain (4.2). □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let the assumptions (1.8), (1.10) and (1.13) be in force. To derive the convergence rate, we shall estimate the difference of the solutions $(u,\boldsymbol{v})$ and u^* to System (1.1) and (1.2), respectively. Defining $(\delta u,\delta\boldsymbol{v})\triangleq(u-u^*,\boldsymbol{v}-\boldsymbol{v}^*)$ and using (1.19), we have

$$
\begin{cases}
\partial_t\delta u-\displaystyle\sum_{i=1}^d\frac{\partial}{\partial x_i}(A_i\frac{\partial}{\partial x_i}\delta u)=-\sum_{i=1}^d\frac{\partial}{\partial x_i}Z_i-\sum_{i=1}^d\frac{\partial}{\partial x_i}\big(f_i(u)-f_i(u^*)\big),\\
\delta v_i=-A_i\dfrac{\partial}{\partial x_i}\delta u+f_i(u)-f_i(u^*)+Z_i,\quad i=1,2,...,d
\end{cases}\tag{4.8}
$$

with $\boldsymbol{Z}=((Z_1,Z_2,\cdots,Z_d)$ defined in (1.20). In the high-frequency region, due to $2^{-J_\varepsilon}\lesssim\varepsilon$, the uniform bounds (1.7) and (4.1) give directly

$$
\begin{cases}
\|\delta u\|^h_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}-1}_{p,1})}\lesssim\varepsilon\|(u,u^*)\|^h_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim\varepsilon(\|u_0^*\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}}+\mathcal{X}_{p,0}),\\
\|\delta\boldsymbol{v}\|^h_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim\varepsilon\Big(\dfrac{1}{\varepsilon}\|v\|^h_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}+\|v^*\|^h_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+1}_{p,1})}\Big)\lesssim\varepsilon(\|u_0^*\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}}+\mathcal{X}_{p,0}).
\end{cases}\tag{4.9}
$$

In the low-frequency region, applying Lemma 6.8 to $(4.8)_1$, we deduce

$$
\|\delta u^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|\delta u^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+1}_{p,1})}\lesssim\|u_0-u_0^*\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}+\|\big(f(u)-f(u^*)\big)^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}
$$

which, together with (2.3) and the key uniform bound $\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim\varepsilon$ in (4.2), leads to

$$
\|\delta u^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|\delta u^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+1}_{p,1})}\lesssim\|u_0-u_0^*\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\varepsilon+\|f(u)-f(u^*)\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}.\tag{4.10}
$$

It follows from (1.7), $(3.10)_2$, (4.1), the interpolation in Lemma 6.2 and the composition estimate (6.5) in Corollary 6.7 that

$$
\begin{aligned}
&\|f(u)-f(u^*)\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\quad\lesssim\|(u,u^*)\|_{\widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}\|\delta u\|_{\widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\quad\lesssim\|(u,u^*)\|_{\widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}\Big(\|\delta u^\ell\|_{\widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}+\|u\|^h_{\widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\frac{d}{2}}_{2,1})}+\varepsilon\|u^*\|^h_{\widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+1}_{p,1})}\Big)\\
&\quad\lesssim\Big(\mathcal{X}_{p,0}+\|u_0^*\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}}\Big)\Big(\|\delta u^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|\delta u^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}\Big)\\
&\qquad+\Big(\mathcal{X}_{p,0}+\|u_0^*\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}}\Big)^2.
\end{aligned}\tag{4.11}
$$

Inserting (4.11) into (4.10) and using that both $\mathcal{X}_{p,0}$ and $\|u_0^*\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}}$ are suitably small, we obtain

$$\|\delta u^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|\delta u^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+1}_{p,1})}\lesssim\|u_0-u_0^*\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\varepsilon. \tag{4.12}$$

Thanks to (4.2), $(4.8)_2$, (4.11) and (4.12), we recover the information on $\delta\boldsymbol{v}$ in low frequencies as follows

$$\begin{aligned}\|\delta\boldsymbol{v}^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}&\lesssim\|\delta u^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+1}_{p,1})}+\|f(u)-f(u^*)\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}+\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}\\&\lesssim\|u_0-u_0^*\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\varepsilon.\end{aligned} \tag{4.13}$$

Hence, by (4.9), (4.12) and (4.13), we obtain (1.14) which concludes the proof of Theorem 1.3.

5 Uniform time asymptotics

5.1 Time-decay of the solution

Proposition 5.1. *Assume that p satisfy (1.12) and $(u_0,\boldsymbol{v}_0)$ satisfies (1.10) and (1.15). Let $(u,\boldsymbol{v})$ be the global solution to System (1.1) subject to the initial datum $(u_0,\boldsymbol{v}_0)$ obtained in Theorem 1.2. Then, it holds that*

$$\|(u,\varepsilon\boldsymbol{v})(t)\|_{\dot{B}^{\sigma}_{p,1}}\lesssim(1+t)^{-\frac{1}{2}(\sigma-\sigma_1)}\mathcal{D}_{p,0},\qquad\sigma_1<\sigma\le\frac{d}{p} \tag{5.1}$$

for $\mathcal{D}_{p,0}=\|(u_0^\ell,\varepsilon\boldsymbol{v}_0^\ell)\|_{\dot{B}^{\sigma_1}_{p,\infty}}+\mathcal{X}_{p,0}$.

The proof of Proposition 5.1 is divided into the following four steps.

- *Step 1: Estimates of the solution in $\dot{B}^{\sigma_1}_{p,\infty}$.*

We first establish the uniform evolution of the $\dot{B}^{\sigma_1}_{p,\infty}$-regularity.

Lemma 5.1. *Let p be given by (1.12), and $(u,\boldsymbol{v})$ be the global solution to (1.1) satisfying (1.11) for $\varepsilon\in(0,1)$. In addition to (1.10), assume further that (1.15) holds. Then, we have*

$$\|u^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}+\|u^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1+2}_{p,\infty})}+\varepsilon\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}+\frac{1}{\varepsilon}\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}\lesssim\mathcal{D}_{p,0} \tag{5.2}$$

with $\boldsymbol{Z}$ defined by (1.20) and $\mathcal{D}_{p,0}=\|(u_0^\ell,\varepsilon\boldsymbol{v}_0^\ell)\|_{\dot{B}^{\sigma_1}_{p,\infty}}+\mathcal{X}_{p,0}$.

Proof. We recall that $(u,\boldsymbol{Z})$ satisfies (1.21) and (4.3). Applying the low-frequency cut-off operator $\dot{S}_{J_\varepsilon}$ to (1.21) and (4.3) yields

$$\begin{cases}\dfrac{\partial}{\partial t}u^\ell-\displaystyle\sum_{i=1}^d\frac{\partial}{\partial x_i}(A_i\frac{\partial}{\partial x_i}u^\ell)=-\sum_{i=1}^d\frac{\partial}{\partial x_i}Z_i^\ell+\sum_{i=1}^d\frac{\partial}{\partial x_i}\dot{S}_{J_\varepsilon}f_i(u),\\ \dfrac{\partial}{\partial t}Z_k^\ell+\dfrac{1}{\varepsilon^2}Z_k^\ell=A_k\dfrac{\partial}{\partial x_k}\Big(\displaystyle\sum_{i=1}^d\frac{\partial}{\partial x_i}(A_i\frac{\partial}{\partial x_i}u^\ell)\Big)-A_k\frac{\partial}{\partial x_k}\sum_{i=1}^d\frac{\partial}{\partial x_i}Z_i^\ell\\ \qquad\qquad+A_k\dfrac{\partial}{\partial x_k}\dot{S}_{J_\varepsilon}\Big(\displaystyle\sum_{i=1}^d\frac{\partial}{\partial x_i}f_i(u)\Big)-\sum_{i,j=1}^d\dot{S}_{J_\varepsilon}\Big(\frac{\partial}{\partial_{u_i}}f_k(u)\frac{\partial}{\partial x_i}v_j^i\Big),\quad k=1,2,\cdots,d.\end{cases} \tag{5.3}$$

Applying Lemma 6.8 to $(5.3)_1$ yields

$$\|u^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}+\|u^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1+2}_{p,\infty})}\lesssim\|u_0^\ell\|_{\dot{B}^{\sigma_1}_{p,\infty}}+2^{J_\varepsilon}\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}+\|f(u)\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1+1}_{p,\infty})}.$$

In addition, one deduces from applying Lemma 6.9 to $(5.3)_2$ that

$$\begin{aligned}\varepsilon\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}+\frac{1}{\varepsilon}\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}&\lesssim\varepsilon\|\boldsymbol{Z}_0^\ell\|_{\dot{B}^{\sigma_1}_{p,\infty}}+\varepsilon2^{J_\varepsilon}\|u^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1+2}_{p,\infty})}+\varepsilon2^{J_\varepsilon}\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}\\&\quad+\varepsilon\|f(u)\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1+2}_{p,\infty})}+\sum_{k,i,j=1}^d\varepsilon\|\frac{\partial}{\partial_{u_i}}f_k(u)\frac{\partial}{\partial x_i}v_j^i\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})},\end{aligned}$$

where $\boldsymbol{Z}_0=(Z_{0,i},\cdots,Z_{0,d})$ with $Z_{0,k}\triangleq A_k\frac{\partial}{\partial x_k}u_0+v_{0,i}+f_k(u_0)$. Since $2^{J_\varepsilon}\le 2^{k_0}\varepsilon^{-1}$ with k_0 small enough, we have

$$\begin{aligned}&\|u^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}+\|u^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1+2}_{p,\infty})}+\varepsilon\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}+\frac{1}{\varepsilon}\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}\\&\lesssim\|u_0^\ell\|_{\dot{B}^{\sigma_1}_{p,\infty}}+\varepsilon\|\boldsymbol{Z}_0^\ell\|_{\dot{B}^{\sigma_1}_{p,\infty}}+\|f(u)\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1+1}_{p,\infty})}+\sum_{k,i,j=1}^d\varepsilon\|\frac{\partial}{\partial_{u_i}}f_k(u)\frac{\partial}{\partial x_i}v_j^i\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}.\end{aligned}\tag{5.4}$$

Now, we bound the terms on r.h.s of (5.4). We denote by $\widetilde{f}_i$ the smooth function such that $f_i(u)=\widetilde{f}_i(u)u$ and $\widetilde{f}_i(0)=0$. Due to the facts that $2\le p\le 2d$, $-\frac{d}{p}\le\sigma_1\le\frac{d}{p}-1$, $\varepsilon\le1$ and $2^{J_\varepsilon}\sim\varepsilon^{-1}$, one deduces from (2.2), (6.3) and Lemma 6.6 that

$$\|f_i(u_0)\|_{\dot{B}^{\sigma_1}_{p,\infty}}\lesssim\|\widetilde{f}_i(u_0)\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|u_0\|_{\dot{B}^{\sigma_1}_{p,\infty}}\lesssim(\|u_0\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}}+\|u_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}})(\|u_0\|^\ell_{\dot{B}^{\sigma_1}_{p,\infty}}+\varepsilon^{\frac{d}{2}-\sigma_1}\|u_0\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}).$$

