# ASYMPTOTIC-PRESERVING FINITE DIFFERENCE METHOD FOR PARTIALLY DISSIPATIVE HYPERBOLIC SYSTEMS

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Abstract. We analyze the preservation of asymptotic properties of partially dissipative hyperbolic systems when switching to a fully discrete setting. We prove that one of the simplest consistent and unconditionally stable numerical methods – the implicit central finite difference scheme – preserves both the large time asymptotic behaviour and the parabolic relaxation limit of one-dimensional partially dissipative hyperbolic systems that satisfy the Kalman rank condition.

The large time asymptotic-preserving property is achieved by conceiving time-weighted perturbed energy functionals in the spirit of the hypocoercivity theory. For the relaxation-preserving property, drawing inspiration from the observation that, in the continuous case, solutions are shown to exhibit distinct behaviour in low and high frequencies, we introduce a novel discrete Littlewood-Paley decomposition tailored to the central finite difference scheme. This allows us to prove Bernstein-type estimates for discrete differential operators and leads to new diffusive limit results such as the strong convergence of the discrete linearized compressible Euler system with damping towards the discrete heat equation, uniformly with respect to the spatial mesh parameter.

## Contents



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## 1. Introduction and informal results

Extensive literature exists on the analysis of partially dissipative hyperbolic models, particularly focusing on their asymptotic behaviour and singular limits using a combination of Fourier and hypocoercivity techniques. While in the continuous setting there is growing progress in understanding these phenomena, a persistent challenge arises when transitioning to a numerical context and seeking to preserve such properties in a grid-uniform manner.

In this context, our research contains both theoretical and experimental evidence for the fact that hypocoercivity and relaxation properties inherent to partially dissipative hyperbolic systems can be effectively captured by one of the simplest and unconditionally stable numerical schemes: the implicit central finite difference method.

### 1.1. Partially dissipative systems – propagation of damping through hyperbolic dynamics.

We are concerned with the numerical analysis of partially dissipative hyperbolic systems of the form

$$\partial_t U + A\partial_x U = -BU, \quad (t,x) \in (0,\infty)\times\mathbb{R}, \tag{1.1}$$

where $U = U(t,x) \in \mathbb{R}^N$ ($N \geq 2$) is the vector-valued unknown and $A, B$ are symmetric $N\times N$ matrices. We assume that (1.1) has a *partially dissipative structure*: The matrix $B$ takes the form

$$B = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{B} \end{pmatrix}, \tag{1.2}$$

where $\widetilde{B}$ is a positive definite symmetric $N_2 \times N_2$ matrix ($1 \leq N_2 < N$). Under these conditions, $\widetilde{B}$ satisfies the *strong dissipativity condition*: there exists a constant $\lambda > 0$ such that, for all $X \in \mathbb{R}^{N_2}$,

$$\langle \widetilde{B}X, X\rangle \geq \lambda |X|^2, \tag{1.3}$$

where $\langle , \rangle$ denotes the inner product on $\mathbb{R}^{N_2}$. Based on this definition, we decompose the solution as $U = (U_1, U_2)$ where $U_1 \in \mathbb{R}^{N_1}$, for $N_1 := N - N_2$, corresponds to the conserved components and $U_2 \in \mathbb{R}^{N_2}$ to the dissipated ones. According to this decomposition, we have the block form

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix},$$

where the dimensions of the components $A_{1,1}$, $A_{1,2}$, $A_{2,1}$ and $A_{2,2}$ are $N_1\times N_1$, $N_1\times N_2$, $N_2\times N_1$ and $N_2\times N_2$, respectively.

In general, the $L^2$-stability of these systems is unclear, as the dissipative operator $\widetilde{B}$ only acts on the component $U_2$. Shizuta & Kawashima (1985) observed that if the eigenvectors of $A$ avoid the kernel of the dissipative matrix $B$ (this requirement is called the *SK condition*), then the solutions are stable in $L^2$. More recently, Beauchard & Zuazua (2011) established a link between the SK condition, control theory and the theory of hypocoercivity (Villani 2010). In particular, they constructed perturbed energy functionals permitting to recover the asymptotic behaviour of the solutions of (1.1) under the Kalman rank condition:

**Definition 1.1.** A pair of matrices $(A, B)$ verifies the *Kalman* rank condition if

$$\text{the matrix } \mathcal{K}(A,B) := (B|AB|\dots|A^{N-1}B) \quad \text{has full rank } N. \tag{K}$$

In practice, the condition (K) means that the partially dissipative effects of $B$ can be propagated to the other components through the hyperbolic dynamics of the system. In Beauchard & Zuazua (2011), the authors showed that the condition (K) is equivalent to the (SK) condition. Under these conditions, in numerous references on the topic (e.g. Yong (2004), Bianchini et al. (2007), Hanouzet & Natalini (2003), Crin-Barat & Danchin (2022*a*,*b*)), authors rely on Fourier techniques to justify the global well-posedness and study the large time behaviour of solutions of nonlinear systems. Recently, Crin-Barat et al. (2024) developed a Fourier-free method prone to tackle situations where the Fourier transform cannot be employed such as e.g. bounded domains, time and space-dependent matrices or Riemannian manifolds. Their method leads to the following *natural* time-decay estimates for the solution of (1.1).

**Theorem 1.2** ((Crin-Barat et al. 2024, Theorem 2.1))**.** *Let $U^0 \in (H^1(\mathbb{R}))^N$, $A$ and $B$ be symmetric $N \times N$ matrices, with $B$ as in* (1.2)*, satisfying the Kalman rank condition. Then, for all $t > 0$, the solution $U$ of* (1.1) *with the initial datum $U^0$ satisfies*

$$\|U_2(t)\|_{L^2(\mathbb{R})} + \|\partial_x U(t)\|_{L^2(\mathbb{R})} \leq C(1+t)^{-\frac{1}{2}} \|U^0\|_{H^1(\mathbb{R})}, \tag{1.4}$$

*where $C > 0$ is a constant independent of time and $U^0$.*

This theorem highlights the hypocoercive nature inherent to partially dissipative systems. Although the damping term does not directly influence every component of the system, the whole solution decays in time due to the cross-influence between the matrices $A$ and $B$ resulting from the Kalman condition (K). The decay rate resembles that of the heat equation for $L^2$ data, but, in this hyperbolic framework, we need to assume additionally that the initial datum is in $H^1$, due to the lack of parabolic regularising effects. Moreover, in Crin-Barat et al. (2024), it is shown that the decay (1.4) is optimal for $H^1$ initial data. The lack of exponential decay essentially comes from the fact that the hyperbolic and dissipative operators in the system are of different orders.

*Remark* 1.3. A classical system fitting the description (1.1)-(1.2) and verifying the Kalman rank condition (K) is the compressible Euler equations with damping. Indeed, linearizing this system around the constant equilibrium $(\rho^*, u^*) = (\rho^*, 0)$, with $\rho^* > 0$, one obtains

$$\begin{cases} \partial_t \rho + \partial_x u = 0, \\ \partial_t u + \partial_x \rho = -u, \end{cases} \qquad (t,x) \in (0,\infty) \times \mathbb{R}, \tag{1.5}$$

where $\rho = \rho(x,t) \geq 0$ denotes the fluid density and $u = u(x,t) \in \mathbb{R}$ stands for the fluid velocity.

1.2. **Large time asymptotic-preserving schemes for partially dissipative systems.** The first purpose of this paper is to prove that one of the simplest structure-preserving (i.e. consistent and stable) numerical schemes for (1.1), namely the implicit scheme based on the central finite difference discrete operator on a uniform $h$-sized spatial grid:

$$(\mathcal{D}_h U)_n = \frac{U_{n+1} - U_{n-1}}{2h}, \; n \in \mathbb{Z}, \tag{1.6}$$

preserves the large-time asymptotics derived in Theorem 1.2. The choice of this particular scheme is justified by its unconditional stability for hyperbolic systems, see the reasoning in (Strikwerda 2004, Sections 1.6 and 7.1) and (Jovanović & Süli 2014, Section 3.2).

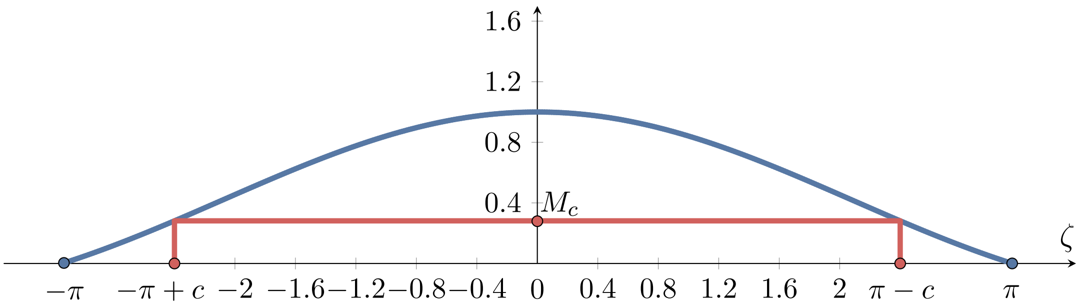


FIGURE 1. In blue: Plot of the function $\zeta \to \frac{\sin(\zeta)}{\zeta}$. $M_c$ is the value of the function at the high-frequency thresholds $\pm(\pi - c)$, where $c$ is a constant in $(0, \frac{\pi}{2})$. Below this value, i.e below the red line, the analysis needs special treatment compared to the continuous setting.

Next, we present an informal version of our asymptotic behaviour result:

*The central finite difference scheme is large time asymptotic-preserving for the system* (1.1) *in the sense that we recover the time-decay* (1.4) *for the fully discretized version of* (1.1)*, uniformly with respect to both spatial and temporal mesh-size parameters, when the Kalman rank condition holds.*

The complete version of this result can be found in Theorem 2.1.

1.3. **Relaxation-preserving scheme.** Up to this point, we have at hand a structure-preserving and large time asymptotic-preserving numerical scheme. One of the natural further steps is to analyse whether this scheme behaves well with respect to another type of asymptotics: singular perturbations. More precisely, Yong (1999) showed that the system (1.1) can be relaxed to a parabolic one in a diffusive scaling. We will prove that, under certain regularity conditions on the initial data, the same relaxation behaviour is observed in the discrete setting, uniformly with respect to the spatial grid width $h$.

We first introduce such approximations in a simple, yet illustrative case: the linear compressible Euler system (1.5) with relaxation, which reads

$$\begin{cases} \partial_t \rho^\varepsilon + \partial_x u^\varepsilon = 0, \\ \varepsilon^2 \partial_t u^\varepsilon + \partial_x \rho^\varepsilon = -u^\varepsilon, \end{cases} \tag{1.7}$$

where $\rho^\varepsilon, u^\varepsilon : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ and $\varepsilon > 0$ is the relaxation parameter. As $\varepsilon \to 0$, the solutions of (1.7) converge, at least formally, to the solutions of the discrete heat equation:

$$\begin{cases} \partial_t \rho - \partial_{xx}^2 \rho = 0, \\ u = -\partial_x \rho, \end{cases} \tag{1.8}$$

where the second equation corresponds to the discrete Darcy law.

In Orive & Zuazua (2006) and Crin-Barat & Danchin (2023) showed that it is essential to analyze separately the low and high frequencies of the solutions to derive strong convergence results for such relaxation procedures. In particular, in Crin-Barat & Danchin (2023), the authors introduce a hybrid Littlewood-Paley decomposition to justify the strong convergence of the nonlinear compressible Euler system with damping toward the porous media equation in any dimension and for ill-prepared data. In the Sobolev framework, no $\mathcal{O}(\varepsilon)$ relaxation rate are available in the literature, to the best of our knowledge. This seems to be due to the necessity of splitting the frequency analysis, as the behaviour of the low and high frequencies are fundamentally different.

Following the frequency-splitting approach, Danchin (2023) justified the following type of relaxation limit for a generalized version of (1.7): the solutions of

$$\begin{cases} \partial_t U_1 + A_{1,2}\partial_x U_2 = 0, \\ \varepsilon^2 \partial_t U_2 + A_{2,1}\partial_x U_1 = -\widetilde{B}U_2 \end{cases} \tag{1.9}$$

converge, as $\varepsilon \to 0$, toward the solution of

$$\partial_t U_1 - A_{1,2}\widetilde{B}^{-1}A_{2,1}\partial_{xx}U_1 = 0. \tag{1.10}$$

It is this limit that we analyze in a fully discrete setting in the present paper.

Note that it was proven in Danchin (2023) that the matrix $A_{1,2}\widetilde{B}^{-1}A_{2,1}$ is symmetric and positive definite if $A$ is symmetric, $\widetilde{B}$ is symmetric and positive definite and $(A, B)$ satisfy the Kalman rank condition, see Lemma A.1.

Inspired by this, in the present paper, in order to justify relaxation-preserving properties of the numerical scheme, we introduce a novel and numerically suited Littlewood-Paley decomposition. In this regard, the main challenge that arises is that the Fourier symbol of the discrete operator, which is:

$$\widehat{(\mathcal{D}_h v)}(\xi) = i\frac{\sin(\xi h)}{h}\hat{v}(\xi), \quad \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right], \tag{1.11}$$

becomes very small at high frequencies $\xi \sim \pm\frac{\pi}{h}$. Therefore, we are not able to uniformly approximate $\frac{\sin(\xi h)}{h}$ by $\xi$, since, in the high-frequency regime, one has $\left|\frac{\sin(\xi h)}{\xi h}\right| \ll 1$ (see Figure 1). To tackle this issue, we develop a non-standard dyadic decomposition tailored to the central finite difference operator $\mathcal{D}_h$. More precisely, whereas in the continuous Littlewood-Paley theory, the dyadic decomposition of the frequency domain is done in logarithmically equidistant annuli

$$|\xi| \in \left[\frac{3}{4}2^j, \frac{4}{3}2^{j+1}\right], \tag{1.12}$$

in our case, we work with a numerically adapted dyadic decomposition based on non-uniform annuli of the form

$$F_h(j) := \left\{\xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right] : \left|\frac{\sin(\xi h)}{h}\right| \in \left[\frac{3}{4}2^j, \frac{4}{3}2^{j+1}\right]\right\}. \tag{1.13}$$

See Figure 2 for a comparison between the decomposition intervals in (1.12) and (1.13). On that figure, we remark that the numerically adapted dyadic decomposition showcases a pseudo-low-frequency regime near the boundary of the frequency domain (i.e. for $\xi \sim \pm\frac{\pi}{h}$), which will need special treatment in our analysis (for example, in the proof of Proposition 2.5). Moreover, we observe that the discrete high-frequency regime does not align with the continuous one, since the former is bounded for a fixed mesh width $h$. Nevertheless, since, as $h$ approaches zero, we recover the full high-frequency range, we employ a spectrum-based approach to obtain convergence results independent of $h$.

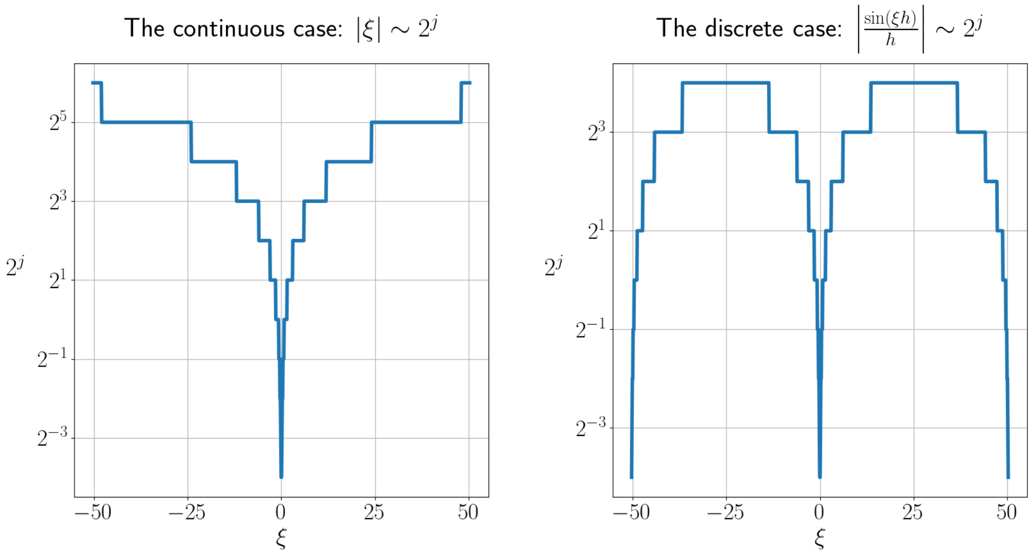


FIGURE 2. The decomposition of the frequency space in the continuous case (1.12) (left) and in the discrete setting (1.13) (right), for $h = 2^{-4}$.

We also note that our approach differs from the discrete Littlewood-Paley decomposition outlined in Hong & Yang (2019), which utilizes the dyadic decomposition (1.12). One significant improvement is that our numerically adapted decomposition (1.13) enables us to establish Bernstein-type estimates for the operator $\mathcal{D}_h$ in the following sense:

$$c\, 2^j \|\delta_h^j v\|_{\ell_h^2} \leq \|\mathcal{D}_h \delta_h^j v\|_{\ell_h^2} \leq C\, 2^j \|\delta_h^j v\|_{\ell_h^2},$$

where $\delta_h^j v$ is the localization of the grid function $v$ to the frequency band (1.13) – see Definition 4.4 – and $c, C$ are universal positive constants. We also refer to Sections 3 and 4 for more information about the functional framework we use.

The above Bernstein estimate leads to the following definition of homogeneous discrete Besov norms: for a regularity index $s \geq 0$ and $v \in \ell_h^2$, we define

$$\|v\|_{\dot{B}_h^s} := \sum_{j \in \mathbb{Z}} 2^{js} \|\delta_h^j v\|_{\ell_h^2}. \tag{1.14}$$

Within this framework, in order to split our analysis into low and high frequencies, we simply need to apply the frequency-localization linear operator $\delta_h^j$ to the system and study the solution separately for $j \ll 1/\varepsilon$ and $j \gg 1/\varepsilon$ respectively. Furthermore, the norm defined in (1.14) is related to Sobolev and Lebesgue discrete functional norms, see Proposition 2.5.

Next, we present an informal version of our relaxation-preserving result:

*The implicit central finite difference scheme is relaxation-preserving for the system* (1.9) *in the sense that, the solutions of the full discretized version of* (1.9) *converge to the solutions of the discretization of* (1.10) *in the supremum, Sobolev and Besov norms at the rate* $\mathcal{O}(\varepsilon^2)$*, uniformly with respect to the mesh-size parameters, under the condition* $\tau \lesssim \varepsilon^2$*.*

The rigorous form of this result can be found in Theorem 2.6 and Corollary 2.8.

## 2. Main results

2.1. **Large time asymptotic-preserving property of the central finite difference scheme.** In this section, we establish the counterpart of (1.4) for discrete hyperbolic systems, demonstrating the preservation of the hypocoercivity property when transitioning to a numerical context. We consider the following fully discrete implicit scheme, where $\tau \in (0,1)$ is the time step and $h \in (0,1)$ is the width of the spatial grid:

$$\begin{cases} \dfrac{U^{k+1}-U^k}{\tau} + A\mathcal{D}_h U^{k+1} = -BU^{k+1}. \end{cases} \tag{2.1}$$

where, for every $k \in \{0,1,\dots,K\}$, the bilateral sequence $U^k = (U^k_n)_{n\in\mathbb{Z}}$ belongs to the space $(\ell^2_h)^N$. Decomposing the solution into its conserved and dissipated quantities and using (1.2), we have

$$\begin{cases} \dfrac{U_1^{k+1}-U_1^k}{\tau} + A_{1,1}\mathcal{D}_h U_1^{k+1} + A_{1,2}\mathcal{D}_h U_2^{k+1} = 0, \\ \dfrac{U_2^{k+1}-U_2^k}{\tau} + A_{2,1}\mathcal{D}_h U_1^{k+1} + A_{2,2}\mathcal{D}_h U_2^{k+1} = -\widetilde{B} U_2^{k+1}, \end{cases} \tag{2.2}$$

where, for every $k \in \{0,1,\dots,K\}$, the bilateral sequences $U_1^k = (U^k_{1;n})_{n\in\mathbb{Z}}, U_2^k = (U^k_{2;n})_{n\in\mathbb{Z}}$ belong to the space $(\ell^2_h)^{N_1}$ and $(\ell^2_h)^{N_2}$ respectively.

We are now in position to state our first result regarding time-decay estimates for (2.1), uniformly with respect to the mesh width. The discrete Lebesgue and Sobolev norms $\|\cdot\|_{\ell^2_h}$ and $\|\cdot\|_{\mathfrak{h}^1_h}$ used below are introduced in Definition 3.1.

**Theorem 2.1** (Numerical hypocoercivity for hyperbolic systems)**.** *Let $U^0 \in (\ell^2_h)^N$, $A$ and $B$ be symmetric $N\times N$ matrices with $B$ as in* (1.2)*-*(1.3) *and such that $(A,B)$ satisfies the Kalman rank condition* (K)*. Then, for all $k \in \mathbb{N}^*$, the solution $U$ of* (2.1) *with the initial datum $U^0$ satisfies*

$$\|U_2^k\|_{\ell^2_h} + \|\mathcal{D}_h U^k\|_{\ell^2_h} \leq C(1+t^k)^{-\frac{1}{2}}\|U^0\|_{\mathfrak{h}^1_h}, \tag{2.3}$$

*where $t^k := k\tau$ and $C>0$ is a constant independent of the mesh-size parameter $h$, the time-step $\tau$ and $U^0$.*

*Remark* 2.2. The decay rate obtained in (2.3) is the same as the one derived in the continuous case by Crin-Barat et al. (2024). Applying Theorem 2.1 to the linearized compressible Euler system (1.7), we obtain, for all $k \in \mathbb{N}^*$,

$$\|u^k\|_{\ell^2_h} + \|(\mathcal{D}_h\rho^k, \mathcal{D}_h u^k)\|_{\ell^2_h} \leq C(1+t^k)^{-\frac{1}{2}}\|(\rho^0,u^0)\|_{\mathfrak{h}^1_h}, \tag{2.4}$$

where $C>0$ is a constant independent of $h$, $\tau$ and $(\rho_0, u_0)$. The sharpness of the decay rate is also validated by simulations in Section 7.1.

*Remark* 2.3. Passing to the limit $\tau \to 0$ in (2.3) and using the same procedure as in (Jovanović & Süli 2014, Section 3.2), one can prove an equivalent decay result for the semi-discrete approximation of the hyperbolic system (1.1), namely:

$$\begin{cases} U : [0,\infty) \to \ell^2_h \\ \partial_t U(t) + A\mathcal{D}_h U(t) = -BU(t), \quad t \geq 0, \\ U(0) = U^0 \in \ell^2_h. \end{cases} \tag{2.5}$$

The decay result for (2.5) reads

$$\|U_2(t)\|_{\ell^2_h} + \|\mathcal{D}_h U(t)\|_{\ell^2_h} \leq C(1+t)^{-\frac{1}{2}}\|U^0\|_{\mathfrak{h}^1_h}, \quad t > 0, \tag{2.6}$$

where $C > 0$ is a constant independent of $h$, the time $t$ and $U_0$.

**Strategy of proof and comparison with the literature.** To establish Theorem 2.1, we construct time-weighted Lyapunov functionals inspired by the recent work Crin-Barat et al. (2024). Their approach, employing various tools to analyze partially dissipative systems without relying on the Fourier transform, broadens the scope of applicability beyond standard methods, such as Shizuta & Kawashima (1985), Beauchard & Zuazua (2011), Bianchini et al. (2007) and Crin-Barat & Danchin (2022*a*). The construction of the time-weighted Lyapunov functionals is influenced by the works Hérau (2007) and Hérau & Nier (2004) on the asymptotic behaviour of hypocoercive kinetic models and Beauchard & Zuazua (2011) concerning the hypocoercivity phenomenon for hyperbolic systems. In the present paper, the Lyapunov functionals we use closely resemble the one in Crin-Barat et al. (2024) and are tailored for the central finite difference approximation of the partially dissipative system (1.1). Differentiating these functionals with respect to time and employing the Kalman rank condition (K), we derive the desired time-decay rates.

Regarding the discrete asymptotic stability of partially dissipative systems, numerous studies are dedicated to formulating large-time asymptotic-preserving numerical schemes for hypocoercive phenomena. Closely connected to our work, we highlight the contributions of Porretta & Zuazua (2016) and Georgoulis (2021), where time-decay estimates are derived for discretized versions of the two-dimensional Kolmogorov equation, employing finite difference and finite element schemes, respectively. In the broader context of kinetic models, particularly emphasizing the Fokker-Planck equation, we refer to Blaustein & Filbet (2024), Dujardin et al. (2020), Filbet & Rodrigues (2017), Filbet et al. (2021), Foster et al. (2017), Bessemoulin-Chatard et al. (2020) and references therein.