This implies that

$$\varepsilon\|\boldsymbol{Z}_0^\ell\|_{\dot{B}^{\sigma_1}_{p,\infty}}\lesssim\varepsilon\|\boldsymbol{v}_0^\ell\|_{\dot{B}^{\sigma_1}_{p,\infty}}+\varepsilon\|\nabla u_0^\ell\|_{\dot{B}^{\sigma_1}_{p,\infty}}+\varepsilon\|f(u_0)\|^\ell_{\dot{B}^{\sigma_1}_{p,\infty}}\lesssim\|(u_0^\ell,\varepsilon\boldsymbol{v}_0^\ell)\|_{\dot{B}^{\sigma_1}_{p,\infty}}+\mathcal{X}_{p,0}.\tag{5.5}$$

It also holds from (1.11), (2.2), Lemma 6.6 and (6.3) with $2\le p\le 2d$ that

$$\begin{aligned}\varepsilon\|f(u)\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1+2}_{p,\infty})}&\lesssim\|\nabla f(u)\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}\\&\lesssim\sum_i^d\|\frac{\partial}{\partial u_i}f(u)\|_{\widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}\|\nabla u\|_{\widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}\\&\lesssim\|u\|_{\widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}(\|u^\ell\|_{\widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\sigma_1+1}_{p,\infty})}+\varepsilon^{\frac{d}{p}-1-\sigma_1}\|u\|^h_{\widetilde{L}^2(\mathbb{R}^+;\dot{B}^{\frac{d}{2}}_{2,1})}),\end{aligned}$$

which, together with the uniform bound (1.11) and Corollary 6.3, yields

$$\varepsilon\|f(u)\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1+2}_{p,\infty})}\lesssim\mathcal{X}_{p,0}(\|u^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}+\|u^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1+2}_{p,\infty})})+\mathcal{X}_{p,0}.\tag{5.6}$$

Similarly, the last term on r.h.s of (5.4) is estimated as follows

$$\begin{aligned}\varepsilon\|\frac{\partial}{\partial_{u_i}}f_k(u)\frac{\partial}{\partial x_i}v_j^i\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}&\lesssim\varepsilon\|\frac{\partial}{\partial_{u_i}}f_k(u)\frac{\partial}{\partial x_i}(v_j^i)^\ell\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}+\varepsilon\|\frac{\partial}{\partial_{u_i}}f_k(u)\frac{\partial}{\partial x_i}(v_j^i)^h\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}\\&\lesssim\varepsilon\|u\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}\|\nabla v\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}+\varepsilon\|u\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}\|\nabla v\|^h_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}\\&\lesssim\varepsilon\Big(\|u^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}+\varepsilon^{\frac{d}{p}-\varepsilon}\|u\|^h_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{2}}_{2,1})}\Big)\|v\|^\ell_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{p}+1}_{p,1})}\\&\quad+\|u\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{p}}_{p,1})}\varepsilon^{\frac{d}{p}-1}\|v\|^h_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\frac{d}{2}}_{2,1})}\\&\lesssim\mathcal{X}_{p,0}\|u^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}+\mathcal{X}_{p,0}.\end{aligned}\tag{5.7}$$

where one used $\varepsilon\le 1$. Hence, it follows by (5.4)-(5.7) that

$$
\begin{aligned}
&\|u^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}+\|u^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1+2}_{p,\infty})}+\varepsilon\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}+\frac{1}{\varepsilon}\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}\\
&\lesssim \mathcal{X}_{p,0}+\mathcal{X}_{p,0}\Big(\|u^\ell\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot{B}^{\sigma_1}_{p,\infty})}+\|u^\ell\|_{\widetilde{L}^1(\mathbb{R}^+;\dot{B}^{\sigma_1+2}_{p,\infty})}\Big),
\end{aligned}
$$

which, combined with the smallness of $\mathcal{X}_{p,0}$, gives (5.2). The proof of Lemma 5.1 is complete. □

- *Step 2: Time-weighted estimates in the low-frequency region*

Inspired by [18,29,38], we perform time-weighted energy estimates for $(u,\boldsymbol{Z})$. Let $\alpha>1$ be any given constant. Multiplying (5.3) by t^α, we obtain

$$
\begin{cases}
\dfrac{\partial}{\partial t}(t^\alpha u^\ell)-\displaystyle\sum_{i=1}^d\frac{\partial}{\partial x_i}\Big(A_i\frac{\partial}{\partial x_i}(t^\alpha u^\ell)\Big)=\alpha t^{\alpha-1}u^\ell-\sum_{i=1}^d\frac{\partial}{\partial x_i}(t^\alpha Z_i)+t^\alpha\sum_{i=1}^d\frac{\partial}{\partial x_i}\dot{S}_{J_\varepsilon}f_i(u),\\
\dfrac{\partial}{\partial t}(t^\alpha Z_k^\ell)+\dfrac{1}{\varepsilon^2}(t^\alpha Z_k^\ell)=\alpha t^{\alpha-1}Z_k^\ell+A_k\dfrac{\partial}{\partial x_k}\Big(\displaystyle\sum_{i=1}^d\frac{\partial}{\partial x_i}\Big(A_i\frac{\partial}{\partial x_i}(t^\alpha u^\ell)\Big)\Big)\\
\qquad -A_k\dfrac{\partial}{\partial x_k}\displaystyle\sum_{i=1}^d\frac{\partial}{\partial x_i}(t^\alpha Z_i^\ell)+t^\alpha A_k\frac{\partial}{\partial x_k}\Big(\sum_{i=1}^d\frac{\partial}{\partial x_i}\dot{S}_{J_\varepsilon}f_i(u)\Big)-t^\alpha\sum_{i,j=1}^d\dot{S}_{J_\varepsilon}\Big(\frac{\partial}{\partial_{u_i}}f_k(u)\frac{\partial}{\partial x_i}v_j^i\Big)
\end{cases}
\tag{5.8}
$$

for $k=1,2,\cdots,d$.

By similar arguments as in Subsection 3.1, one deduces from Lemmas 6.8-6.9 to (5.8) and $2^{J_\varepsilon}\le 2^{k_0}\varepsilon^{-1}$ with $2^{k_0}\ll 1$ that

$$
\begin{aligned}
&\|\tau^\alpha u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|\tau^\alpha u^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}+\varepsilon\|\tau^\alpha\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\frac{1}{\varepsilon}\|\tau^\alpha\boldsymbol{Z}^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\lesssim\int_0^t\tau^{\alpha-1}\|u^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\,d\tau+\varepsilon\int_0^t\tau^{\alpha-1}\|\boldsymbol{Z}^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\,d\tau\\
&\quad+\|\tau^\alpha f(u)\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}+\sum_{k,i,j=1}^d\varepsilon\|\tau^\alpha\frac{\partial}{\partial_{u_i}}f_k(u)\frac{\partial}{\partial x_i}v_j^i\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\sigma_1}_{p,\infty})}.
\end{aligned}
\tag{5.9}
$$

The key ingredient for the gain of decay rate is to control the first term on r.h.s of (5.19) by time-space interpolation arguments. Let $\theta=\frac{2}{d/p+2-\sigma_1}$ such that $d/p=\sigma_1\theta+(d/p+2)(1-\theta)$. Then, applying the interpolation inequality in Lemma 6.2 yields

$$
\begin{aligned}
\int_0^t\tau^{\alpha-1}\|u^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\,d\tau&\lesssim\int_0^t\tau^{\alpha-1}(\|u^\ell\|_{\dot{B}^{\sigma_1}_{p,\infty}})^\theta(\|u^\ell\|_{\dot{B}^{\frac{d}{p}+2}_{p,\infty}})^{1-\theta}\,d\tau\\
&\lesssim\Big(t^{\alpha-\frac12(\frac{d}{p}-\sigma_1)}\|u^\ell\|_{L^\infty_t(\dot{B}^{\sigma_1}_{p,\infty})}\Big)^\theta\|\tau^\alpha u^\ell\|^{1-\theta}_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\\
&\lesssim\kappa\|\tau^\alpha u^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}+\frac{1}{\kappa}t^{\alpha-\frac12(\frac{d}{p}-\sigma_1)}\|u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\sigma_1}_{p,\infty})}
\end{aligned}
\tag{5.10}
$$

for some small constant $\kappa>0$ to be chosen. One also has

$$
\begin{aligned}
\varepsilon\int_0^t\tau^{\alpha-1}\|\boldsymbol{Z}^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\,d\tau&\lesssim\varepsilon\Big(t^{\alpha-\frac12(\frac{d}{p}-\sigma_1)}\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\Big)^\theta\|\tau^\alpha\boldsymbol{Z}^\ell\|^{1-\theta}_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\lesssim\varepsilon^{2\theta}\Big(t^{\alpha-\frac12(\frac{d}{p}-1-\sigma_1)}\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\sigma_1}_{p,\infty})}\Big)^\theta\Big(\frac{1}{\varepsilon}\|\tau^\alpha\boldsymbol{Z}^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\Big)^{1-\theta}\\
&\lesssim\frac{\kappa}{\varepsilon}\|\tau^\alpha\boldsymbol{Z}^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\frac{1}{\kappa}t^{\alpha-\frac12(\frac{d}{p}-\sigma_1)}\varepsilon\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\sigma_1}_{p,\infty})},
\end{aligned}
\tag{5.11}
$$

where we used $\varepsilon\le 1$ and

$$\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim 2^{J_\varepsilon(\frac{d}{p}-\sigma_1)}\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\sigma_1}_{p,\infty})}\lesssim \varepsilon^{-(\frac{d}{p}-\sigma_1)}\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\sigma_1}_{p,\infty})}.$$

Concerning the estimation of nonlinear terms, from (1.11) and Lemma 6.9 with $(s,\sigma)=(d/p+1,d/2)$, we have

$$\begin{aligned}
&\|\tau^\alpha f(u)\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\\
&\lesssim \int_0^t \Big(\big(\tau^\alpha\|u^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}}+\tau^\alpha\|u\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big)\|u\|^\ell_{\dot{B}^{\frac{d}{p}+1}_{p,1}}+2^{J_\varepsilon}\big(2^{J_\varepsilon}\|u^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|u\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big)\tau^\alpha\|u\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\Big)\,d\tau\\
&\lesssim \|\tau^\alpha u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|\tau^\alpha u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})})\|u\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\\
&\quad+\|u\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\frac{1}{\varepsilon^2}\|\tau^\alpha u\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\|u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\frac{1}{\varepsilon}\|\tau^\alpha u\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\\
&\lesssim \mathcal{X}_{p,0}(\|\tau^\alpha u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|\tau^\alpha u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\frac{1}{\varepsilon^2}\|\tau^\alpha u\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}).
\end{aligned}\tag{5.12}$$

Due to $p\ge 2$ and $d/p-1\ge -d/p$, one infers from (1.11), (2.3), (6.3), Lemma 6.6 and Bernstein's inequality that

$$\begin{aligned}
\varepsilon\|\tau^\alpha\frac{\partial}{\partial_{u_i}}f_k(u)\frac{\partial}{\partial x_i}v^i_j\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}&\lesssim \|\tau^\alpha\frac{\partial}{\partial_{u_i}}f_k(u)\frac{\partial}{\partial x_i}v^i_j\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}-1}_{p,\infty})}\\
&\lesssim \|\tau^\alpha u\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|v\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\lesssim (\|\tau^\alpha u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|\tau^\alpha u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})})(\|v\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|v\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})})\\
&\lesssim \mathcal{X}_{p,0}(\|\tau^\alpha u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|\tau^\alpha u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}).
\end{aligned}\tag{5.13}$$