## 2.2. Strong relaxation limit in the fully discrete setting.

2.2.1. *A new frequency-based discrete framework.* The proof of our second main result – the discrete relaxation limit – is inspired by Crin-Barat & Danchin (2023) pertaining to the continuous setting. In this reference, it is shown that the solutions of the nonlinear compressible Euler system converge strongly, in suitable norms, as the relaxation parameter $\varepsilon$ approaches zero, to the solutions of the porous media equation. There, the authors use a frequency-splitting method and treat the low and high frequencies in two different manners. Importantly, in their approach, the threshold between low and high frequencies is located at $1/\varepsilon$, which implies that the high-frequency regime disappears in the limit $\varepsilon \to 0$.

Drawing upon these insights, to obtain new results related to hyperbolic relaxation procedures for discrete hyperbolic systems, we employ the novel construction of Besov norms roughly described in Section 1.3 and rigorously introduced in Section 4.2.

Another key concept, inspired from (Strikwerda 2004, Section 10.1), is the truncation operator: $\mathcal{T}_h : L^2(\mathbb{R}) \to \ell^2_h$,

$$(\mathcal{T}_h v)_n = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{i\xi nh} \hat{v}(\xi)\, \mathrm{d}\xi, \tag{2.7}$$

where $\hat{v}$ is the continuous Fourier transform of $v \in L^2(\mathbb{R})$. It is notable that the discrete Fourier transform $\widehat{\mathcal{T}_h v}(\xi)$, as introduced in Definition 4.1, coincides with the continuous Fourier transform $\hat{v}(\xi)$ for any $\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]$. Thus, the purpose of this truncation operator is to transfer functions defined on the real line to a grid of width $h$, while preserving the Fourier transform. For a comprehensive understanding of the suitability of this operator in accurately projecting functions onto the $h$-grid, interested readers can refer to (Strikwerda 2004, Theorems 10.1.3 and 10.1.4). The forthcoming result concerning the truncation operator will ultimately ensure that the constants in our relaxation result are uniform with respect to the grid width $h$. Its proof can be found in Section 6.

**Theorem 2.4** (Uniform Besov estimates with respect to the grid width)**.** *For every $s' > 0$ and $s \in (0, s')$ there exists a constant $C_{s',s} > 0$ depending only on $s$ and $s'$ such that, for every $h > 0$ and every $v \in H^{s'}(\mathbb{R})$, we have*

$$\|\mathcal{T}_h v\|_{\dot{B}^s_h} \leq C_{s',s} \|v\|_{H^{s'}(\mathbb{R})}. \tag{2.8}$$

Furthermore, the discrete Besov norm that we employ can be related to discrete Sobolev norm $\|\cdot\|_{\dot{\mathfrak{h}}^s_h}$ and the discrete Lebesgue norm $\|\cdot\|_{\ell^\infty_h}$ (refer to Sections 3 and 4 for the definitions) as follows.

**Proposition 2.5.** *For every $s \in \mathbb{R}$, there exists a constant $C_s > 0$ depending only on $s$ such that, for every $v \in \ell^2_h$, the following inequality holds true:*

$$\|v\|_{\dot{\mathfrak{h}}^s_h} \leq C_s \|v\|_{\dot{B}^s_h}. \tag{2.9}$$

*Moreover, in the particular case $s = \frac{1}{2}$, the discrete Besov norm controls the supremum norm:*

$$\|v\|_{\ell^\infty_h} \leq C \|v\|_{\dot{B}^{\frac{1}{2}}_h}, \tag{2.10}$$

*where $C$ is a universal constant.*

The proof of Proposition 2.5 can be found Section 4.2.

2.2.2. *Discrete strong relaxation.* We consider the following relaxed system, which is of the form (2.2), but with $A_{1,1} = 0$:

$$\begin{cases} \dfrac{U_1^{\varepsilon,k+1} - U_1^{\varepsilon,k}}{\tau} + A_{1,2}\mathcal{D}_h U_2^{\varepsilon,k+1} = 0, \\ \varepsilon^2 \left( \dfrac{U_2^{\varepsilon,k+1} - U_2^{\varepsilon,k}}{\tau} \right) + A_{2,1}\mathcal{D}_h U_1^{\varepsilon,k+1} + \widetilde{B} U_2^{\varepsilon,k+1} = 0 \end{cases} \tag{2.11}$$

and its associated limit problem

$$\frac{U_1^{k+1} - U_1^k}{\tau} + A_{1,2} \widetilde{B}^{-1} A_{2,1} \mathcal{D}_h^2 U_1^{k+1} = 0. \tag{2.12}$$

We recall that in Crin-Barat & Danchin (2023) and Danchin (2023) it was shown that when $(A,B)$ satisfy the Kalman rank condition, the matrix $A_{1,2}\widetilde{B}^{-1}A_{2,1}$ is positive definite, see Lemma A.1. This ensures the parabolicity of (2.12). We also assume that the first component $U_1^0$ of the initial data $(U_1^0, U_2^0)$ of the hyperbolic system (2.11) coincides with the initial datum of the parabolic problem (2.12).

Before stating our result, we define, for $K \in \mathbb{N}^*$, $\tau > 0$ and $X$ a normed space, the spaces $\ell^1_{\tau,K}(X)$ and $\ell^\infty_{\tau,K}(X)$ of time sequences $v = (v^k)_{k=1}^K : \{1,\dots,K\} \to X$ associated, respectively, to the norms

$$\|v\|_{\ell^1_{\tau,K}(X)} := \tau \sum_{k=0}^{K-1} \|v^{k+1}\|_X \quad \text{and} \quad \|v\|_{\ell^\infty_{\tau,K}(X)} := \sup_{k\in\{0,..,K-1\}} \|v^{k+1}\|_X. \tag{2.13}$$

**Theorem 2.6** (Numerical relaxation limit). *Assume that the matrices $(A,B)$ satisfy the Kalman rank condition (K). Let $M > 0$, $s' > 2$ and the functions defined on the real line: $(\widetilde{U}_1^0, \widetilde{U}_2^0) \in (H^{s'}(\mathbb{R}))^{N_1} \times (H^{s'}(\mathbb{R}))^{N_2}$. Let $h > 0$ and the initial data for the systems (2.11) and (2.12) be obtained by truncation: $U_i^0 = \mathcal{T}_h(\widetilde{U}_i^0)$, $i = \overline{1,2}$.*

*Then, for every $K \in \mathbb{N}^*$, $s \in (2, s')$, $\varepsilon > 0$ and every $\tau \in (0, M\varepsilon^2)$, the solutions of (2.11) and (2.12) satisfy the following strong convergence result:*

$$\|U_1^\varepsilon - U_1\|_{\ell^\infty_{\tau,K}(\dot{B}_h^{s-2})} + \|U_1^\varepsilon - U_1\|_{\ell^1_{\tau,K}(\dot{B}_h^{s})} + \|\widetilde{B}^{-1}A_{2,1}\mathcal{D}_h U_1^\varepsilon + U_2^\varepsilon\|_{\ell^1_{\tau,K}(\dot{B}_h^{s-1})} \lesssim C\varepsilon^2, \tag{2.14}$$

*where $C = C_1\left(1 + \|(\widetilde{U}_1^0, \varepsilon\widetilde{U}_2^0)\|_{H^{s'}} + \|\widetilde{U}_2^0\|_{H^{s'-1}}\right)$, with $C_1 > 0$ a constant depending only on $M$, $s'$, $s$ and the matrices $A$ and $B$.*

*Remark* 2.7. As an application, under the conditions of Theorem 2.6, we have that the solutions $(\rho^\varepsilon, u^\varepsilon)$ and $\rho$ of the discretized versions of (1.7) and (1.8), respectively, verify

$$\|\rho^\varepsilon - \rho\|_{\ell^\infty_{\tau,K}(\dot{B}_h^{s-2})} + \|\rho^\varepsilon - \rho\|_{\ell^1_{\tau,K}(\dot{B}_h^{s})} + \|\mathcal{D}_h\rho^\varepsilon + u^\varepsilon\|_{\ell^1_{\tau,K}(\dot{B}_h^{s-1})} \lesssim C\varepsilon^2, \tag{2.15}$$

Combining Proposition 2.5 and Theorem 2.6, we obtain the strong convergence in the $\dot{\mathfrak{h}}_h^s$ and $\ell_h^\infty$ norms, uniformly in $h$ and $K$.

**Corollary 2.8.** *Let all the assumptions of Theorem 2.6 be in force. The following statements hold true:*

- *(i) The solution sequence $U_1^\varepsilon$ of (2.11) converges strongly, as $\varepsilon \to 0$, to the solution $U_1$ of (2.11) in $\ell^\infty_{\tau,K}(\dot{\mathfrak{h}}_h^{s-2})$ and $\ell^1_{\tau,K}(\dot{\mathfrak{h}}_h^{s})$ at the rate $\mathcal{O}(\varepsilon^2)$. Furthermore, the quantity $\widetilde{B}^{-1}A_{2,1}\mathcal{D}_h U_1^\varepsilon + U_2^\varepsilon$ converges to $0$ in $\ell^1_{\tau,K}(\dot{\mathfrak{h}}_h^{s-1})$ at the rate $\mathcal{O}(\varepsilon^2)$.*
- *(ii) If we additionally assume $s' > 5/2$, the solution sequence $U_1^\varepsilon$ of (2.11) converges strongly, as $\varepsilon \to 0$, to the solution $U_1$ of (2.12) in $\ell^\infty_{\tau,K}(\ell_h^\infty)$ and $\ell^1_{\tau,K}(\ell_h^\infty)$ at the rate $\mathcal{O}(\varepsilon^2)$. Furthermore, the quantity $\widetilde{B}^{-1}A_{2,1}\mathcal{D}_h U_1^\varepsilon + U_2^\varepsilon$ converges to $0$ in $\ell^1_{\tau,K}(\ell_h^\infty)$ at the rate $\mathcal{O}(\varepsilon^2)$.*

*All the convergences above are uniform with respect to $h > 0$ and $K \in \mathbb{N}^*$.*

*Remark* 2.9. The convergence rate we obtain in this linear setting is one order higher than the one obtained in Crin-Barat & Danchin (2023) and Danchin (2023). This is at the cost of stronger regularity requirements for the initial data. The result we obtain seems to be sharp since it is consistent with the rate observed in the numerical simulations in Section 7.2.

*Remark* 2.10. The condition $s < s'$ in Theorem 2.6 is imposed in order for the truncation inequality (2.8) to take place, which in turn follows from the convergence of the series (6.6).

*Remark* 2.11. As in Remark 2.3, one can obtain the relaxation result for the semi-discrete system (2.5) by passing $\tau \to 0$ in (2.14):

$$\|U_1^\varepsilon - U_1\|_{L^\infty_T(\dot{B}_h^{s-2})} + \|U_1^\varepsilon - U_1\|_{L^1_T(\dot{B}_h^{s})} + \|\widetilde{B}^{-1}A_{2,1}\mathcal{D}_h U_1^\varepsilon + U_2^\varepsilon\|_{L^1_T(\dot{B}_h^{s-1})} \lesssim \varepsilon^2 C, \quad \forall T > 0,$$

where the initial data for the semi-discrete system and the constant $C$ are the same as in Theorem 2.6, and where, for any Banach space $X$, time $T > 0$ and $p \in [1, \infty]$, we denoted by $L^p_T(X)$ the set of measurable functions $g : [0, T] \to X$ such that $t \mapsto \|g(t)\|_X$ is in $L^p(0, T)$.

**Comments and comparison with the literature.** In our approach, a distinctive advantage of our discrete Littlewood-Paley decomposition, in contrast to existing literature (such as the work Hong & Yang (2019) on Strichartz estimates for discrete Schrödinger and Klein-Gordon equations), lies in the adaptation of the localization annuli to the precise form of the discrete differential operator $\mathcal{D}_h$. This adaptation allows us to obtain Bernstein-type estimates necessary for proving the relaxation property. The justification of our results differs from previous endeavors related to similar hyperbolic approximation procedures, e.g. Boscarino & Russo (2009), Jin (2012), Hu & Shu (2024), Boscarino & Russo (2024) where implicit–explicit (IMEX) Runge–Kutta schemes are used. Here, we establish the large time asymptotic-preserving property of the implicit central finite difference scheme within a refined frequency-based functional framework, strategically constructed to approach stiff relaxation procedures for hyperbolic systems. In a related context, we highlight Degond (2013), Dimarco & Pareschi (2014), Jin (2022), Lemou & Mieussens (2008), Jin (2010), Jin et al. (2000), Bessemoulin-Chatard et al. (2020), Goudon et al. (2013), Blaustein & Filbet (2024), Ma et al. (2023), Bessemoulin-Chatard & Mathis (2024) where authors delved into the relaxation limit of kinetic and hyperbolic models.

In particular, in the recent work Blaustein & Filbet (2024), the authors craft a discrete framework for studying the Vlasov-Poisson-Fokker-Planck system, first rewriting the equations as a partially dissipative hyperbolic system with stiff relaxation terms, using Hermite polynomials in terms of the velocity. Then, in line with the continuous approach by Dolbeault et al. (2015), they justify the relaxation limit of such hyperbolic systems (which shares similarities with the one studied in the present paper), revealing the diffusion limit at the discrete level of the kinetic model. A crucial difference in our current scenario is that we tackle the full-space case, as opposed to the torus setting. In the full-space case, there is a lack of a spectral gap in low frequencies due to the absence of a Poincaré-type inequality, thus leading to a dichotomous behaviour in low and high frequencies which, in turn, requires the development of a functional framework tailored to deal with this polarity.

## 3. Discrete hypocoercivity for hyperbolic systems

### 3.1. **Notations and discrete framework.**

This section is dedicated to the proof of the large-time asymptotic result (2.3). Across the paper, the notations $E \sim F$ and $E \lesssim F$ signify that there exists a constant $C > 1$ depending only on the matrices $A$ and $B$ such that $\frac{1}{C}F \leq E \leq CF$ and $E \leq CF$, respectively.

First, we recall the definition of discrete Lebesgue and Sobolev norms.

**Definition 3.1** (Discrete Lebesgue and Sobolev norms)**.** We consider $p \in [1, \infty)$, $(v_n)_{n\in\mathbb{Z}}$ a bilateral infinite complex-valued sequence and $h > 0$ the width of an equidistant spatial grid. We say that $v \in \ell_h^p$ if

$$\|v\|_{\ell_h^2}^p := h \sum_{n\in\mathbb{Z}} |v_n|^p < \infty, \tag{3.1}$$

and $v \in \ell_h^\infty$ if

$$\|v\|_{\ell_h^\infty} := \sup_{n\in\mathbb{Z}} |v_n| < \infty. \tag{3.2}$$

The discrete Sobolev $\mathfrak{h}_h^1$ norm of a bilateral sequence $v \in \ell_h^2$ is given by

$$\|v\|_{\mathfrak{h}_h^1}^2 := \|v\|_{\ell_h^2}^2 + \|\mathcal{D}_h v\|_{\ell_h^2}^2. \tag{3.3}$$

Next, we state a well-known integration by parts formula for the operator $\mathcal{D}_h$ defined in (1.6), which will be useful in our computations.

**Proposition 3.2.** *Let $u, v \in \ell_h^2$. The following integration by parts formula holds:*

$$(u, \mathcal{D}_h v)_{\ell_h^2} = -(\mathcal{D}_h u, v)_{\ell_h^2},$$

*where the $\ell_h^2$ scalar product associated to the norm (3.1) is given by*

$$(u, v)_{\ell_h^2} = h \sum_{n\in\mathbb{Z}} u_n v_n.$$

*An immediate consequence of the integration by parts formula is that, for every $u \in \ell_h^2$,*

$$(u, \mathcal{D}_h u)_{\ell_h^2} = 0.$$

For a more in-depth exploration of finite difference schemes and their properties, interested readers can consult Strikwerda (2004).

3.2. **Proof Theorem 2.1.** In order to simplify the presentation of the computations, we introduce the following notation: for a sequence $(v^k)_{k=0}^K$ such that $v^k \in \ell_h^2$, $\forall k$, we denote:

$$\delta_\tau v^k := \frac{v^{k+1} - v^k}{\tau}. \tag{3.4}$$

Then, the system (2.2) becomes

$$\begin{cases} \delta_\tau U_1^k + A_{1,1}\mathcal{D}_h U_1^{k+1} + A_{1,2}\mathcal{D}_h U_2^{k+1} = 0; \\ \delta_\tau U_2^k + A_{2,1}\mathcal{D}_h U_1^{k+1} + A_{2,2}\mathcal{D}_h U_2^{k+1} = -\widetilde{B} U_2^{k+1}. \end{cases} \tag{3.5}$$

Next, the following identity plays an important role in studying the decay properties of the system (3.5). For $v \in \ell_h^2$, we have, by direct calculation

$$2(\delta_\tau v^k, v^{k+1})_{\ell_h^2} = \delta_\tau \|v^k\|_{\ell_h^2}^2 + \frac{1}{\tau}\|v^{k+1} - v^k\|_{\ell_h^2}^2. \tag{3.6}$$

Taking the scalar product of (3.5) with $U^{k+1}$, using (3.6), the symmetry of $A$ and the inequality (1.3) we obtain

$$\begin{aligned} \delta_\tau \|U^k\|_{\ell_h^2}^2 + \frac{1}{\tau}\|U^{k+1} - U^k\|_{\ell_h^2}^2 = {} & 2(-A_{1,1}\mathcal{D}_h U_1^{k+1} - A_{1,2}\mathcal{D}_h U_2^{k+1}, U_1^{k+1})_{\ell_h^2} \\ & + 2(-A_{2,1}\mathcal{D}_h U_1^{k+1} - A_{2,2}\mathcal{D}_h U_2^{k+1}, U_2^{k+1})_{\ell_h^2} \\ & - 2(\widetilde{B} U_2^{k+1}, U_2^{k+1})_{\ell_h^2} \end{aligned} \tag{3.7}$$

$$\leq -2\lambda\|U_2^{k+1}\|_{\ell_h^2}^2,$$

where the last equality follows by integration by parts. The same reasoning for the discrete space-derivative of System 3.5 leads to

$$\delta_\tau\|\mathcal{D}_h U^k\|_{\ell_h^2}^2 + \frac{1}{\tau}\|\mathcal{D}_h U^{k+1} - \mathcal{D}_h U^k\|_{\ell_h^2}^2 \leq -2\lambda\|D_h U_2^{k+1}\|_{\ell_h^2}^2, \tag{3.8}$$

Taking the sum of (3.7) and (3.8), we arrive at

$$\delta_\tau\|U^k\|_{\mathfrak{h}_h^1}^2 + \frac{1}{\tau}\|U^{k+1} - U^k\|_{\mathfrak{h}_h^1}^2 \leq -2\lambda\|U_2^{k+1}\|_{\mathfrak{h}_h^1}^2, \tag{3.9}$$

Next, inspired by the works of Beauchard & Zuazua (2011) and Crin-Barat et al. (2024), we define the discrete Lyapunov functional:

$$\mathcal{L}^k = \|U^k\|_{\mathfrak{h}_h^1}^2 + \eta_0 t^k\|\mathcal{D}_h U^k\|_{\ell_h^2}^2 + \mathcal{I}^k, \tag{3.10}$$

where the correction term $\mathcal{I}^k$ takes the following form:

$$\mathcal{I}^k := \sum_{q=1}^{N-1} \varepsilon_q (BA^{q-1}U^k, BA^q\mathcal{D}_h U^k)_{\ell_h^2}, \tag{3.11}$$

and the positive parameters $\eta_0$ and $\varepsilon_q$ will be made precise later.

Our aim is to estimate $\delta_\tau\mathcal{L}^k$. Since, for the discrete time derivative of the first term in $\mathcal{L}^k$, we already have the equality (3.9), we focus on the second term, for which a direct calculation implies

$$\delta_\tau\left[t^k\|\mathcal{D}_h U^k\|_{\ell_h^2}^2\right] = t^k\delta_\tau\|\mathcal{D}_h U^k\|_{\ell_h^2}^2 + \|\mathcal{D}_h U^{k+1}\|_{\ell_h^2}^2 \tag{3.12}$$

Next, we take the discrete-time derivative of $\mathcal{I}^k$. For every $1 \leq q \leq N-1$, we have

$$\begin{aligned}
\delta_\tau(BA^{q-1}U^k, BA^q\mathcal{D}_h U^k)_{\ell_h^2} &= \frac{(BA^{q-1}U^{k+1}, BA^q\mathcal{D}_h U^{k+1})_{\ell_h^2}}{\tau} - \frac{(BA^{q-1}U^k, BA^q\mathcal{D}_h U^k)_{\ell_h^2}}{\tau} \\
&= (BA^{q-1}\delta_\tau U^k, BA^q\mathcal{D}_h U^{k+1})_{\ell_h^2} + (BA^{q-1}U^{k+1}, BA^q\delta_\tau\mathcal{D}_h U^k)_{\ell_h^2} \\
&\quad - \frac{1}{\tau}(BA^{q-1}U^{k+1} - BA^{q-1}U^k, BA^q\mathcal{D}_h U^{k+1} - BA^q\mathcal{D}_h U^k)_{\ell_h^2}
\end{aligned}$$

Employing Cauchy-Schwarz inequality in the last term, we arrive at:

$$\delta_\tau\mathcal{I}^k \leq \sum_{q=1}^{N-1} \varepsilon_q (BA^{q-1}\delta_\tau U^k, BA^q\mathcal{D}_h U^{k+1})_{\ell_h^2} + (BA^{q-1}U^{k+1}, BA^q\delta_\tau\mathcal{D}_h U^k)_{\ell_h^2} \tag{3.13}$$

$$+ \sum_{q=1}^{N-1} \varepsilon_q \frac{1}{2\tau}\left(\|BA^{q-1}(U^{k+1} - U^k)\|_{\ell_h^2}^2 + \|BA^q(\mathcal{D}_h U^{k+1} - \mathcal{D}_h U^k)\|_{\ell_h^2}^2\right). \tag{3.14}$$

We take now the quantities $(\varepsilon_q)_{q=1}^{N-1}$ small enough such that

$$\sum_{q=1}^{N-1} \varepsilon_q \frac{1}{2\tau}\left(\|BA^{q-1}(U^{k+1} - U^k)\|_{\ell_h^2}^2 + \|BA^q(\mathcal{D}_h U^{k+1} - \mathcal{D}_h U^k)\|_{\ell_h^2}^2\right) \leq \frac{1}{2\tau}\|U^{k+1} - U^k\|_{\mathfrak{h}_h^1}^2. \tag{3.15}$$

Using (3.15) and that $\delta_\tau U^k = -A\mathcal{D}_h U^{k+1} - BU^{k+1}$, we obtain

$$\begin{aligned}\delta_\tau \mathcal{I}^k + \sum_{q=1}^{N-1} \varepsilon_q \|BA^q \mathcal{D}_h U^{k+1}\|_{\ell_h^2}^2 \leq &- \sum_{q=1}^{N-1} \varepsilon_q (BA^{q-1}BU^{k+1}, BA^q \mathcal{D}_h U^{k+1})_{\ell_h^2} \\ &- \sum_{q=1}^{N-1} \varepsilon_q (BA^{q-1}U^{k+1}, BA^q B\mathcal{D}_h U^{k+1})_{\ell_h^2} \\ &- \sum_{q=1}^{N-1} \varepsilon_q (BA^{q-1}U^{k+1}, BA^{q+1}\mathcal{D}_h^2 U^{k+1})_{\ell_h^2} \\ &+ \frac{1}{2\tau}\|U^{k+1}-U^k\|_{\mathfrak{h}_h^1}^2.\end{aligned} \tag{3.16}$$

Concerning the discrete time derivative of $\mathcal{I}^k$ We have the following lemma whose proof is relegated to the Appendix A.1.