Inserting (5.10)-(5.13) into (5.9) and making use of the evolution of the low-frequency $\dot{B}^{\sigma_1}_{p,\infty}$-regularity in Lemma 5.1, we obtain

$$\begin{aligned}
&\|\tau^\alpha u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|\tau^\alpha u^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}+\varepsilon\|\tau^\alpha\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\frac{1}{\varepsilon}\|\tau^\alpha\boldsymbol{Z}^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\lesssim (\mathcal{X}_{p,0}+\kappa)(\|\tau^\alpha u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|\tau^\alpha u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\frac{1}{\varepsilon^2}\|\tau^\alpha u\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\frac{1}{\varepsilon}\|\tau^\alpha\boldsymbol{Z}^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})})\\
&\quad+\frac{1}{\kappa}t^{\alpha-\frac{1}{2}(\frac{d}{p}-\sigma_1)}(\|u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\sigma_1}_{p,\infty})}+\varepsilon\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\sigma_1}_{p,\infty})}).
\end{aligned}\tag{5.14}$$

- *Step 3: Time-weighted estimates in the high-frequency region*

We perform similar computations as in Subsection 3.2. Multiplying (1.1) with t^α leads to

$$\begin{cases}
\dfrac{\partial}{\partial t}(t^\alpha u)+\displaystyle\sum_{i=1}^d\frac{\partial}{\partial x_i}(t^\alpha v_i)=\alpha t^{\alpha-1}u,\\
\varepsilon^2\dfrac{\partial}{\partial t}(t^\alpha v_i)+A_i\dfrac{\partial}{\partial x_i}(t^\alpha u)+v_i=\varepsilon^2t^{\alpha-1}v_i+(t^\alpha f_i(u)),\\
(t^\alpha u,t^\alpha v_i)|_{t=0}=(0,0).
\end{cases}$$

Repeating the same argument as in (3.15)-(3.23), we get

$$\|\tau^\alpha(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\frac{1}{\varepsilon^2}\|\tau^\alpha(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\lesssim\int_0^t\tau^{\alpha-1}\|(u,\varepsilon\boldsymbol{v})\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\,d\tau+\frac{1}{\varepsilon}\|\tau^\alpha f(u)\|^h_{\widetilde{L}^1_t(B^{\frac{d}{2}}_{2,1})}. \tag{5.15}$$

Let θ be given in Step 2. It is easy to deduce

$$\begin{aligned}\int_0^t\tau^{\alpha-1}\|(u,\varepsilon\boldsymbol{v})\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\,d\tau&\lesssim\int_0^t\tau^{\alpha-1}(\|(u,\varepsilon\boldsymbol{v})\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}})^\theta(\|(u,\varepsilon\boldsymbol{v})\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}})^{1-\theta}\,d\tau\\&\lesssim(t^{\alpha-\frac12(\frac{d}{p}-1-\sigma_1)}\|(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})})^\theta(\|\tau^\alpha(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})})^{1-\theta}\\&\lesssim\frac{\kappa}{\varepsilon^2}\|\tau^\alpha(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\frac{1}{\kappa}t^{\alpha-\frac12(\frac{d}{p}-1-\sigma_1)}\|(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}.\end{aligned}\tag{5.16}$$

Thanks to the composition estimate in Lemma 6.11, there holds that

$$\begin{aligned}\|\tau^\alpha f(u)\|^h_{\widetilde{L}^1_t(B^{\frac{d}{2}}_{2,1})}&\lesssim\big(\|u\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\big)\frac{1}{\varepsilon}\|\tau^\alpha u\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\\&\quad+\big(\|u\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\varepsilon\|u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\big)\|\tau^\alpha u\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\\&\lesssim\mathcal{X}_{p,0}\Big(\|\tau^\alpha u^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}+\frac{1}{\varepsilon^2}\|\tau^\alpha u\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\Big),\end{aligned}\tag{5.17}$$

where we used that

$$\|\tau^\alpha u\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\lesssim\|\tau^\alpha u^\ell\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}+\|\tau^\alpha u^h\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\lesssim\|\tau^\alpha u^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}+\frac{1}{\varepsilon^2}\|\tau^\alpha u\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}.$$

Hence, from (5.15)-(5.17) we have

$$\begin{aligned}&\|\tau^\alpha(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\frac{1}{\varepsilon^2}\|\tau^\alpha(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\\&\lesssim(\kappa+\mathcal{X}_{p,0})\Big(\|\tau^\alpha u^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}+\frac{1}{\varepsilon^2}\|\tau^\alpha u\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\Big)+\frac{1}{\kappa}t^{\alpha-\frac12(\frac{d}{p}-1-\sigma_1)}\|(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}.\end{aligned}\tag{5.18}$$

- *Step 4: Time-decay rates*

Adding (5.14) and (5.18) together, we derive

$$\begin{aligned}&\|\tau^\alpha u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|\tau^\alpha u^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}+\varepsilon\|\tau^\alpha\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\frac{1}{\varepsilon}\|\tau^\alpha\boldsymbol{Z}^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\&\quad+\|\tau^\alpha(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\frac{1}{\varepsilon^2}\|\tau^\alpha(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\\&\lesssim(\mathcal{X}_{p,0}+\kappa)\Big(\|\tau^\alpha u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|\tau^\alpha u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\frac{1}{\varepsilon^2}\|\tau^\alpha u\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\frac{1}{\varepsilon}\|\tau^\alpha\boldsymbol{Z}^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\Big)\\&\quad+\frac{1}{\kappa}t^{\alpha-\frac12(\frac{d}{p}-\sigma_1)}\Big(\|u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\sigma_1}_{p,\infty})}+\varepsilon\|\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\sigma_1}_{p,\infty})}+\|(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\Big).\end{aligned}$$

Choosing the constant κ small enough and noticing that $\mathcal{X}_{p,0}$ satisfies (1.10), we arrive at

$$\begin{aligned}&\|\tau^\alpha u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|\tau^\alpha u^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}+\varepsilon\|\tau^\alpha\boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\frac{1}{\varepsilon}\|\tau^\alpha\boldsymbol{Z}^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\&\quad+\|\tau^\alpha(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\frac{1}{\varepsilon^2}\|\tau^\alpha(u,\varepsilon\boldsymbol{v})\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\lesssim\mathcal{X}_{p,0}t^{\alpha-\frac12(\frac{d}{p}-\sigma_1)},\end{aligned}\tag{5.19}$$

where we have used the uniform bound (1.11) and the low-frequency evolution (5.2). In addition, thanks to $Z_i = A_i \dfrac{\partial}{\partial x_i} u + v_i - f_i(u)$ (5.19) and Lemma 6.6, we recover the time-weighted estimate of $\boldsymbol{v}^\ell$ as follows

$$
\begin{aligned}
\varepsilon\|\tau^\alpha \boldsymbol{v}^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} &\lesssim \varepsilon\|\tau^\alpha \boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \|\tau^\alpha u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \varepsilon\|\tau^\alpha f(u)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} \\
&\lesssim \varepsilon\|\tau^\alpha \boldsymbol{Z}^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \|\tau^\alpha u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \varepsilon\|\tau^\alpha u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} \lesssim \mathcal{X}_{p,0} t^{\alpha-\frac{1}{2}(\frac{d}{p}-\sigma_1)}.
\end{aligned} \tag{5.20}
$$

Dividing (5.19) and (5.20) by t^α and making use of (1.11), we have

$$
\|(u^\ell, \varepsilon\boldsymbol{v}^\ell)(t)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \varepsilon\|(u, \varepsilon\boldsymbol{v})(t)\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}} \lesssim \mathcal{D}_{p,0}(1+t)^{-\frac{1}{2}(\frac{d}{p}-\sigma_1)}, \qquad t>0, \tag{5.21}
$$

which, together with the real interpolation between (5.2) and (5.21), implies

$$
\begin{aligned}
\|(u^\ell, \varepsilon\boldsymbol{v}^\ell)(t)\|_{\dot{B}^{\sigma}_{p,1}} &\lesssim \|(u^\ell, \varepsilon\boldsymbol{v}^\ell)(t)\|_{\dot{B}^{\sigma_1}_{p,\infty}}^{\frac{\frac{d}{p}-\sigma}{\frac{d}{p}-\sigma_1}} \|(u^\ell, \varepsilon\boldsymbol{v}^\ell)(t)\|_{\dot{B}^{\frac{d}{p}}_{p,1}}^{\frac{\sigma-\sigma_1}{\frac{d}{p}-\sigma_1}} \\
&\lesssim \mathcal{D}_{p,0}(1+t)^{-\frac{1}{2}(\sigma-\sigma_1+1)}, \qquad t>0, \quad \sigma_1 < \sigma < \frac{d}{p}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\|(u, \varepsilon\boldsymbol{v})(t)\|_{\dot{B}^{\sigma}_{p,1}} &\lesssim \|(u^\ell, \varepsilon\boldsymbol{v}^\ell)(t)\|_{\dot{B}^{\sigma}_{p,1}} + \varepsilon^{\frac{d}{p}-\sigma}\|(u, \varepsilon\boldsymbol{v})(t)\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}} \\
&\lesssim \mathcal{D}_{p,0}(1+t)^{-\frac{1}{2}(\sigma-\sigma_1)}, \qquad t>0, \quad \sigma_1 < \sigma \le \frac{d}{p}.
\end{aligned}
$$

The proof of Proposition 5.1 is complete.

5.2 Improved decay of the difference

Similarly to Lemma 5.1, we first establish the evolution of the lower-order $\dot{B}^{\sigma_1-1}_{p,\infty}$-regularity of the difference.

Lemma 5.2. *Let $(u, \boldsymbol{v})$ be the global solution to (1.1) subject to the initial datum $(u_0, \boldsymbol{v}_0)$, and u^* be the global solution to (1.2) with the initial datum u_0. Then, under the assumptions (1.10) and (1.15), it holds that*

$$
\|\delta u\|_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{B}^{\sigma_1-1}_{p,\infty})} \lesssim \varepsilon\mathcal{D}_{p,0} \tag{5.22}
$$

with $\mathcal{D}_{p,0} = \|(u^\ell_0, \varepsilon\boldsymbol{v}^\ell_0)\|_{\dot{B}^{\sigma_1}_{p,\infty}} + \mathcal{X}_{p,0}$.