**Lemma 3.3** (Discrete time derivative of $\mathcal{I}$)**.** *For any positive constant $\varepsilon_0$, there exists a sequence $\{\varepsilon_q\}_{q=1}^{N-1}$ of small positive constants such that*

$$\delta_\tau \mathcal{I}^k + \frac{1}{2}\sum_{q=1}^{N-1} \varepsilon_q \|BA^q \mathcal{D}_h U^{k+1}\|_{\ell_h^2}^2 \leq \varepsilon_0 \|U_2^{k+1}\|_{\mathfrak{h}_h^1}^2 + \frac{1}{2\tau}\|U^{k+1}-U^k\|_{\mathfrak{h}_h^1}^2. \tag{3.17}$$

Next, we might further decrease the positive quantities $(\varepsilon_q)_{q=1}^{N-1}$ provided by the lemma such that, by applying the Cauchy-Schwarz inequality in (3.11), one has

$$\mathcal{L}^k \sim \|U^k\|_{\mathfrak{h}_h^1}^2 + \eta_0 t^k \|\mathcal{D}_h U^k\|_{\ell_h^2}^2. \tag{3.18}$$

Using (3.17), we obtain:

$$\begin{aligned}&\delta_\tau \mathcal{L}^k + 2\lambda \|U_2^{k+1}\|_{\ell_h^2}^2 + \lambda(1+2\eta_0 t^k)\|\mathcal{D}_h U_2^{k+1}\|_{\ell_h^2}^2 + \frac{1}{2}\sum_{q=1}^{N-1}\varepsilon_q \|BA^q\mathcal{D}_h U^{k+1}\|_{\ell_h^2}^2 \\ &\leq \eta_0 \|\mathcal{D}_h U^{k+1}\|_{\ell_h^2}^2 + \varepsilon_0 \|U_2^{k+1}\|_{\ell_h^2}^2 + \varepsilon_0 \|\mathcal{D}_h U_2^{k+1}\|_{\ell_h^2}^2.\end{aligned} \tag{3.19}$$

From (Beauchard & Zuazua 2011, Lemma 1), we have that, for $y \in \mathbb{C}^N$, the function

$$\mathcal{N}(y) := \left(\sum_{q=0}^{N-1} |BA^q y|^2\right)^{\frac{1}{2}} \quad \text{defines a norm on } \mathbb{C}^N, \tag{3.20}$$

which, by standard properties of finite-dimensional spaces, is equivalent to any other norm, in particular to the Euclidean one. Using this norm equivalence, we obtain

$$\frac{\lambda}{4}\|\mathcal{D}_h U_2^{k+1}\|_{\ell_h^2}^2 + \frac{1}{2}\sum_{q=1}^{N-1}\varepsilon_q \|BA^q \mathcal{D}_h U^{k+1}\|_{\ell_h^2}^2 \geq \frac{\varepsilon_*}{C_2}\|\mathcal{D}_h U^{k+1}\|_{\ell_h^2}^2,$$

with $\varepsilon_* := \min\{\lambda/2, \varepsilon_0, \varepsilon_1, ..., \varepsilon_{N-1}\}$ and $C_2 > 0$ a constant depending only on $(A, B)$ and $N$. Therefore, to ensure the coercivity of (3.19), we adjust the coefficients appropriately as

$$0 < \eta_0 < \frac{\varepsilon_*}{4C_2}, \qquad 0 < \varepsilon_0 < \frac{\lambda}{4},$$

and therefore get

$$\delta_\tau \mathcal{L}^k + \lambda \|U_2^{k+1}\|_{\ell_h^2}^2 + \lambda\left(\frac{1}{2} + \eta_0 t^k\right)\|\mathcal{D}_h U_2^{k+1}\|_{\ell_h^2}^2 + \frac{\varepsilon_*}{4C_2}\|\mathcal{D}_h U^{k+1}\|_{\ell_h^2}^2 \leq 0. \tag{3.21}$$

Therefore, by (3.21), we have $\mathcal{L}^k \leq \mathcal{L}^0$, which by the equivalence in (3.18) leads to

$$\|U^k\|_{\ell_h^2} + (1+t^k)^{\frac{1}{2}}\|\mathcal{D}_h U^k\|_{\ell_h^2} \leq C\|U^0\|_{\mathfrak{h}_h^1}. \tag{3.22}$$

3.3. **Improved decay for the component $U_2$.** Multiplying the second equation of (3.5) with $U_2^{k+1}$, we obtain by (3.6) and the Cauchy-Schwarz inequality that

$$\delta_\tau \|U_2^k\|_{\ell_h^2}^2 + \frac{1}{\tau}\|U_2^{k+1} - U_2^k\|_{\ell_h^2}^2 + 2\lambda\|U_2\|_{\ell_h^2}^2 \leq C\|\mathcal{D}_h U^{k+1}\|_{\ell_h^2}\|U_2^{k+1}\|_{\ell_h^2}, \tag{3.23}$$

where $C > 0$ is a constant depending only on the matrices $A$ and $B$. Inspired by the continuous version of this result (see Bianchini et al. (2007) for instance), we consider the following expression, for a fixed $\lambda' > 0$ which will be made precise later:

$$E^k = \delta_\tau \left[ e^{\lambda' t^k}\|U_2^k\|_{\ell_h^2}\right].$$

By direct computation, we derive that, provided that $\|U^{k+1}\|_{\ell_h^2}$ and $\|U^k\|_{\ell_h^2}$ are not both zero,

$$E^k \leq e^{\lambda' t^k}\left[\lambda' e^{\lambda'\tau}\|U_2^{k+1}\|_{\ell_h^2} + \frac{\delta_\tau\|U_2^k\|_{\ell_h^2}^2}{\|U_2^{k+1}\|_{\ell_h^2} + \|U_2^k\|_{\ell_h^2}}\right],$$

which, by (3.23), leads to:

$$E^k \leq e^{\lambda' t^k}\left[\lambda' e^{\lambda'\tau}\|U_2^{k+1}\|_{\ell_h^2} - \frac{2\lambda\|U_2^{k+1}\|_{\ell_h^2}^2 + \frac{1}{\tau}\|U_2^{k+1} - U_2^k\|_{\ell_h^2}^2}{\|U_2^{k+1}\|_{\ell_h^2} + \|U_2^k\|_{\ell_h^2}} + C\frac{\|\mathcal{D}_h U^{k+1}\|_{\ell_h^2}\|U_2^{k+1}\|_{\ell_h^2}}{\|U_2^{k+1}\|_{\ell_h^2} + \|U_2^k\|_{\ell_h^2}}\right], \tag{3.24}$$

Next, taking $\gamma = \min\{\frac{2}{3}\lambda, \frac{1}{\tau}\}$ and $\lambda' > 0$ such that $\lambda' e^{\lambda'} \leq \gamma$, we obtain

$$\begin{aligned}2\lambda\|U_2^{k+1}\|_{\ell_h^2}^2 + \frac{1}{\tau}\|U_2^{k+1} - U_2^k\|_{\ell_h^2}^2 &\geq \gamma\left[3\|U_2^{k+1}\|_{\ell_h^2}^2 + \|U_2^{k+1} - U_2^k\|_{\ell_h^2}^2\right] \geq \gamma\left[2\|U_2^{k+1}\|_{\ell_h^2}^2 + \frac{1}{2}\|U_2^k\|_{\ell_h^2}^2\right] \\ &\geq \gamma\left[\|U_2^{k+1}\|_{\ell_h^2}^2 + \|U_2^{k+1}\|_{\ell_h^2}\|U_2^k\|_{\ell_h^2}\right] = \gamma\|U_2^{k+1}\|_{\ell_h^2}\left[\|U_2^{k+1}\|_{\ell_h^2} + \|U_2^k\|_{\ell_h^2}\right].\end{aligned}$$

Therefore, (3.24) implies:

$$E^k \leq Ce^{\lambda' t^k}\frac{\|\mathcal{D}_h U^{k+1}\|_{\ell_h^2}\|U_2^{k+1}\|_{\ell_h^2}}{\|U_2^{k+1}\|_{\ell_h^2} + \|U_2^k\|_{\ell_h^2}} \leq Ce^{\lambda' t^k}\|\mathcal{D}_h U^{k+1}\|_{\ell_h^2} \lesssim e^{\lambda' t^k}(1+t^k)^{-\frac{1}{2}}\|U^0\|_{\mathfrak{h}_h^1}, \tag{3.25}$$

where the last inequality follows from (3.22). We note that the conclusion of (3.25) is still valid if $\|U_2^{k+1}\|_{\ell_h^2}$ and $\|U_2^k\|_{\ell_h^2}$ both vanish. Summing up (3.25) for $k = \overline{0, K-1}$, we obtain

$$e^{\lambda' t^K}\|U_2^K\|_{\ell_h^2} \lesssim \|U_2^0\|_{\ell_h^2} + \|U^0\|_{\mathfrak{h}_h^1}\tau\sum_{k=1}^{K-1} e^{\lambda' t^k}(1+t^k)^{-\frac{1}{2}},$$

or, equivalently,

$$\|U_2^K\|_{\ell_h^2} \lesssim e^{-\lambda' t^K}\|U_2^0\|_{\ell_h^2} + \|U^0\|_{\mathfrak{h}_h^1}\tau\sum_{k=1}^{K-1} e^{\lambda'(t^k - t^K)}(1+t^k)^{-\frac{1}{2}}.$$

We notice that the sum above is comparable to a Darboux sum of the integral $I(t^K)$ Lemma A.2, so it follows that

$$\|U_2^K\|_{\ell_h^2} \lesssim \|U^0\|_{\mathfrak{h}_h^1}(1+t^K)^{-\frac{1}{2}}.$$

The proof of Theorem 2.1 is concluded. ∎

## 4. Fourier-based numerical framework

4.1. **Discrete Fourier Transform.** Within this section, we introduce the discrete one-dimensional Fourier transform and revisit some fundamental properties such as invertibility and Parseval's equality. The following definition is essentially taken from (Trefethen 1994, Section 2.2).

**Definition 4.1** (Discrete Fourier transform)**.** We consider a bilateral infinite real sequence $(v_n)_{n\in\mathbb{Z}}$ and a grid width $h>0$. Assume that $v\in\ell^2_h$. The discrete Fourier transform of $v$ is defined as $\hat{v}:[-\frac{\pi}{h},\frac{\pi}{h}]\to\mathbb{R}$,

$$\hat{v}(\xi):=\frac{h}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}e^{-i\xi nh}v_n.$$

The inverse Fourier transform $\mathcal{F}^{-1}:L^2\left([-\frac{\pi}{h},\frac{\pi}{h}]\right)\to\ell^2_h$ has the following form

$$\left(\mathcal{F}^{-1}(g)\right)_n=\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}e^{i\xi nh}g(\xi)\,\mathrm{d}\xi. \tag{4.1}$$

*Remark* 4.2. In order to rigorously define the Discrete Fourier transform, one has to follow the same pathway as for the continuous one (see, for example, (Rudin 1987, Chapter 9)): define the Fourier transform of for summable sequences (i.e. in $\ell^1_h$) and then extending them by density to $\ell^2_h$.

The next proposition, taken from (Trefethen 1994, Theorem 2.5), summarizes some basic properties of the discrete Fourier transform.

**Proposition 4.3.** *Let $v\in\ell^2_h$. The following properties hold:*

*(1) $\hat{v}\in L^2\left([-\frac{\pi}{h},\frac{\pi}{h}]\right)$ and $\|\hat{v}\|_{L^2([-\frac{\pi}{h},\frac{\pi}{h}])}=\|v\|_{\ell^2_h}$. (Parseval's equality)*

*(2) The sequence $v\in\ell^2_h$ can be recovered from its discrete Fourier transform by the equality:*

$$v=\mathcal{F}^{-1}(\hat{v}).$$

*(3) Let $w\in\ell^1_h$. Then, the convolution product of $u$ and $v$ defined as*

$$(v*w)_n:=h\sum_{m\in\mathbb{Z}}v_m w_{n-m}$$

*belongs to $\ell^2_h$ and*

$$\widehat{v*w}=\sqrt{2\pi}\,\hat{v}\hat{w}.$$

In order to study discrete hyperbolic systems such as (2.1), we derive the Fourier symbol of the discrete central finite difference operator $\mathcal{D}_h$:

$$\widehat{(\mathcal{D}_h v)}(\xi)=i\frac{\sin(\xi h)}{h}\hat{v}(\xi), \tag{4.2}$$

This allows us to define homogeneous and inhomogeneous discrete fractional Sobolev norms: for $v\in\ell^2_h$ and $s>0$,

$$\|v\|^2_{\dot{\mathfrak{h}}^s_h}:=\|\mathcal{D}^s_h v\|_{\ell^2_h}:=\left\|\hat{v}(\xi)\left|\frac{\sin(\xi h)}{h}\right|^s\right\|_{L^2([-\frac{\pi}{h},\frac{\pi}{h}])}\quad\text{and}\quad\|v\|^2_{\mathfrak{h}^s_h}:=\|v\|^2_{\ell^2_h}+\|\mathcal{D}^s_h v\|^2_{\ell^2_h}. \tag{4.3}$$

4.2. **Discrete Besov norms.** In this section, we establish an analogous framework for the standard Besov norms (associated with the continuous Fourier transform) within the discrete setting introduced in the preceding sections. The definition of these discrete Besov norms is guided by our objective of localizing the frequencies of a bilateral sequence $(v_n)_{n\in\mathbb{Z}}$ in such a manner that, for each $j \in \mathbb{Z}$, the localization $\delta_h^j v$ is designed to satisfy a Bernstein-type estimate:

$$\|\mathcal{D}_h(\delta_h^j v)\|_{\ell_h^2} \sim 2^j \|\delta_h^j v\|_{\ell_h^2}, \tag{4.4}$$

where $\mathcal{D}_h$ is the central finite difference operator. We will formulate the rigorous form of the Bernstein estimate in Section 4.3. Also, the interested reader could refer to (Bahouri et al. 2011, Chapter 2) for an introduction to continuous Besov spaces and their basic properties.

The form (4.2) of the central finite difference operator in Fourier variables suggests the following notation:

$$F_h(j) := \left\{ \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right] : \left|\frac{\sin(\xi h)}{h}\right| \in \mathcal{C}_j \right\}, \tag{4.5}$$

where, for every $j \in \mathbb{Z}$, we denote

$$\mathcal{C}_j := \left[\frac{3}{4}2^j, \frac{4}{3}2^{j+1}\right]. \tag{4.6}$$

Inspired by the dyadic decomposition used to construct the continuous Besov spaces (Bahouri et al. 2011, Sections 2.2 and 2.3), we consider a family of functions $(\varphi_j)_{j\in\mathbb{Z}}$ depending on $h$ with the following properties:

$$\varphi_j : \left[-\frac{\pi}{h}, \frac{\pi}{h}\right] \to [0,1], \quad \forall j \in \mathbb{Z}, \tag{4.7}$$

$$\operatorname{supp}(\varphi_j) \subseteq F_h(j), \quad \forall j \in \mathbb{Z}, \tag{4.8}$$

$$\sum_{j\in\mathbb{Z}} \varphi_j(\xi) = 1, \quad \forall \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]. \tag{4.9}$$

We note that, since the family of sets $(\mathcal{C}_j)_{j\in\mathbb{Z}}$ is locally finite, the above sum makes sense for every $\xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$. Now, we can define the $j$-th frequency localization of a sequence $(v_n)_{n\in\mathbb{Z}}$ and the discrete homogeneous Besov norms.

**Definition 4.4** (Discrete localization operators)**.** Let $v \in \ell_h^2$ and $j \in \mathbb{Z}$. We define the $j$-th frequency localization of $v$ as

$$\delta_h^j v := \mathcal{F}^{-1}(\hat{v}\varphi_j).$$

**Definition 4.5** (Discrete Besov Norms – refer to (Bahouri et al. 2011, Definition 2.15) for the continuous case)**.** Let $s \geq 0$ and $v \in \ell_h^2$. The discrete Besov $s$-norm of $v$ is defined as

$$\|v\|_{\dot{B}_h^s} := \sum_{j\in\mathbb{Z}} 2^{js} \|\delta_h^j v\|_{\ell_h^2}. \tag{4.10}$$

We note that this norm is finite for every $v \in \ell_h^2$, by an argument similar to the proof of Theorem 2.4.

4.3. **Basic properties of discrete Besov norms.** First, we revisit and rigorously formulate the Bernstein estimate (4.4):

**Proposition 4.6** (Bernstein estimate for central finite difference operator)**.** *Let $\mathcal{D}_h$ be the central finite difference operator. There exist two universal positive constants $C, c > 0$ such that, for every $h > 0$, every bilateral sequence $v \in \ell^2_h$ and every integer $j$,*

$$c\,2^j \|\delta^j_h v\|_{\ell^2_h} \leq \|\mathcal{D}_h \delta^j_h v\|_{\ell^2_h} \leq C\,2^j \|\delta^j_h v\|_{\ell^2_h},$$

*where $\delta^j_h$ is the localization operator introduced in Definition 4.4.*

*Proof.* Taking into account (4.2) and Definition 4.4, we obtain

$$\widehat{(\mathcal{D}_h \delta^j_h v)}(\xi) = i\frac{\sin(\xi h)}{h}\varphi_j(\xi)\hat{v}(\xi).$$

From (4.5) and (4.8) we get that, for every $\xi \in \mathrm{supp}(\varphi_j)$,

$$\left|\frac{\sin(\xi h)}{h}\right| \in \mathcal{C}_j.$$

Then, the conclusion follows from Parseval's equality. □

**Definition 4.7** (Frequency-restricted discrete Besov norms)**.** Let $s \in \mathbb{R}$ and $\kappa$ a small enough positive constant that will be precisely fixed in the proof of Theorem 2.6. For $J_\varepsilon := \log_2 \frac{\kappa}{\varepsilon}$, i.e. $2^{J_\varepsilon} = \frac{\kappa}{\varepsilon}$, we define

$$\|v\|^L_{\dot{B}^s_h} := \sum_{j \leq J_\varepsilon} 2^{js} \|\delta^j_h v\|_{\ell^2_h} \quad \text{and} \quad \|v\|^H_{\dot{B}^s_h} := \sum_{j \geq J_\varepsilon} 2^{js} \|\delta^j_h v\|_{\ell^2_h}. \tag{4.11}$$

From Proposition 4.6, using that $2^{J_\varepsilon} = \frac{\kappa}{\varepsilon}$, we immediately deduce the following low-high frequencies Bernstein-type inequalities.

**Proposition 4.8.** *Let $v \in \ell^2_h$, $s \geq 0$ and $s' > 0$. The following Bernstein-type inequalities hold:*

$$\|v\|^L_{\dot{B}^s_h} \leq C\frac{\kappa^{s'}}{\varepsilon^{s'}} \|v\|^L_{\dot{B}^{s-s'}_h}, \text{ provided that } s \geq s'; \tag{4.12}$$

$$\|v\|^H_{\dot{B}^s_h} \leq C\frac{\varepsilon^{s'}}{\kappa^{s'}} \|v\|^H_{\dot{B}^{s+s'}_h} \tag{4.13}$$

*where $C > 0$ is a universal constant.*

Next we prove Proposition 2.5. One of the important implications of this norm inequality is that the estimates obtained for discrete Besov norms (4.10) lead to results in well-known norms. We refer to (Bahouri et al. 2011, Proposition 2.39) for a more general embedding result in the continuous framework.

*Proof of Proposition 2.5.* First of all, the inequality (2.9) follows immediately by the definition of the $\dot{\mathfrak{h}}^s_h$ norm and by (4.8)-(4.9), using Minkowski's inequality.

In the sequel, we focus on proving the estimate (2.10). Indeed, the property (4.9) implies that, for every $\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]$,

$$\hat{v}(\xi) = \sum_{j \in \mathbb{Z}} \widehat{(\delta^j_h v)}(\xi),$$

where the sum above is finite for any particular $\xi$. As a result, for any $n \in \mathbb{Z}$,

$$v_n = \sum_{j\in\mathbb{Z}} (\delta_h^j v)_n,$$

which implies that

$$\|v\|_{\ell_h^\infty} \le \sum_{j\in\mathbb{Z}} \|\delta_h^j v\|_{\ell_h^\infty}.$$

From the definition (4.10) of the Besov norm, it is enough to prove that:

$$\|\delta_h^j v\|_{\ell_h^\infty} \le C\, 2^{\frac{j}{2}} \|\delta_h^j v\|_{\ell_h^2}. \tag{4.14}$$

Indeed, from (4.8), it follows that $\varphi_j \cdot \chi_{F_h(j)} = \varphi_j$, where $\chi_S$ stands for the characteristic function of a set $S$. The discrete Fourier inverse formula (4.1) implies that:

$$\begin{aligned}(\delta_h^j v)_n &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{i\xi hn} \hat{v}(\xi)\varphi_j(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{i\xi hn} \hat{v}(\xi)\varphi_j(\xi)\chi_{F_h(j)}\,\mathrm{d}\xi.\end{aligned}$$

Since $\left|e^{i\xi hn}\right| = 1$, the Cauchy-Schwarz inequality and Parseval's equality further imply that

$$\begin{aligned}\left|(\delta_h^j v)_n\right| &\le \frac{1}{\sqrt{2\pi}} \|\hat{v}\varphi_j\|_{L^2([-\frac{\pi}{h},\frac{\pi}{h}])} \|\chi_{F_h(j)}\|_{L^2([-\frac{\pi}{h},\frac{\pi}{h}])} \\ &= \frac{1}{\sqrt{2\pi}} \|\delta_h^j v\|_{\ell_h^2} \, |F_h(j)|^{\frac{1}{2}}\,,\end{aligned}$$

where $|S|$ stands for the Lebesgue measure of the set $S$. Therefore, we are left to prove that:

$$|F_h(j)| \le C \cdot 2^j. \tag{4.15}$$

In order to prove this claim, we observe that an element $\xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$ belongs to $F_h(j)$ if and only if

$$\frac{\sin(\xi h)}{\xi h}\xi \in \left[\frac{3}{4}2^j, \frac{4}{3}2^{j+1}\right]. \tag{4.16}$$

Next, we fix a constant $c \in \left(0, \frac{\pi}{2}\right)$ and notice from the plot in Figure 1 that there exists another constant $M_c \in (0,1)$ such that, for every $x \in [-\pi + c, \pi + c]$,

$$M_c \le \frac{\sin(x)}{x} \le 1. \tag{4.17}$$

Therefore, if $\xi h \in [-\pi + c, \pi - c]$, then (4.16) implies that $\xi \in \left[\frac{3}{4}2^j, \frac{4}{3M_c}2^{j+1}\right]$. We can now estimate the Lebesgue measure of a part of $F_h(j)$:

$$\left|F_h(j) \cap \left[\frac{-\pi + c}{h}, \frac{\pi - c}{h}\right]\right| \le 2^j \left(\frac{8}{3M_c} - \frac{3}{4}\right). \tag{4.18}$$

Then, we consider the case $\xi h \in [\pi - c, \pi]$, which means that $\pi - \xi h \in [0, c] \subset [-\pi + c, \pi - c]$. In this case, we have

$$\frac{\sin(\pi - \xi h)}{\pi - \xi h} \in [M_c, 1].$$

Therefore, if $\xi$ is such that (4.16) holds, then the equality $\sin(x) = \sin(\pi - x)$ implies that

$$\frac{\pi}{h} - \xi = \frac{\sin(\xi h)}{\xi h}\xi \frac{\pi - \xi h}{\sin(\pi - \xi h)} \in \left[\frac{3}{4}2^j, \frac{4}{3M_c}2^{j+1}\right].$$

This leads us to an estimate of the Lebesgue measure of a second part of $F_h(j)$:

$$\left| F_h(j) \cap \left[ \frac{\pi - c}{h}, \frac{\pi}{h} \right] \right| \leq 2^j \left( \frac{8}{3M_c} - \frac{3}{4} \right). \tag{4.19}$$

By matters of symmetry, we arrive also to an estimate regarding the third part of $F_h(j)$:

$$\left| F_h(j) \cap \left[ \frac{-\pi}{h}, \frac{-\pi + c}{h} \right] \right| \leq 2^j \left( \frac{8}{3M_c} - \frac{3}{4} \right). \tag{4.20}$$

Combining (4.18), (4.19) and (4.20) we obtain the claim (4.15), which finishes the proof. □

## 5. Proof Theorem 2.6: Numerical relaxation limit

We rewrite the system (2.11) using the notation in (3.4) as

$$\begin{cases} \delta_\tau U_1^{\varepsilon,k} + A_{1,2}\mathcal{D}_h U_2^{\varepsilon,k+1} = 0, \\ \varepsilon^2 \delta_\tau U_2^{\varepsilon,k} + A_{2,1}\mathcal{D}_h U_1^{\varepsilon,k+1} + \widetilde{B} U_2^{\varepsilon,k+1} = 0, \end{cases} \tag{5.1}$$

Applying the localization operator $\delta_h^j$ to the system (5.1), we obtain

$$\begin{cases} \delta_\tau U_{1,j}^{\varepsilon,k} + A_{1,2}\mathcal{D}_h U_{2,j}^{\varepsilon,k+1} = 0, \\ \varepsilon^2 \delta_\tau U_{2,j}^{\varepsilon,k} + A_{2,1}\mathcal{D}_h U_{1,j}^{\varepsilon,k+1} + \widetilde{B} U_{2,j}^{\varepsilon,k+1} = 0, \end{cases} \tag{5.2}$$

where we used the notation $f_j := \delta_h^j f$ for any $f \in \ell_h^2$. From here, the analysis is inspired by the computations done in Crin-Barat & Danchin (2023) and Danchin (2023), but with certain modifications aimed to sharpen, in this linear setting, the convergence ratio to $\mathcal{O}(\varepsilon^2)$, instead of $\mathcal{O}(\varepsilon)$.