Proof. We recall that the difference $\delta u = u - u^*$ solve (1.22). Applying the low-frequency cutoff operator $\dot{S}_{J_\varepsilon}$ to (1.22) and taking advantage of Lemma 6.8, we deduce

$$
\begin{aligned}
&\|\delta u\|^\ell_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{B}^{\sigma_1-1}_{p,\infty})} + \|\delta u\|^\ell_{\widetilde{L}^2(\mathbb{R}_+;\dot{B}^{\sigma_1}_{p,\infty})} + \|\delta u\|^\ell_{\widetilde{L}^1(\mathbb{R}_+;\dot{B}^{\sigma_1+1}_{p,\infty})} \\
&\quad\lesssim \|\boldsymbol{Z}\|^\ell_{\widetilde{L}^1(\mathbb{R}_+;\dot{B}^{\sigma_1}_{p,\infty})} + \|f(u)-f(u^*)\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\sigma_1}_{p,\infty})}.
\end{aligned} \tag{5.23}
$$

Recalling that (5.2) holds, we have the key observation:

$$
\|\boldsymbol{Z}\|^\ell_{\widetilde{L}^1(\mathbb{R}_+;\dot{B}^{\sigma_1}_{p,\infty})} \lesssim \varepsilon\mathcal{D}_{p,0}. \tag{5.24}
$$

In light of (1.7), (4.1), (2.2) and Corollary 6.7, the nonlinear term in (5.23) can be estimated as follows

$$\|f(u)-f(u^*)\|^{\ell}_{\widetilde{L}^1(\mathbb{R}_+;\dot{B}^{\sigma_1}_{p,\infty})}\lesssim \|(u,u^*)\|_{\widetilde{L}^2(\mathbb{R}_+;\dot{B}^{\frac{d}{p}}_{p,1})}\|\delta u\|_{\widetilde{L}^2(\mathbb{R}_+;\dot{B}^{\sigma_1}_{p,\infty})}\lesssim \mathcal{X}_{p,0}\|\delta u\|_{\widetilde{L}^2(\mathbb{R}_+;\dot{B}^{\sigma_1}_{p,\infty})}. \tag{5.25}$$

According to (1.7), (1.11) and $p\geq 2$, the high frequencies of δu also satisfies

$$\begin{aligned}&\|\delta u\|^h_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{B}^{\sigma_1-1}_{p,\infty})}+\|\delta u\|^h_{\widetilde{L}^2(\mathbb{R}_+;\dot{B}^{\sigma_1}_{p,\infty})}\\&\quad\lesssim \varepsilon^{\frac{d}{p}-\sigma_1+1}\|(u,u^*)\|^h_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{B}^{\frac{d}{p}}_{p,1})}+\varepsilon^{\frac{d}{p}-\sigma_1}\|(u,u^*)\|^h_{\widetilde{L}^2(\mathbb{R}_+;\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim\varepsilon\mathcal{D}_{p,0},\end{aligned}\tag{5.26}$$

where we noted $\frac{d}{p}-\sigma_1>1$. Combining (5.23)-(5.25) together and using the smallness of $\mathcal{X}_{p,0}$, we end up with (5.22). □

Finally, to complete the proof of Theorem 1.4, we prove enhanced time-decay for the difference.

Proposition 5.2. *Let p be given by (1.12). Assume that $(u_0,\boldsymbol{v}_0)$ satisfies (1.10) and (1.15). Let $(u,\boldsymbol{v})$ be the global solution to (1.1) subject to the initial datum $(u_0,\boldsymbol{v}_0)$, and u^* be the global solution to (1.2) subject to the initial datum u_0. Then, it holds that*

$$\|(u-u^*)(t)\|_{\dot{B}^{\sigma}_{p,1}}\lesssim\varepsilon(1+t)^{-\frac12(\sigma-\sigma_1+1)}\mathcal{D}_{p,0},\quad \sigma_1<\sigma\leq\frac{d}{p}-1$$

with $\mathcal{D}_{p,0}=\|(u_0^\ell,\varepsilon\boldsymbol{v}_0^\ell)\|_{\dot{B}^{\sigma_1}_{p,\infty}}+\mathcal{X}_{p,0}$.

Proof. Remember that $(u,\boldsymbol{v})$ satisfies the heat-like decay estimates (5.1). Following the low-frequency analysis of Proposition 5.1, we have

$$\|u^*(t)\|_{\dot{B}^{\sigma}_{p,1}}\lesssim(1+t)^{-\frac12(\sigma-\sigma_1)}\mathcal{D}_{p,0},\qquad t>0,\quad \sigma_1<\sigma\leq\frac{d}{p}. \tag{5.27}$$

Hence, using Bernstein inequality, the decay estimates (5.1) and (5.27) enable us to derive the enhanced decay of δu in high-frequency:

$$\|\delta u(t)\|^h_{\dot{B}^{\sigma}_{p,1}}\lesssim\varepsilon^{\frac{d}{p}-\sigma}\|(u,u^*)(t)\|^h_{\dot{B}^{\frac{d}{p}}_{p,1}}\lesssim\varepsilon(1+t)^{-\frac12(\sigma+\sigma_1+1)},\quad t>0,\quad \sigma_1<\sigma\leq\frac{d}{p}-1. \tag{5.28}$$

In order to derive decay in low frequencies, from (1.22) we have

$$\begin{aligned}\partial_t(t^\alpha\delta u^\ell)-\sum_{i=1}^d\Big(t^\alpha\frac{\partial}{\partial x_i}(A_i\frac{\partial}{\partial x_i}\delta u^\ell)\Big)&=\alpha(t^{\alpha-1}\delta u^\ell)\\-\sum_{i=1}^d\Big(t^\alpha\frac{\partial}{\partial x_i}Z_i^\ell\Big)&-\sum_{i=1}^d\Big(t^\alpha\frac{\partial}{\partial x_i}\dot{S}_{J_\varepsilon}\big(f_i(u)-f_i(u^*)\big)\Big),\end{aligned}\tag{5.29}$$

where $\alpha>1$ is a given constant. Since $\delta u|_{t=0}=0$, using (5.32) and applying Lemma 6.8 to (5.29), it follows that

$$\begin{aligned}&\|\tau^\alpha\delta u^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|\tau^\alpha\delta u^\ell\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|\tau^\alpha\delta u^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\\&\quad\lesssim\int_0^t\tau^{\alpha-1}\|\delta u^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\,d\tau+\|\tau^\alpha\boldsymbol{Z}^\ell\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|\tau^\alpha(f(u)-f(u^*))\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}.\end{aligned}\tag{5.30}$$

Let $\theta \in (0,1)$ be given by $d/p-1=\theta(\sigma_1-1)+(1-\theta)(d/p+1)$. In view of real interpolation, the key estimate (5.22) guarantees that

$$\begin{aligned}\int_0^t \tau^{\alpha-1}\|\delta u^\ell\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\,d\tau &\lesssim \int_0^t \tau^{\alpha-1}\|\delta u^\ell\|^{\theta}_{\dot B^{\sigma_1-1}_{p,\infty}}\|\delta u\|^{1-\theta}_{\dot B^{\frac{d}{p}+1}_{p,1}}\,d\tau\\ &\lesssim \Big(t^{\alpha-\frac12(\frac{d}{p}-\sigma_1)}\|\delta u\|_{\dot B^{\sigma_1-1}_{p,\infty}}\Big)^{\theta}\|\delta u\|^{1-\theta}_{\widetilde L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}\\ &\lesssim \kappa_1\|\delta u\|_{\widetilde L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}+\frac{\varepsilon}{\kappa_1}t^{\alpha-\frac12(\frac{d}{p}-\sigma_1)}\mathcal{D}_{p,0}\end{aligned}\tag{5.31}$$

for some constant κ_1 to be chosen later. To control the second term on the r.h.s of $(5.29)_1$, from (5.16) and (5.19), we have

$$\begin{aligned}\|\tau^\alpha \boldsymbol{Z}\|_{\widetilde L^1_t(\dot B^{\frac{d}{p}}_{p,1})} &\lesssim \|\tau^\alpha \boldsymbol{Z}^\ell\|_{\widetilde L^1_t(\dot B^{\frac{d}{p}}_{p,1})}+\|\tau^\alpha u\|^h_{\widetilde L^1_t(\dot B^{\frac{d}{2}}_{2,1})}+\varepsilon\|\tau^\alpha v\|^h_{\widetilde L^1_t(\dot B^{\frac{d}{2}}_{2,1})}+\varepsilon\|\tau^\alpha f(u)\|^h_{\widetilde L^1_t(\dot B^{\frac{d}{2}}_{2,1})}\\ &\lesssim t^{\alpha-\frac12(\frac{d}{p}-\sigma_1)}\mathcal{D}_{p,0}.\end{aligned}\tag{5.32}$$

The estimates (1.7) and (4.1) as well as Corollary 6.7 ensure that

$$\|\tau^\alpha(f(u)-f(u^*))\|_{\widetilde L^1_t(\dot B^{\frac{d}{p}}_{p,1})}\lesssim \|(u,u^*)\|_{\widetilde L^2_t(\dot B^{\frac{d}{p}}_{p,1})}\|\delta u\|_{\widetilde L^2_t(\dot B^{\frac{d}{p}}_{p,1})}\lesssim \mathcal{D}_{p,0}\|\delta u\|_{\widetilde L^2_t(\dot B^{\frac{d}{p}}_{p,1})}.\tag{5.33}$$

From (5.30)-(5.33), one infers, for some suitably small $\kappa_1>0$, that

$$\|\tau^\alpha\delta u\|_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})}+\|\tau^\alpha\delta u\|_{\widetilde L^2_t(\dot B^{\frac{d}{p}}_{p,1})}+\|\tau^\alpha\delta u\|_{\widetilde L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}\lesssim \varepsilon t^{\alpha-\frac12(\frac{d}{p}-\sigma_1)}\mathcal{D}_{p,0}.\tag{5.34}$$

Since $\alpha>1$ is any given constant, dividing two sides of (5.34) by t^α and using the bounds (1.7) and (4.1), one has

$$\|\delta u(t)\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\lesssim \varepsilon(1+t)^{-\frac12(\frac{d}{p}-\sigma_1)}\mathcal{D}_{p,0},\quad t>0.\tag{5.35}$$

By virtue of the real interpolation between (5.22) and (5.35), we get

$$\|\delta u(t)\|_{\dot B^{\sigma}_{p,1}}\lesssim \|\delta u(t)\|^{\frac{\frac{d}{p}-1-\sigma}{\frac{d}{p}-\sigma_1}}_{\dot B^{\sigma_1-1}_{p,\infty}}\|\delta u(t)\|^{\frac{\sigma-\sigma_1+1}{\frac{d}{p}-\sigma_1}}_{\dot B^{\frac{d}{p}-1}_{p,1}}\lesssim \varepsilon(1+t)^{-\frac12(\sigma-\sigma_1+1)}\mathcal{D}_{p,0}\tag{5.36}$$

for $t>0$ and $-\frac{d}{p}<\sigma<\frac{d}{p}-1$. By (5.35) and (5.36), the proof of Proposition 5.2 is concluded. □

Proof of Theorem 1.4. Under the assumptions of Theorem 1.4, we conclude from Propositions 5.1 and 5.2 that $(u,\boldsymbol{v})$ satisfies the time-decay estimates (1.16), and the difference $u-u^*$ verifies the enhanced time-decay estimates (1.17). □

6 Appendix

6.1 Some analysis tools in Besov spaces

We state some properties of Besov spaces and related estimates which we repeatedly used in the paper. The first lemma is devoted to the classical Bernstein's inequalities.