In this section, the positive constant inherent to the "$\lesssim$" notation may also depend on the parameter $M$.

**Low-frequency analysis:** $j \leq J_\varepsilon = \log_2 \frac{\kappa}{\varepsilon}$**.**

Defining the damped mode $W^{\varepsilon,k} = \widetilde{B}^{-1} A_{2,1}\mathcal{D}_h U_1^{\varepsilon,k} + U_2^{\varepsilon,k}$ and inserting it in (5.2), we have

$$\begin{cases} \delta_\tau U_{1,j}^{\varepsilon,k} - A_{1,2}\widetilde{B}^{-1}A_{2,1}\mathcal{D}_h^2 U_{1,j}^{\varepsilon,k+1} = -A_{1,2}\mathcal{D}_h W_j^{\varepsilon,k+1}, \\ \delta_\tau W_j^{\varepsilon,k} + \dfrac{\widetilde{B}}{\varepsilon^2} W_j^{\varepsilon,k+1} = \widetilde{B}^{-1}A_{2,1}A_{1,2}\widetilde{B}^{-1}A_{2,1}\mathcal{D}_h^3 U_{1,j}^{\varepsilon,k+1} - \widetilde{B}^{-1}A_{2,1}A_{1,2}\mathcal{D}_h^2 W_j^{\varepsilon,k+1} \end{cases} \tag{5.3}$$

Taking the scalar product of the first equation of (5.3) with $U_{1,j}^{\varepsilon,k+1}$, we obtain, by Lemma A.1, the Cauchy-Schwarz inequality and (3.6), that

$$\delta_\tau \|U_{1,j}^{\varepsilon,k}\|_{\ell_h^2}^2 + \frac{1}{\tau}\|U_{1,j}^{\varepsilon,k+1} - U_{1,j}^{\varepsilon,k}\|_{\ell_h^2}^2 + 2\lambda_0 \|\mathcal{D}_h U_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2 \lesssim 2\|\mathcal{D}_h W_j^{\varepsilon,k+1}\|_{\ell_h^2} \|U_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2}. \tag{5.4}$$

Using the Bernstein estimate in Proposition 4.6, we have

$$\delta_\tau \|U_{1,j}^{\varepsilon,k}\|_{\ell_h^2}^2 + \frac{1}{\tau}\|U_{1,j}^{\varepsilon,k+1} - U_{1,j}^{\varepsilon,k}\|_{\ell_h^2}^2 + 2\lambda_0 2^{2j} \|U_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2 \lesssim \|\mathcal{D}_h W_j^{\varepsilon,k+1}\|_{\ell_h^2} \|U_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2}. \tag{5.5}$$

We can now apply Lemma A.3 which yields

$$\|U_{1,j}^{\varepsilon,K}\|_{\ell_h^2} + 2^{2j}\tau \sum_{k=0}^{K-1} \|U_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2} \lesssim \|U_{1,j}^0\|_{\ell_h^2} + \tau \sum_{k=0}^{K-1} \|\mathcal{D}_h W_j^{\varepsilon,k+1}\|_{\ell_h^2}, \tag{5.6}$$

since we have that $\tau \leq M\varepsilon^2 = M\kappa^2 2^{-2J_\varepsilon} \leq M\kappa^2 \cdot 2^{-2j}$.

Then, for $s \in \mathbb{R}$, multiplying (5.6) by $2^{js}$ and summing on $j \leq J_\varepsilon$, we obtain, with the notations in Definition 4.7, that

$$\|U_1^{\varepsilon,K}\|^L_{\dot{B}^s_h} + \|U_1^\varepsilon\|^L_{\ell^1_{\tau,K}(\dot{B}^{s+2}_h)} \lesssim \|U_1^0\|^L_{\dot{B}^s_h} + \|W^\varepsilon\|^L_{\ell^1_{\tau,K}(\dot{B}^{s+1}_h)}. \tag{5.7}$$

Performing similar estimates for $W_j^\varepsilon$, we obtain

$$\|W^{\varepsilon,K}\|^L_{\dot{B}^{s-1}_h} + \frac{1}{\varepsilon^2}\|W^\varepsilon\|^L_{\ell^1_{\tau,K}(\dot{B}^{s-1}_h)} \lesssim \|W^0\|^L_{\dot{B}^{s-1}_h} + \|U_1^\varepsilon\|^L_{\ell^1_{\tau,K}(\dot{B}^{s+2}_h)} + \|W^\varepsilon\|^L_{\ell^1_{\tau,K}(\dot{B}^{s+1}_h)}, \tag{5.8}$$

since $\tau \leq M\varepsilon^2$. Using the low-frequency Bernstein inequality (4.12), we have

$$\|W^\varepsilon\|^L_{\ell^1_{\tau,K}(\dot{B}^{s+1}_h)} \lesssim \frac{\kappa^2}{\varepsilon^2}\|W^\varepsilon\|^L_{\ell^1_{\tau,K}(\dot{B}^{s-1}_h)}. \tag{5.9}$$

Summing (5.7) and (5.8), using (5.9) and fixing $\kappa$ suitably small, we obtain

$$\begin{aligned}\|U_1^{\varepsilon,K}\|^L_{\dot{B}^s_h} + \|W^{\varepsilon,K}\|^L_{\dot{B}^{s-1}_h} + \frac{1}{\varepsilon^2}\|W^\varepsilon\|^L_{\ell^1_{\tau,K}(\dot{B}^{s-1}_h)} &\lesssim \|U_1^0\|^L_{\dot{B}^s_h} + \|W^0\|^L_{\dot{B}^{s-1}_h} \\ &\lesssim \|U_1^0\|^L_{\dot{B}^s_h} + \|U_2^0\|^L_{\dot{B}^{s-1}_h}.\end{aligned} \tag{5.10}$$

**High-frequency analysis:** $j > J_\varepsilon = \log_2 \frac{\kappa}{\varepsilon}$**.**

We define the following Lyapunov functional

$$\mathcal{L}_j^{\varepsilon,k} = \|(U_{1,j}^{\varepsilon,k}, \varepsilon U_{2,j}^{\varepsilon,k})\|^2_{\ell^2_h} + 2^{-2j}\mathcal{I}_j^{\varepsilon,k}, \tag{5.11}$$

where the corrector term $\mathcal{I}_j^{\varepsilon,k}$ reads

$$\mathcal{I}_j^{\varepsilon,k} := \eta(U_{2,j}^{\varepsilon,k}, \widetilde{B}^{-1}A_{2,1}\mathcal{D}_h U_{1,j}^{\varepsilon,k})_{\ell^2_h}, \tag{5.12}$$

where $\eta$ is a constant that will be chosen small enough in the computations.

*Remark* 5.1. Compared to Section 3, the corrector term can be simplified in this setting as we assume $A_{1,1} = 0$.

By Bernstein's inequality (Proposition 4.6), Young's inequality and using that $2^{-j} \leq \dfrac{\varepsilon}{\kappa}$, we obtain

$$\begin{aligned}2^{-2j}\mathcal{I}_j^{\varepsilon,k} &\leq \eta 2^{-2j}\|U_{2,j}^{\varepsilon,k}\|_{\ell^2_h}\|\widetilde{B}^{-1}A_{2,1}\mathcal{D}_h U_{1,j}^{\varepsilon,k}\|_{\ell^2_h} \\ &\leq \frac{\eta}{2}2^{-2j}\|U_{2,j}^{\varepsilon,k}\|^2_{\ell^2_h} + \frac{\eta}{2}2^{-2j}\|\widetilde{B}^{-1}A_{2,1}\mathcal{D}_h U_{1,j}^{\varepsilon,k}\|^2_{\ell^2_h} \\ &\lesssim \eta\frac{\varepsilon^2}{\kappa^2}\|U_{2,j}^{\varepsilon,k}\|^2_{\ell^2_h} + \eta\|\widetilde{B}^{-1}A_{2,1}U_{1,j}^{\varepsilon,k}\|^2_{\ell^2_h} \\ &\lesssim \eta\|(U_{1,j}^{\varepsilon,k}, \varepsilon U_{2,j}^{\varepsilon,k})\|^2_{\ell^2_h}.\end{aligned}$$

It is then clear that choosing $\eta$ sufficiently small we have

$$\mathcal{L}_j^{\varepsilon,k} \sim \|(U_{1,j}^{\varepsilon,k}, \varepsilon U_{2,j}^{\varepsilon,k})\|^2_{\ell^2_h}. \tag{5.13}$$

We now compute the discrete-time derivative of $\mathcal{L}_j^{\varepsilon,k}$. Concerning the first term, we take the scalar product of (5.2) with $(U_{1,j}^{\varepsilon,k+1}, U_{2,j}^{\varepsilon,k+1})$ and obtain

$$\delta_\tau\|(U_{1,j}^{\varepsilon,k}, \varepsilon U_{2,j}^{\varepsilon,k})\|^2_{\ell^2_h} + \frac{1}{\tau}\|[U_{1,j}^{\varepsilon,k+1} - U_{1,j}^{\varepsilon,k}, \varepsilon(U_{2,j}^{\varepsilon,k+1} - U_{2,j}^{\varepsilon,k})]\|^2_{\ell^2_h} + 2\lambda\|U_{2,j}^{\varepsilon,k+1}\|^2_{\ell^2_h} \leq 0. \tag{5.14}$$

Concerning the discrete-time derivative of $\mathcal{I}_j^{\varepsilon,k}$, we have the following lemma.

**Lemma 5.2** (Discrete time-derivative of $\mathcal{I}_j^{\varepsilon,k}$)**.** *One has*

$$2^{-2j}\delta_\tau \mathcal{I}_j^{\varepsilon,k} + \frac{\eta\lambda_0}{\varepsilon^2}\|U_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2 \lesssim \eta\|U_2^{\varepsilon,k+1}\|_{\ell_h^2}^2 + \frac{\eta}{\tau}\|[U_{1,j}^{\varepsilon,k+1} - U_{1,j}^{\varepsilon,k}, \varepsilon(U_{2,j}^{\varepsilon,k+1} - U_{2,j}^{\varepsilon,k})]\|_{\ell_h^2}^2. \tag{5.15}$$

*Proof of the lemma.* Analogously to the computations leading to (3.16), we obtain

$$\begin{aligned}\delta_\tau \mathcal{I}_j^{\varepsilon,k} + \frac{\eta}{\varepsilon^2}\left(A_{2,1}\mathcal{D}_h U_{1,j}^{\varepsilon,k+1}, \widetilde{B}^{-1}A_{2,1}\mathcal{D}_h U_{1,j}^{\varepsilon,k+1}\right)_{\ell_h^2} \leq & -\frac{\eta}{\varepsilon^2}\left(\widetilde{B}U_{2,j}^{\varepsilon,k+1}, \widetilde{B}^{-1}A_{2,1}\mathcal{D}_h U_{1,j}^{\varepsilon,k+1}\right)_{\ell_h^2} \\ & + \eta\left(\mathcal{D}_h U_{2,j}^{\varepsilon,k+1}, \widetilde{B}^{-1}A_{2,1}A_{1,2}\mathcal{D}_h U_{2,j}^{\varepsilon,k+1}\right)_{\ell_h^2} \\ & + \frac{\eta}{2\tau}\|B^{-1}A_{2,1}\mathcal{D}_h U_{1,j}^{\varepsilon,k+1} - B^{-1}A_{2,1}\mathcal{D}_h U_{1,j}^{\varepsilon,k}\|_{\ell_h^2}^2 \\ & + \frac{\eta}{2\tau}\|U_{2,j}^{\varepsilon,k+1} - U_{2,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2.\end{aligned} \tag{5.16}$$