Lemma 6.1 ([2]). *Let $0 < r < R, 1 \le p \le q \le \infty$ and $k \in \mathbb{N}$. Then, for any function $u \in L^p$ and $\lambda_1 > 0$, it holds*

$$\begin{cases}\text{Supp } \mathcal{F}(u) \subset \{\xi \in \mathbb{R}^d \mid |\xi| \le \lambda_1 R\} \Rightarrow \|D^k u\|_{L^q} \lesssim \lambda_1^{k+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p},\\ \text{Supp } \mathcal{F}(u) \subset \{\xi \in \mathbb{R}^d \mid \lambda_1 r \le |\xi| \le \lambda_1 R\} \Rightarrow \|D^k u\|_{L^p} \sim \lambda_1^k \|u\|_{L^p}.\end{cases}$$

The next lemma states the classical interpolation inequalities.

Lemma 6.2 ([2,38]). *Let $1 \le p, r, r_1, r_2 \le \infty$.*

- *If $u \in \dot{B}^s_{p,r_1} \cap \dot{B}^{\tilde{s}}_{p,r_2}$ and $s \neq \tilde{s}$ then, $u \in \dot{B}^{\theta s+(1-\theta)\tilde{s}}_{p,r}$ for all $\theta \in (0,1)$ and*

$$\|u\|_{\dot{B}^{\theta s+(1-\theta)\tilde{s}}_{p,r}} \lesssim \|u\|^{\theta}_{\dot{B}^s_{p,r}} \|u\|^{1-\theta}_{\dot{B}^{\tilde{s}}_{p,r}}$$

  with $\frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}$.

- *If $u \in \dot{B}^s_{p,\infty} \cap \dot{B}^{\tilde{s}}_{p,\infty}$ and $s < \tilde{s}$, then, $u \in \dot{B}^{\theta s+(1-\theta)\tilde{s}}_{p,1}$ for all $\theta \in (0,1)$ and*

$$\|u\|_{\dot{B}^{\theta s+(1-\theta)\tilde{s}}_{p,1}} \le \frac{C}{\theta(1-\theta)(\tilde{s}-s)} \|u\|^{\theta}_{\dot{B}^s_{p,\infty}} \|u\|^{1-\theta}_{\dot{B}^{\tilde{s}}_{p,\infty}}.$$

The following interpolation inequalities for high and low frequencies is also used in this paper.

Corollary 6.3. *Let $s_1 \le s_2$, $q, r \in [1, +\infty]$, $\theta \in (0,1)$ and $1 \le \alpha_1 \le \alpha \le \alpha_2 \le \infty$ such that $\frac{1}{\alpha} = \frac{\theta}{\alpha_1} + \frac{1-\theta}{\alpha_2}$, then*

$$\|u\|^{\ell}_{\widetilde{L}^{\alpha}_T(\dot{B}^{\theta s_1+(1-\theta)s_2}_{q,r})} \le \left(\|u\|^{\ell}_{\widetilde{L}^{\alpha_1}_T(\dot{B}^{s_1}_{q,r})}\right)^{\theta} \left(\|u\|^{\ell}_{\widetilde{L}^{\alpha_2}_T(\dot{B}^{s_2}_{q,r})}\right)^{1-\theta},$$
$$\|u\|^{h}_{\widetilde{L}^{\alpha}_T(\dot{B}^{\theta s_1+(1-\theta)s_2}_{q,r})} \le \left(\|u\|^{h}_{\widetilde{L}^{\alpha_1}_T(\dot{B}^{s_1}_{q,r})}\right)^{\theta} \left(\|u\|^{h}_{\widetilde{L}^{\alpha_2}_T(\dot{B}^{s_2}_{q,r})}\right)^{1-\theta}.$$

The following lemma pertains to classical product laws.

Lemma 6.4. [2,16] *Let $s > 0$, $1 \le p, r \le \infty$, then, we have*

$$\|ab\|_{\dot{B}^s_{p,r}} \lesssim \|a\|_{L^\infty} \|b\|_{\dot{B}^s_{p,r}} + \|b\|_{L^\infty} \|a\|_{\dot{B}^s_{p,r}}. \tag{6.1}$$

For $d \ge 1$ and $-\min\{d/p, d/p'\} < s \le d/p$ for $1/p + 1/p' = 1$, the following inequality holds:

$$\|ab\|_{\dot{B}^s_{p,1}} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|b\|_{\dot{B}^s_{p,1}}. \tag{6.2}$$

Finally, if $d \ge 1$ and $-\min\{d/p, d/p'\} \le s < d/p$ for $1/p + 1/p' = 1$, then, we have

$$\|ab\|_{\dot{B}^s_{p,\infty}} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|b\|_{\dot{B}^s_{p,\infty}}. \tag{6.3}$$

We also show a new product law to handle some nonlinear terms in the proof of the uniqueness.

Lemma 6.5. *Let $s_1 > 0$, $2 \le p \le 4$. Then, it holds that*

$$\|ab\|^h_{\dot{B}^{s_1}_{2,1}} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|b\|^h_{\dot{B}^{s_1}_{2,1}} + \|b\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a\|^h_{\dot{B}^{s_1}_{2,1}} + \|b\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a\|^{\ell}_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|b\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a\|^{\ell}_{\dot{B}^{s_1-\frac{d}{2}+\frac{d}{p}}_{p,1}}. \tag{6.4}$$

Proof. We use Bony's paraproduct decomposition for two tempered distributions a and b:

$$ab = T_a b + R[a,b] + T_b a \quad \text{with} \quad T_a b \triangleq \sum_{j'\in\mathbb{Z}} \dot{S}_{j'-1} a \dot{\Delta}_{j'} b \quad \text{and} \quad R[a,b] \triangleq \sum_{|j'-j''|\leq 1} \dot{\Delta}_{j''} a \dot{\Delta}_{j'} b.$$

First, we bound $T_a b$. It is clear that

$$\|T_a b\|^h_{\dot{B}^{s_1}_{2,1}} \leq \sum_{\substack{j\geq J-1\\ |j-j'|\leq 1}} 2^{s_1 j} \|\dot{S}_{j'-1} a \dot{\Delta}_j \dot{\Delta}_{j'} b\|_{L^2} + \sum_{\substack{j\geq J-1\\ |j-j'|\leq 4}} 2^{s_1 j} \|[\dot{\Delta}_j, \dot{S}_{j'-1} a] \dot{\Delta}_{j'} b\|_{L^2}.$$

The embedding $\dot{B}^{\frac{d}{p}}_{p,1} \hookrightarrow L^\infty$ leads to

$$\sum_{\substack{j\geq J-1\\ |j-j'|\leq 1}} 2^{s_1 j} \|\dot{S}_{j'-1} a \dot{\Delta}_j \dot{\Delta}_{j'} b\|_{L^2} \lesssim \|\dot{S}_{j'-1} a\|_{L^\infty} \sum_{j\geq J-1} 2^{s_1 j} \|\dot{\Delta}_j b\|_{L^2} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|b\|^h_{\dot{B}^{s_1}_{2,1}}.$$

Note that

$$\sum_{\substack{j\geq J-1\\ |j-j'|\leq 4}} 2^{s_1 j} \|[\dot{\Delta}_j, \dot{S}_{j'-1} a] \dot{\Delta}_{j'} b\|_{L^2} \lesssim \Big(\sum_{j'\geq J-1} 2^{s_1 j'} + \sum_{J-5\leq j'\leq J-2} \Big) 2^{s_1 j'} \|[\dot{\Delta}_j, S_{j'-1} a] \Delta_{j'} b\|_{L^2}.$$

By the commutator estimate in [2, Lemma 2.97], we have

$$\begin{aligned}
\sum_{j'\geq J-1} 2^{s_1 j'} \|[\dot{\Delta}_j, S_{j'-1} a] \Delta_{j'} b\|_{L^2} &\lesssim \sum_{j'\geq J-1} (2^{s_1 j'} \|\Delta_{j'} b\|_{L^2})(2^{-j'} \|\nabla S_{j'-1} a\|_{L^\infty}) \\
&\lesssim \|\nabla a\|_{\dot{B}^{-1}_{\infty,1}} \|b\|^h_{\dot{B}^{s_1}_{2,1}} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|b\|^h_{\dot{B}^{s_1}_{2,1}}.
\end{aligned}$$

Similarly, as $\frac{2p}{p-2} \geq p$ due to $2 \leq p \leq 4$, one has

$$\begin{aligned}
&\sum_{J-5\leq j'\leq J-2} 2^{s_1 j'} \|[\dot{\Delta}_j, S_{j'-1} a] \Delta_{j'} b\|_{L^2} \\
&\lesssim 2^{(s_1-\frac{d}{2})J} \sum_{J-5\leq j'\leq J-2} (2^{\frac{d}{p} j'} \|\Delta_{j'} b\|_{L^p})(2^{(\frac{d}{2}-\frac{d}{p}-1)j'} \|\nabla S_{j'-1} a\|_{L^{\frac{2p}{p-2}}}) \\
&\lesssim 2^{(s_1-\frac{d}{2})J} \|b\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a\|^\ell_{\dot{B}^{\frac{d}{2}-\frac{d}{p}-1}_{\frac{2p}{p-2},1}} \lesssim 2^{(s_1-\frac{d}{2})J} \|b\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}}.
\end{aligned}$$

Hence, it follows that

$$\|T_a b\|^h_{\dot{B}^{s_1}_{2,1}} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|b\|^h_{\dot{B}^{s_1}_{2,1}} + \|b\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a\|^\ell_{\dot{B}^{\frac{d}{p}}_{p,1}}.$$

The term $\|T_b a\|^h_{\dot{B}^{s_1}_{2,1}}$ can be treated similarly. Finally, classical remainder estimates (see [2, Theorem 2.85]) implies

$$\begin{aligned}
\|R[a,b]\|^h_{\dot{B}^{s_1}_{2,1}} &\leq \|R[a^h,b]\|^h_{\dot{B}^{s_1}_{2,1}} + \|R[a^\ell,b]\|^h_{\dot{B}^{s_1}_{2,1}} \\
&\lesssim \|b\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a\|^h_{\dot{B}^{s_1}_{2,1}} + \|b\|_{\dot{B}^{\frac{d}{2}-\frac{d}{p}}_{\frac{2p}{p-2},1}} \|a\|^\ell_{\dot{B}^{s_1-\frac{d}{2}+\frac{d}{p}}_{p,1}} \\
&\lesssim \|b\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a\|^h_{\dot{B}^{s_1}_{2,1}} + \|b\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|a\|^\ell_{\dot{B}^{s_1-\frac{d}{2}+\frac{d}{p}}_{p,1}}.
\end{aligned}$$

This completes the proof of Lemma 6.5. □

Next, we introduce classical estimates for the composition of functions.

Lemma 6.6 ([2,22]). *Assume $d \geq 1$ and $F(m)$ be a smooth function such that $F(0)=0$. For any $s>0$, $p,r \in [1,\infty]$ and real-valued function m in $\dot{B}^s_{p,r} \cap L^\infty$, $F(u)$ belongs to $\dot{B}^s_{p,r}$ and fulfills*

$$\|F(m)\|_{\dot{B}^s_{p,r}} \leq C_m \|m\|_{\dot{B}^s_{p,r}},$$

where $C_m>0$ denotes a constant dependent of $\|m\|_{L^\infty}$, F', s, p, r and d.

We have the following corollary.

Corollary 6.7. *Assume that $F(m)$ is a smooth function satisfying $F'(0)=0$. Let $1 \leq p \leq \infty$. For any couple (m_1,m_2) of functions in $\dot{B}^s_{p,1} \cap L^\infty$, there exists a constant $C_{m_1,m_2}>0$ depending on F'' and $\|(m_1,m_2)\|_{L^\infty}$ such that*

- *Let $-\min\{\frac{d}{p}, d(1-\frac{1}{p})\} < s \leq \frac{d}{p}$ and $1 \leq r \leq \infty$. Then, we have*

$$\|F(m_1)-F(m_2)\|_{\dot{B}^s_{p,r}} \leq C\|(m_1,m_2)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|m_1-m_2\|_{\dot{B}^s_{p,r}}. \tag{6.5}$$

- *In the limiting case $r=\infty$, for any $-\min\{\frac{d}{p}, d(1-\frac{1}{p})\} \leq s < \frac{d}{p}$, it holds that*

$$\|F(m_1)-F(m_2)\|_{\dot{B}^s_{p,\infty}} \leq C\|(m_1,m_2)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|m_1-m_2\|_{\dot{B}^s_{p,\infty}}. \tag{6.6}$$

Proof. From

$$F(m_1)-F(m_2) = (m_1-m_2)\int_0^1 F'(m_1+\tau(m_2-m_1))\, d\tau,$$

and (6.2), Lemma 6.6 and the embedding $\dot{B}^{\frac{d}{p}}_{p,1} \to L^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}\|F(m_1)-F(m_2)\|_{\dot{B}^s_{p,r}} &\lesssim \|m_1-m_2\|_{\dot{B}^s_{p,r}} \sup_{\tau\in[0,1]} \|f'(m_1+\tau(m_2-m_1))\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\\ &\lesssim \|m_1-m_2\|_{\dot{B}^s_{p,r}} \|(m_1,m_2)\|_{\dot{B}^{\frac{d}{p}}_{p,1}}.\end{aligned}$$

This yields (6.5). In order to prove (6.6), one can follow a similar argument replacing (6.2) by (6.3). □

We now consider the following Cauchy problem of the parabolic equations:

$$\begin{cases}\partial_t u - \displaystyle\sum_{i=1}^d \frac{\partial}{\partial x_i}(A_i \frac{\partial}{\partial x_i} u) = F, & t>0, \quad \boldsymbol{x} \in \mathbb{R}^d,\\ u(0,\boldsymbol{x}) = u_0(\boldsymbol{x}),\end{cases} \tag{6.7}$$

where the unknown is $u = u(\boldsymbol{x},t) \in \mathbb{R}^n$.

Lemma 6.8. *Let $\varepsilon>0$, $d,n \geq 1$, $s_1,s_2 \in \mathbb{R}$, $1 \leq \rho_1,\rho_2,p_1,p_2 \leq \infty$ and $T>0$ be given time, and J_ε be the threshold between low and high frequencies. Assume $u_0^\ell \in \dot{B}^{s_1}_{p_1,1}$, $u_0^h \in \dot{B}^{s_2}_{p_2,1}$, $F^\ell \in \widetilde{L}^{\rho_1}_T(\dot{B}^{s_1-2+\frac{2}{\rho_1}}_{p_1,1})$ and $F^h \in \widetilde{L}^{\rho_2}_T(\dot{B}^{s_2-2+\frac{2}{\rho_2}}_{p_2,1})$. If u is a solution to the Cauchy problem (6.7), then, for all $\tilde{\rho}_1 \in [\rho_1,\infty]$ and $\tilde{\rho}_2 \in [\rho_2,\infty]$, u satisfies*

$$\|u\|^\ell_{\widetilde{L}^{\tilde{\rho}_1}_T(\dot{B}^{s_1+\frac{2}{\tilde{\rho}_1}}_{p_1,1})} \leq C\Big(\|u_0\|^\ell_{\dot{B}^{s_1}_{p_1,1}} + \|F\|^\ell_{\widetilde{L}^{\rho_1}_T(\dot{B}^{s_1-2+\frac{2}{\rho_1}}_{p_1,1})}\Big),$$

and

$$\|u\|^h_{\widetilde{L}^{\tilde{\rho}_2}_T(\dot{B}^{s_2+\frac{2}{\tilde{\rho}_2}}_{p_2,1})} \leq C\Big(\|u_0\|^h_{\dot{B}^{s_2}_{p_2,1}} + \|F\|^h_{\widetilde{L}^{\rho_2}_T(\dot{B}^{s_2-2+\frac{2}{\rho_2}}_{p_2,1})}\Big),$$

where $C>0$ is a constant independent of T and ε.

Proof. Note that from (6.7), $\dot{\Delta}_j u$ can be represented by

$$\dot{\Delta}_j u(t) = e^{t\sum_{i=1}^{d} \frac{\partial}{\partial x_i}(A_i \frac{\partial}{\partial x_i})} \dot{\Delta}_j u_0 + \int_0^t e^{\sum_{i=1}^{d} \frac{\partial}{\partial x_i}(A_i \frac{\partial}{\partial x_i})(t-\tau)} \dot{\Delta}_j F(\tau)\, d\tau.$$

As in [2, Lemma 2.4], one can show the localized semigroup estimate

$$\|\dot{\Delta}_j u(t)\|_{L^p} \lesssim e^{-\tilde{a}2^{2j}t}\|\dot{\Delta}_j u_0\|_{L^p} + \int_0^t e^{-\tilde{a}2^{2j}(t-\tau)}\|\dot{\Delta}_j F(\tau)\|_{L^p}\, d\tau$$

for some constant $\tilde{a} > 0$. Setting $p = p_1$ and applying Young's inequality, we get

$$\|\dot{\Delta}_j u\|_{L_T^{\tilde{\rho}_1} L^{p_1}} \lesssim \Big(\frac{1-e^{-\tilde{a}\tilde{\rho}_1 T 2^{2j}}}{\tilde{a}\tilde{\rho}_1 2^{2j}}\Big)^{\frac{1}{\tilde{\rho}_1}} \|\dot{\Delta}_j u_0\|_{L^{p_1}} + \Big(\frac{1-e^{-\tilde{a}\rho_1' T 2^{2j}}}{\tilde{a}\rho_1' 2^{2j}}\Big)^{\frac{1}{\rho_1'}} \|\dot{\Delta}_j F\|_{L_T^{\rho_1} L^{p_1}}$$

with $\frac{1}{\rho_1'} = 1 + \frac{1}{\tilde{\rho}_1} - \frac{1}{\rho_1}$. Summing the above inequalities over $j \leq J_\varepsilon$, we have

$$\|u\|^{\ell}_{L_T^{\tilde{\rho}_1}(\dot{B}^{s_1+\frac{2}{\tilde{\rho}_1}}_{p_1,1})} \lesssim \|u_0\|^{\ell}_{\dot{B}^{s_1}_{p_1,1}} + \|F\|^{\ell}_{\widetilde{L}_t^{\rho_1}(\dot{B}^{s_1-2+\frac{2}{\rho_1}}_{p_1,1})}.$$

The second estimate is similar. □

We also study the Cauchy problem of the damped equation

$$\begin{cases} \partial_t u + \dfrac{1}{\varepsilon^2} u = F, \\ u(0,\boldsymbol{x}) = u_0(\boldsymbol{x}), \end{cases} \qquad t > 0, \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{6.8}$$

By direct computations, we have the following lemma.

Lemma 6.9. *Let $d, n \geq 1$, $s \in \mathbb{R}$, $s_1, s_2 \in \mathbb{R}$, $1 \leq \rho_1, \rho_2, p_1, p_2 \leq \infty$ and $T > 0$ be given time, and J_ε be the threshold between low and high frequencies. Assume $u_0^\ell \in \dot{B}^{s_1}_{p_1,1}$, $u_0^h \in \dot{B}^{s_2}_{p_2,1}$, $F^\ell \in \widetilde{L}^{\rho_1}_T(\dot{B}^{s_1}_{p_1,1})$ and $F^h \in \widetilde{L}^{\rho_1}_T(\dot{B}^{s_2}_{p_2,1})$. If u is a solution to the Cauchy problem (6.8), then, for all $\tilde{\rho}_1 \in [\rho_1, \infty]$ and $\tilde{\rho}_2 \in [\rho_2, \infty]$, it holds that*

$$\varepsilon^{-\frac{2}{\tilde{\rho}_1}} \|u\|^{\ell}_{\widetilde{L}^{\tilde{\rho}_1}_T(\dot{B}^{s_1}_{p_1,1})} \leq C\Big(\|u_0\|^{\ell}_{\dot{B}^{s_1}_{p_1,1}} + \varepsilon^{2-\frac{2}{\rho_1}} \|F\|^{\ell}_{\widetilde{L}^{\rho_1}_T(\dot{B}^{s_1}_{p_1,1})}\Big),$$

and

$$\varepsilon^{-\frac{2}{\tilde{\rho}_2}} \|u\|^{h}_{\widetilde{L}^{\tilde{\rho}_2}_T(\dot{B}^{s_2}_{p_2,1})} \leq C\Big(\|u_0\|^{h}_{\dot{B}^{s_2}_{p_2,1}} + \varepsilon^{2-\frac{2}{\rho_1}} \|F\|^{h}_{\widetilde{L}^{\rho_2}_T(\dot{B}^{s_2}_{p_2,1})}\Big),$$

where $C > 0$ is a constant independent of T and ε.

6.2 New estimates of composition functions in hybrid Besov spaces

We develop some new composition estimates for functions in $L^p - L^2$ hybrid Besov spaces which generalize the previous related estimates in [11, 18, 39] and play a key role in our nonlinear analysis. We denote by J the general threshold between low and high frequencies (not necessarily defined by (1.9)).

Lemma 6.10. *Let $s > 0$, $\sigma \in \mathbb{R}$, and $p \geq \max\{1, \frac{2d}{d+2}\}$. Then, for any smooth function $F(m)$ satisfying $F(0) = F'(0) = 0$, there is a constant $C_m > 0$ depending only on $\|m\|_{L^\infty}$, s, σ and d such that*

$$\begin{aligned} \|F(m)\|^{\ell}_{\dot{B}^{s}_{p,1}} &\leq C_m\big(\|m^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big)\|m\|^{\ell}_{\dot{B}^{s}_{p,1}} \\ &\quad + C_m 2^{(s-\sigma+\frac{d}{2}-\frac{d}{p})J}\big(2^J\|m^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big)\|m\|^h_{\dot{B}^{\sigma}_{2,1}}. \end{aligned} \tag{6.9}$$

Proof. As in [2, Theorem 2.61], we use $F(0)=F'(0)=0$ to decompose $F(m)$ as

$$F(m)=\sum_{k'\in\mathbb{Z}}F(\dot{S}_{k'+1}m)-F(\dot{S}_{k'}m)=\sum_{k'\in\mathbb{Z}}\dot{\Delta}_{k'}mM_{k'}=\dot{\Delta}_{k'}m(\dot{S}_{k'-1}m+\dot{\Delta}_{k'}m)\widetilde{M}_{k'} \tag{6.10}$$

with

$$\widetilde{M}_{k'}\triangleq\int_0^1 M'_{k'}(\tau(\dot{S}_{k'-1}m+\dot{\Delta}_{k'}m))\,d\tau\quad\text{and}\quad M_{k'}\triangleq\int_0^1 F'(\dot{S}_{k'}m+\tau\dot{\Delta}_{k'}m)\,d\tau.$$

Thanks to Bernstein's inequality and Leibniz's formula, we have

$$\begin{aligned}&\|\dot{\Delta}_k(\dot{\Delta}_{k'}m\widetilde{M}_{k'}\dot{S}_{k'-1}m)\|_{L^{p^1}}+\|\dot{\Delta}_k(\dot{\Delta}_{k'}m\widetilde{M}_{k'}\dot{\Delta}_{k'}m)\|_{L^{p^1}}\\&\quad\lesssim 2^{(k'-k)|\beta|}\|\dot{\Delta}_{k'}m\|_{L^{p^2}}\|m\|_{L^{p^3}}(1+\|m\|_{L^\infty})^{|\beta|}\end{aligned} \tag{6.11}$$

with $\frac{1}{p^3}+\frac{1}{p^2}=\frac{1}{p^1}$ and $\beta\in\mathbb{N}^n$.

In order to justify (6.9), we decompose the low-frequency region as

$$\Omega_{\ell\ell}\triangleq\{(k,k')\mid k'<k\le J\},\quad\Omega_{\ell m}\triangleq\{(k,k')\mid k\le k'\le J\},\quad\Omega_{\ell h}\triangleq\{(k,k')\mid k\le J<k'\}.$$

Applying (6.11) with $|\beta|=[s]+1$ and $(p^1,p^2,p^3)=(\infty,p,\infty)$, we obtain

$$\begin{aligned}&\sum_{\Omega_{\ell\ell}}2^{ks}\|\dot{\Delta}_k(\dot{\Delta}_{k'}m(\dot{S}_{k'-1}m+\dot{\Delta}_{k'}m)\widetilde{M}_{k'})\|_{L^p}\\&\quad\lesssim(1+\|m\|_{L^\infty})^{[s]+1}\|m\|_{L^\infty}\sum_{k'<J}2^{k's}\|\dot{\Delta}_{k'}m\|_{L^p}\sum_{k>k'}2^{(k-k')(s-[s]-1)}\\&\quad\lesssim(1+\|m\|_{L^\infty})^{[s]+1}\big(\|m^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}}+\|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big)\|m\|^\ell_{\dot{B}^s_{p,1}}.\end{aligned} \tag{6.12}$$

Similarly, it follows by (6.11) with $|\beta|=0$ and $(p^1,p^2,p^3)=(\infty,p,\infty)$ that

$$\begin{aligned}&\sum_{\Omega_{\ell m}}2^{ks}\|\dot{\Delta}_k(\dot{\Delta}_{k'}m(\dot{S}_{k'-1}m+\dot{\Delta}_{k'}m)\widetilde{M}_{k'})\|_{L^p}\\&\quad\lesssim\|m\|_{L^\infty}\sum_{k'\le J}2^{k's}\|\dot{\Delta}_{k'}m\|_{L^p}\sum_{k\le k'}2^{(k-k')s}\lesssim\big(\|m^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}}+\|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big)\|m\|^\ell_{\dot{B}^s_{p,1}}.\end{aligned} \tag{6.13}$$

We analyze the part of the region $\Omega_{\ell h}$ as follows.

- Case 1: $\max\{1,\frac{2d}{d+2}\}\le p<2$ and $s\le\sigma$.

By (6.11) with $|\beta|=0$ and $(p^1,p^2,p^3)=(p,\frac{2p}{2-p},2)$, we have

$$\begin{aligned}&\sum_{\Omega_{\ell h}}2^{ks}\|\dot{\Delta}_k(\dot{\Delta}_{k'}m(\dot{S}_{k'-1}m+\dot{\Delta}_{k'}m)\widetilde{M}_{k'})\|_{L^p}\\&\quad\lesssim\|m\|_{L^{\frac{2p}{2-p}}}\sum_{k'>J}2^{k's}\|\dot{\Delta}_{k'}m\|_{L^2}\sum_{k'>k}2^{(k-k')s}\lesssim\|m\|_{L^{\frac{2p}{2-p}}}\|m\|^h_{\dot{B}^s_{2,1}}.\end{aligned} \tag{6.14}$$

Noticing that $\frac{d}{p}\le d-\frac{d}{p}<\frac{d}{2}$ due to $\max\{1,\frac{2d}{d+2}\}\le p<2$, we conclude from the low-high frequency decomposition, (2.2) and embedding inequalities that

$$\|m\|_{L^{\frac{2p}{2-p}}}\lesssim\|m\|_{\dot{B}^{d-\frac{d}{p}}_{2,1}}\lesssim\|m^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}}+\|m\|^h_{\dot{B}^{d-\frac{d}{p}}_{2,1}}\lesssim 2^{(\frac{d}{2}-\frac{d}{p}+1)J}\|m^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+2^{(\frac{d}{2}-\frac{d}{p})J}\|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}. \tag{6.15}$$

Therefore, by (2.2), (6.14), (6.15) and the fact that $s\le\sigma$, we obtain

$$\sum_{\Omega_{\ell h}}2^{ks}\|\dot{\Delta}_k(\dot{\Delta}_{k'}m(\dot{S}_{k'-1}m+\dot{\Delta}_{k'}m)\widetilde{M}_{k'})\|_{L^p}\lesssim 2^{(s-\sigma+\frac{d}{2}-\frac{d}{p})J}(2^J\|m^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}})\|m\|^h_{\dot{B}^\sigma_{2,1}}.$$

- Case 2: $p \geq 2$ and $s \leq \sigma$.

In this case, one deduces from Bernstein's inequality, (2.2) and (6.11) with $|\beta| = 0$ and $(p^1, p^2, p^3) = (2, \infty, 2)$ that

$$\begin{aligned}&\sum_{\Omega_{\ell h}} 2^{ks} \|\dot{\Delta}_k(\dot{\Delta}_{k'} m(\dot{S}_{k'-1} m + \dot{\Delta}_{k'} m)\widetilde{M}_{k'})\|_{L^p}\\
&\lesssim 2^{(\frac{d}{2}-\frac{d}{p})J} \sum_{\Omega_{\ell h}} 2^{ks} \|\dot{\Delta}_k(\dot{\Delta}_{k'} m(\dot{S}_{k'-1} m + \dot{\Delta}_{k'} m)\widetilde{M}_{k'})\|_{L^2}\\
&\lesssim 2^{(\frac{d}{2}-\frac{d}{p})J} \|m\|_{L^\infty} \|m\|^h_{\dot{B}^s_{2,1}}\\
&\lesssim 2^{(s-\sigma+\frac{d}{2}-\frac{d}{p})J} (2^J \|m^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}) \|m\|^h_{\dot{B}^\sigma_{2,1}}.\end{aligned}$$

- Case 3: $s > \sigma$.

By the low-frequency cut-off and calculations in Cases 1-2, it holds that

$$\begin{aligned}&\sum_{\Omega_{\ell h}} 2^{ks} \|\dot{\Delta}_k(\dot{\Delta}_{k'} m(\dot{S}_{k'-1} m + \dot{\Delta}_{k'} m)\widetilde{M}_{k'})\|_{L^p}\\
&\lesssim 2^{(s-\sigma)J} \sum_{\Omega_{\ell h}} 2^{k\sigma} \|\dot{\Delta}_k(\dot{\Delta}_{k'} m \widetilde{M}_{k'} \dot{S}_{k'-1} m)\|_{L^p} \lesssim 2^{(s-\sigma+\frac{d}{2}-\frac{d}{p})J} (2^J \|m^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}) \|m\|^h_{\dot{B}^\sigma_{2,1}}.\end{aligned}$$

Gathering the above three cases, we derive

$$\begin{aligned}&\sum_{\Omega_{\ell h}} 2^{ks} \|\dot{\Delta}_k(\dot{\Delta}_{k'} m(\dot{S}_{k'-1} m + \dot{\Delta}_{k'} m)\widetilde{M}_{k'})\|_{L^p}\\
&\lesssim 2^{(s-\sigma+\frac{d}{2}-\frac{d}{p})J} (2^J \|m^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}) \|m\|^h_{\dot{B}^\sigma_{2,1}}\end{aligned} \tag{6.16}$$

for all $s > 0$, $\sigma \in \mathbb{R}$ and $p \geq \max\{1, \frac{2d}{d+2}\}$. Adding (6.12), (6.13) and (6.16) together, we arrive at

$$\begin{aligned}&\sum_{k \leq J,\ k' \in \mathbb{Z}} 2^{ks} \|\dot{\Delta}_k(\dot{\Delta}_{k'} m(\dot{S}_{k'-1} m + \dot{\Delta}_{k'} m)\widetilde{M}_{k'})\|_{L^p}\\
&\lesssim (1 + \|m\|_{L^\infty})^{[s]+1} \big(\|m^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big) \|m\|^\ell_{\dot{B}^s_{p,1}} + 2^{(s-\sigma+\frac{d}{2}-\frac{d}{p})J} \big(2^J \|m^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big) \|m\|^h_{\dot{B}^\sigma_{2,1}}.\end{aligned}$$

Similar computations yield

$$\begin{aligned}&\sum_{k \leq J,\ k' \in \mathbb{Z}} 2^{ks} \|\dot{\Delta}_k(\dot{\Delta}_{k'} m(\dot{S}_{k'-1} m + \dot{\Delta}_{k'} m) M_{k'})\|_{L^p}\\
&\lesssim (1 + \|m\|_{L^\infty})^{[s]+1} \big(\|m^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big) \|m\|^\ell_{\dot{B}^s_{p,1}} + 2^{(s-\sigma+\frac{d}{2}-\frac{d}{p})J} \big(2^J \|m^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big) \|m\|^h_{\dot{B}^\sigma_{2,1}}.\end{aligned}$$

These above two estimates, combined with (6.10), implies (6.9). □

Lemma 6.11. *Let $s > 0$, $\sigma \in \mathbb{R}$, $1 \leq p \leq 4$ for $d = 1$ and $1 \leq p \leq \min\{4, \frac{2d}{d-2}\}$ for $d \geq 2$. For any smooth function $F(m)$ satisfying $F(0) = F'(0) = 0$, there is a constant $C_m > 0$ depending only on $\|m\|_{L^\infty}$, s, σ and d such that*

$$\begin{aligned}\|F(m)\|^h_{\dot{B}^s_{2,1}} \leq{}& C_m \big(\|m^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big) \|m\|^h_{\dot{B}^s_{2,1}}\\
&+ C_m 2^{(s-\sigma+\frac{d}{p}-\frac{d}{2})J} \big(2^J \|m^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big) \|m\|^\ell_{\dot{B}^\sigma_{p,1}}\Big).\end{aligned} \tag{6.17}$$

Proof. As in Lemma 6.10, we decompose the high-frequency region as

$$\Omega_{hh} \triangleq \{(k,k') \mid k' > k \geq J\}, \quad \Omega_{hm} \triangleq \{(k,k') \mid k \geq k' \geq J\}, \quad \Omega_{h\ell} \triangleq \{(k,k') \mid k \geq J > k'\}.$$

Recall that $F(m)$ satisfies (6.10). By applying (6.11) with $|\beta| = 0$ and $(p^1,p^2,p^3) = (2,\infty,2)$, we have

$$\begin{aligned}&\sum_{\Omega_{hh}} 2^{ks}\|\dot{\Delta}_k(\dot{\Delta}_{k'}m(\dot{S}_{k'-1}m + \dot{\Delta}_{k'}m)\widetilde{M}_{k'})\|_{L^2}\\ &\lesssim \|m\|_{L^\infty}\sum_{k'>J}2^{k's}\|\dot{\Delta}_{k'}m\|_{L^2}\sum_{k'>k}2^{(k-k')s} \lesssim \big(\|m^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big)\|m\|^h_{\dot{B}^s_{2,1}}.\end{aligned} \tag{6.18}$$

Next, we apply (6.11) with $|\beta| = [s]+1$ and $(p^1,p^2,p^3) = (2,\infty,2)$ to obtain

$$\begin{aligned}&\sum_{\Omega_{hm}} 2^{ks}\|\dot{\Delta}_k(\dot{\Delta}_{k'}m(\dot{S}_{k'-1}m + \dot{\Delta}_{k'}m)\widetilde{M}_{k'})\|_{L^2}\\ &\lesssim (1+\|m\|_{L^\infty})^{[s]+1}\|m\|_{L^\infty}\sum_{k\geq k'\geq J}2^{k's}\|\dot{\Delta}_{k'}m\|_{L^2}\sum_{k\geq k'}2^{(k-k')(s-[s]-1)}\\ &\lesssim (1+\|m\|_{L^\infty})^{[s]+1}\big(\|m^\ell\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big)\|m\|^h_{\dot{B}^s_{2,1}}.\end{aligned} \tag{6.19}$$

For the region $\Omega_{h\ell}$, we consider the following three cases.

- Case 1: $2 \leq p < 4$ for $d=1$, $2 \leq p \leq \min\{\frac{2d}{d-2}, 4\}$ for $d \geq 2$ and $s \geq \sigma$.

Applying (6.11) with $|\beta| = [s]+1$ and $(p^1,p^2,p^3) = (2, \frac{2p}{p-2}, p)$, we have

$$\begin{aligned}&\sum_{\Omega_{h\ell}} 2^{ks}\|\dot{\Delta}_k(\dot{\Delta}_{k'}m(\dot{S}_{k'-1}m + \dot{\Delta}_{k'}m)\widetilde{M}_{k'})\|_{L^2}\\ &\lesssim (1+\|m\|_{L^\infty})^{[s]+1}\|m\|_{L^{\frac{2p}{p-2}}}\sum_{k'<J}2^{k's}\|\dot{\Delta}_{k'}m\|_{L^p}\sum_{k>k'}2^{(k-k')(s-[s]-1)}\\ &\lesssim (1+\|m\|_{L^\infty})^{[s]+1}2^{(s-\sigma)J}\|m\|_{L^{\frac{2p}{p-2}}}\|m\|^\ell_{\dot{B}^\sigma_{p,1}}\end{aligned}$$

Since p satisfies $\frac{2p}{p-2} \geq p \geq 2$ and $\frac{2d}{p} - \frac{d}{2} \geq \frac{d}{p} - 1$, the low-high frequency decomposition together with embedding inequalities and (2.2) implies

$$\|m\|_{L^{\frac{2p}{p-2}}} \lesssim \|m\|_{\dot{B}^0_{\frac{2p}{p-2},1}} \lesssim \|m^\ell\|_{\dot{B}^{\frac{2d}{p}-\frac{d}{2}}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{p}}_{2,1}} \lesssim 2^{(\frac{d}{p}-\frac{d}{2}+1)J}\|m^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + 2^{-(\frac{d}{2}-\frac{d}{p})J}\|m\|^h_{\dot{B}^{\frac{d}{2}}_{p,1}}$$

Therefore, one has

$$\begin{aligned}&\sum_{\Omega_{h\ell}} 2^{ks}\|\dot{\Delta}_k(\dot{\Delta}_{k'}m(\dot{S}_{k'-1}m + \dot{\Delta}_{k'}m)\widetilde{M}_{k'})\|_{L^2}\\ &\lesssim (1+\|m\|_{L^\infty})^{[s]+1}2^{(s-\sigma+\frac{d}{p}-\frac{d}{2})J}\big(2^J\|m^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big)\|m\|^\ell_{\dot{B}^\sigma_{p,1}}\Big).\end{aligned} \tag{6.20}$$

- Case 2: $1 \leq p \leq 2$ and $s \geq \sigma$.

In this case, one has $s + \frac{d}{p} - \frac{d}{2} \geq \sigma$. Taking advantage of Bernstein's inequality and (6.11) with

$|\beta| = [s + \frac{d}{p} - \frac{d}{2}] + 1$ and $(p^1, p^2, p^3) = (p, \infty, p)$, we have

$$
\begin{aligned}
&\sum_{\Omega_{h\ell}} 2^{ks} \|\dot{\Delta}_k(\dot{\Delta}_{k'} m(\dot{S}_{k'-1} m + \dot{\Delta}_{k'} m)\widetilde{M}_{k'})\|_{L^2} \\
&\quad \lesssim \sum_{\Omega_{h\ell}} 2^{k(s+\frac{d}{p}-\frac{d}{2})} \|\dot{\Delta}_k(\dot{\Delta}_{k'} m(\dot{S}_{k'-1} m + \dot{\Delta}_{k'} m)\widetilde{M}_{k'})\|_{L^p} \\
&\quad \lesssim (1 + \|m\|_{L^\infty})^{[s+\frac{d}{p}-\frac{d}{2}]+1} \|m\|_{L^\infty} \sum_{k'<J} 2^{k'(s+\frac{d}{p}-\frac{d}{2})} \|\dot{\Delta}_{k'} m\|_{L^p} \sum_{k>k'} 2^{(k-k')(s+\frac{d}{p}-\frac{d}{2}-[s+\frac{d}{p}-\frac{d}{2}]-1)} \\
&\quad \lesssim (1 + \|m\|_{L^\infty})^{[[s+\frac{d}{p}-\frac{d}{2}]+1} 2^{(s-\sigma+\frac{d}{p}-\frac{d}{2})J} \big(2^J \|m^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big) \|m\|^\ell_{\dot{B}^\sigma_{p,1}}.
\end{aligned}
$$

- Case 3: $s < \sigma$.

In this case, the high-frequency cut-off together with the above estimates in Cases 1-2 implies

$$
\begin{aligned}
&\sum_{\Omega_{h\ell}} 2^{ks} \|\dot{\Delta}_k(\dot{\Delta}_{k'} m(\dot{S}_{k'-1} m + \dot{\Delta}_{k'} m)\widetilde{M}_{k'})\|_{L^2} \\
&\quad \lesssim 2^{(s-\sigma)J} \sum_{\Omega_{h\ell}} 2^{k\sigma} \|\dot{\Delta}_k(\dot{\Delta}_{k'} m \widetilde{M}_{k'} \dot{S}_{k'-1} m)\|_{L^2} \\
&\quad \le C_m 2^{(s-\sigma+\frac{d}{p}-\frac{d}{2})J} \big(2^J \|m^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big) \|m\|^\ell_{\dot{B}^\sigma_{p,1}}.
\end{aligned}
$$

From the estimates in the above three cases, there holds that

$$
\begin{aligned}
&\sum_{\Omega_{h\ell}} 2^{ks} \|\dot{\Delta}_k(\dot{\Delta}_{k'} m(\dot{S}_{k'-1} m + \dot{\Delta}_{k'} m)\widetilde{M}_{k'})\|_{L^2} \\
&\le C_m 2^{(s-\sigma+\frac{d}{p}-\frac{d}{2})J} \big(2^J \|m^\ell\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|m\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\big) \|m\|^\ell_{\dot{B}^\sigma_{p,1}}
\end{aligned}
\tag{6.21}
$$

for all s, σ and p given in Proposition 6.11.

Adding (6.18), (6.19) and (6.21) together, we end up with (6.17) which completes the proof of Proposition 6.11.

□

6.3 Proof of Theorem 1.1

In this subsection we give the proof of Theorem 1.1 on the global well-posedness of the Cauchy problem for the viscous conservation law (1.2).

Proof of Theorem 1.1. Since the local well-posedness can be shown by standard linearization iteration process, we omit the proof of local well-posedness for brevity and focus on giving the necessary a-priori estimates. By virtue of Lemma 6.8 to (1.2), we have

$$
\|u^*\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1} \cap \dot{B}^{\frac{d}{p}}_{p,1})} + \|u^*\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1} \cap \dot{B}^{\frac{d}{p}+2}_{p,1})} \lesssim \|u_0^*\|_{\dot{B}^{\frac{d}{p}-1}_{p,1} \cap \dot{B}^{\frac{d}{p}}_{p,1}} + \|f(u^*)\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1} \cap \dot{B}^{\frac{d}{p}+1}_{p,1})}. \tag{6.22}
$$

Now, we assume $\|u^*\|_{L^\infty_t(L^\infty)} \lesssim 1$. For $i = 1, 2, ..., d$, due to $f_i(0) = \frac{\partial}{\partial_{u_k}} f_i(0) = 0$, there exists a smooth function $\widetilde{f}_i(u)$ such that $f_i(u) = \widetilde{f}_i(u)u$ and $\widetilde{f}_i(0) = 0$. Thus, making use of (6.2) and Lemma 6.6, we deduce for any $1 \le p \le \infty$ that

$$
\|f_i(u^*)\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} \lesssim \|\widetilde{f}_i(u)\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}}_{p,1})} \|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}}_{p,1})} \lesssim \|u\|^2_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}. \tag{6.23}
$$

Similarly, it holds by (6.1), Lemma 6.6 and the embedding $\dot{B}^{\frac{d}{p}}_{p,1}\hookrightarrow L^\infty(\mathbb{R}^d)$ that

$$\begin{aligned}\|f_i(u^*)\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}&\lesssim\|\widetilde{f}_i(u)\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}+\|\widetilde{f}_i(u)\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\&\lesssim\|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}.\end{aligned}\tag{6.24}$$

Inserting (6.23)-(6.24) into (6.22) and taking advantage of the interpolation in Lemma 6.2, we obtain

$$\begin{aligned}&\|u^*\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})}+\|u^*\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1}\cap\dot{B}^{\frac{d}{p}+2}_{p,1})}\\&\lesssim\|u_0^*\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}}+\Big(\|u^*\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})}+\|u^*\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1}\cap\dot{B}^{\frac{d}{p}+2}_{p,1})}\Big)^2.\end{aligned}$$

Thence, by a standard bootstrap argument, one can prove

$$\|u^*\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})}+\|u^*\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1}\cap\dot{B}^{\frac{d}{p}+2}_{p,1})}\lesssim\|u_0^*\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}}\quad\text{for all}\quad t>0.\tag{6.25}$$

This, together with the local well-posedness, shows the global existence of a solution $\boldsymbol{v}^*$ to the Cauchy problem of System (1.2) associated to the initial datum u_0^*. The property $u^*\in C(\mathbb{R}_+;\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})$ follows a similar argument as in [20, p.42]. With the aid of (1.3), (6.23), (6.24) and (6.25) , we recover the information on $\boldsymbol{v}^*$ as follows

$$\|\boldsymbol{v}^*\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}\lesssim\|u^*\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1}\cap\dot{B}^{\frac{d}{p}+2}_{p,1})}+\|f(u^*)\|_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}\lesssim\|u_0^*\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}}.$$

□

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