Using that $A_{1,2}^T = A_{2,1}$ and Lemma A.1, it follows that

$$\begin{aligned}\eta\left(A_{2,1}\mathcal{D}_h U_{1,j}^{\varepsilon,k+1}, \widetilde{B}^{-1}A_{2,1}\mathcal{D}_h U_{1,j}^{\varepsilon,k+1}\right)_{\ell_h^2} &= \eta\left(\mathcal{D}_h U_{1,j}^{\varepsilon,k+1}, A_{1,2}\widetilde{B}^{-1}A_{2,1}\mathcal{D}_h U_{1,j}^{\varepsilon,k+1}\right)_{\ell_h^2} \\ &\geq \eta\lambda_0\|\mathcal{D}_h U_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2.\end{aligned} \tag{5.17}$$

Next, the Cauchy-Schwarz inequality implies that the first term on the right-hand side of (5.16) verifies

$$\begin{aligned}-\frac{\eta}{\varepsilon^2}\left(\widetilde{B}U_{2,j}^{\varepsilon,k+1}, \widetilde{B}^{-1}A_{2,1}\mathcal{D}_h U_{1,j}^{\varepsilon,k+1}\right)_{\ell_h^2} + \eta\left(\mathcal{D}_h U_{2,j}^{\varepsilon,k+1}, \widetilde{B}^{-1}A_{2,1}A_{1,2}\mathcal{D}_h U_{2,j}^{\varepsilon,k+1}\right)_{\ell_h^2} \\ \lesssim \frac{\eta}{\varepsilon^2}\|U_{2,j}^{\varepsilon,k+1}\|_{\ell_h^2}\|\mathcal{D}_h U_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2} + \eta\|\mathcal{D}_h U_{2,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2\end{aligned} \tag{5.18}$$

Using (5.17),(5.18) and Young's inequality in (5.16), we obtain

$$\begin{aligned}\delta_\tau \mathcal{I}_j^{\varepsilon,k} + \frac{\eta\lambda_0}{\varepsilon^2}\|\mathcal{D}_h U_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2 \lesssim & \frac{\eta}{\varepsilon^2 c_2}\|U_{2,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2 + \frac{\eta c_2}{\varepsilon^2}\|\mathcal{D}_h U_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2 + \eta\|\mathcal{D}_h U_{2,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2 \\ & + \frac{\eta}{\tau}\|\mathcal{D}_h U_{1,j}^{\varepsilon,k+1} - \mathcal{D}_h U_{1,j}^{\varepsilon,k}\|_{\ell_h^2}^2 \\ & + \frac{\eta}{\tau}\|U_{2,j}^{\varepsilon,k+1} - U_{2,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2,\end{aligned} \tag{5.19}$$

where the small positive constant $c_2$ is chosen in order for the term $\frac{\eta c_2}{\varepsilon^2}\|\mathcal{D}_h U_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2$ to get absorbed by $\frac{\eta\lambda_0}{\varepsilon^2}\|\mathcal{D}_h U_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2$.

Next, multiplying (5.19) by $2^{-2j}$ and using that $2^{-2j} \leq \frac{\varepsilon^2}{\kappa^2}$ together with the Bernstein estimate in Proposition 4.6, we obtain

$$\begin{aligned}2^{-2j}\delta_\tau \mathcal{I}_j^{\varepsilon,k} + \frac{\eta\lambda_0}{\varepsilon^2}\|U_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2 \lesssim & \frac{\eta}{\kappa^2}\|U_{2,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2 + \frac{\eta}{\varepsilon^2}\|U_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2 + \eta\|U_{2,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2 \\ & + \frac{\eta}{\tau\kappa^2}\|U_{1,j}^{\varepsilon,k+1} - U_{1,j}^{\varepsilon,k}\|_{\ell_h^2}^2 \\ & + \frac{\eta}{\tau}\frac{\varepsilon^2}{\kappa^2}\|U_{2,j}^{\varepsilon,k+1} - U_{2,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2\end{aligned} \tag{5.20}$$

which yields (5.15) and completes the proof of the lemma. □

Combining (5.14) and (5.15) and choosing $\eta$ sufficiently small, we obtain

$$\delta_\tau \mathcal{L}_j^{\varepsilon,k} \lesssim -\frac{1}{\varepsilon^2}\|(U_{1,j}^{\varepsilon,k+1}, \varepsilon U_{2,j}^{\varepsilon,k+1})\|_{\ell_h^2}^2 - \frac{1}{\tau}\|[U_{1,j}^{\varepsilon,k+1} - U_{1,j}^{\varepsilon,k}, \varepsilon(U_{2,j}^{\varepsilon,k+1} - U_{2,j}^{\varepsilon,k})]\|_{\ell_h^2}^2$$

Then, using (5.13) and $\tau \leq M\varepsilon^2$, we employ a similar reasoning as in the proof of Lemma A.3 to get

$$\|(U_{1,j}^{\varepsilon,K}, \varepsilon U_{2,j}^{\varepsilon,K})\|_{\ell_h^2} + \frac{1}{\varepsilon^2}\|(U_{1,j}^{\varepsilon}, \varepsilon U_{2,j}^{\varepsilon})\|_{\ell^1_{\tau,K}(\ell_h^2)} \lesssim \|(U_{1,j}^0, \varepsilon U_{2,j}^0)\|_{\ell_h^2}. \tag{5.21}$$

Multiplying (5.21) by $2^{js}$ and summing the resulting equation for $j \geq J_\varepsilon$, we obtain

$$\|(U_1^{\varepsilon,K}, \varepsilon U_2^{\varepsilon,K})\|_{\dot{B}_h^s}^H + \frac{1}{\varepsilon^2}\|(U_1^\varepsilon, \varepsilon U_2^\varepsilon)\|_{\ell^1_{\tau,K}(\dot{B}_h^s)}^H \lesssim \|(U_1^0, \varepsilon U_2^0)\|_{\dot{B}_h^s}^H. \tag{5.22}$$

Recalling that $W^{\varepsilon,k} = \widetilde{B}^{-1}A_{2,1}\mathcal{D}_h U_1^{\varepsilon,k} + U_2^{\varepsilon,k}$, thanks to Propositions 4.6 and 4.8 it is easy to see that

$$\begin{aligned}\|W^\varepsilon\|_{\ell^1_{\tau,K}(\dot{B}_h^{s-1})}^H &\lesssim \|U_1^\varepsilon\|_{\ell^1_{\tau,K}(\dot{B}_h^s)}^H + \|U_2^\varepsilon\|_{\ell^1_{\tau,K}(\dot{B}_h^{s-1})}^H \\ &\lesssim \|U_1^\varepsilon\|_{\ell^1_{\tau,K}(\dot{B}_h^s)}^H + \frac{\varepsilon}{\kappa}\|U_2^\varepsilon\|_{\ell^1_{\tau,K}(\dot{B}_h^s)}^H \\ &\lesssim \varepsilon^2\|(U_1^0, \varepsilon U_2^0)\|_{\dot{B}_h^s}^H.\end{aligned} \tag{5.23}$$

5.1. **Error estimates analysis.** We can now justify the relaxation estimate (2.14). Recall that $U_1$ is the solution of the fully discrete implicit scheme for the parabolic system

$$\delta_\tau U_1^k - A_{1,2}\widetilde{B}^{-1}A_{2,1}\mathcal{D}_h^2 U_1^{k+1} = 0 \tag{5.24}$$

with the same initial datum $U_1^0$. We define the error unknown $\bar{U}_{1,j}^{\varepsilon,k} := U_{1,j}^{\varepsilon,k} - U_{1,j}^k$, it satisfies

$$\delta_\tau \bar{U}_{1,j}^{\varepsilon,k} - A_{1,2}\widetilde{B}^{-1}A_{2,1}\mathcal{D}_h^2\bar{U}_{1,j}^{\varepsilon,k} = -\mathcal{D}_h W_j^{\varepsilon,k}. \tag{5.25}$$

Multiplying this equation with $\bar{U}_{1,j}^{\varepsilon,k+1}$ and using (3.6), together with the Cauchy-Schwarz inequality, we arrive at

$$\delta_\tau\|\bar{U}_{1,j}^{\varepsilon,k}\|_{\ell_h^2}^2 + \frac{1}{\tau}\|\bar{U}_{1,j}^{\varepsilon,k+1} - \bar{U}_{1,j}^{\varepsilon,k}\|_{\ell_h^2}^2 + 2\lambda_0 2^{2j}\|\bar{U}_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2}^2 \lesssim \|\mathcal{D}_h W_j^{\varepsilon,k+1}\|_{\ell_h^2}\|\bar{U}_{1,j}^{\varepsilon,k+1}\|_{\ell_h^2}, \ \forall j \in \mathbb{Z}. \tag{5.26}$$

Then, we seek to apply Lemma A.3, but, since it requires that $\tau \leq M2^{-2j}$, we need to treat again the low and high frequencies separately. If $j \leq J_\varepsilon$, then $2^{-2j} \geq \frac{\varepsilon^2}{\kappa^2}$, so the condition $\tau \leq M\varepsilon^2$ is enough to obtain, similarly to (5.7), that

$$\|\bar{U}_1^{\varepsilon,K}\|_{\dot{B}_h^{s-2}}^L + \|\bar{U}_1^\varepsilon\|_{\ell^1_{\tau,K}(\dot{B}_h^s)}^L \lesssim \|W^\varepsilon\|_{\ell^1_{\tau,K}(\dot{B}_h^{s-1})}^L, \tag{5.27}$$

where the term corresponding to $k = 0$ vanishes since $\bar{U}^0 = 0$. Then, using (5.10) we obtain

$$\|W^\varepsilon\|_{\ell^1_{\tau,K}(\dot{B}_h^{s-1})}^L \lesssim \varepsilon^2\left(\|U_1^0\|_{\dot{B}_h^s}^L + \|U_2^0\|_{\dot{B}_h^{s-1}}^L\right), \tag{5.28}$$

so, by (5.27),

$$\|\bar{U}_1^{\varepsilon,K}\|_{\dot{B}_h^{s-2}}^L + \|\bar{U}_1^{\varepsilon,K}\|_{\ell^1_{\tau,K}(\dot{B}_h^s)}^L \lesssim \varepsilon^2\left(\|U_1^0\|_{\dot{B}_h^s}^L + \|U_2^0\|_{\dot{B}_h^{s-1}}^L\right), \tag{5.29}$$

For the high-frequency regime $j > J_\varepsilon$, we show that both quantities $U_1^\varepsilon$ and $U_1$ vanish at the rate $\mathcal{O}(\varepsilon^2)$ in the appropriate Besov norms. First, by (5.21) we deduce

$$\|U_{1,j}^\varepsilon\|_{\ell^1_{\tau,K}(\ell_h^2)} \lesssim \varepsilon^2\|(U_{1,j}^0, \varepsilon U_{2,j}^{*,0})\|_{\ell_h^2} \quad \text{and} \quad \|U_{1,j}^{\varepsilon,K}\|_{\ell_h^2} \lesssim \|(U_{1,j}^0, \varepsilon U_{2,j}^0)\|_{\ell_h^2}.$$

Using that $2^{2j} \geq \frac{\kappa^2}{\varepsilon^2}$, we obtain from the last inequality that

$$\|U_{1,j}^{\varepsilon,K}\|_{\ell_h^2} \lesssim \varepsilon^2 2^{2j} \|(U_{1,j}^0, \varepsilon U_{2,j}^0)\|_{\ell_h^2}.$$

Therefore, we get the following estimate for $U_1^\varepsilon$ in Besov norms

$$\|U_1^{\varepsilon,K}\|_{\dot{B}_h^{s-2}}^H + \|U_1^\varepsilon\|_{\ell_{\tau,K}^1(\dot{B}_h^s)}^H \lesssim \varepsilon^2 \|(U_1^0, \varepsilon U_2^0)\|_{\dot{B}_h^s}^H. \tag{5.30}$$

Next, we derive a similar estimate for the solution $U_1$ of the discrete parabolic system (5.24). To that matter, using Lemma A.1, Parseval's equality and the fact that $j > J_\varepsilon$, we obtain

$$\|U_{1,j}^{k+1}\|_{\ell_h^2} \leq \|U_{1,j}^0\|_{\ell_h^2} \lesssim \varepsilon^2 2^{2j} \|U_{1,j}^0\|_{\ell_h^2},$$

which leads to

$$\|U_1^K\|_{\dot{B}_h^{s-2}}^H \lesssim \varepsilon^2 \|U_1^0\|_{\dot{B}_h^s}^H. \tag{5.31}$$

Next, applying the discrete Fourier transform to (5.24) and using the definition of $\delta_\tau$, we have, for every $\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]$,

$$\widehat{U}_{1,j}^{k+1}(\xi) = \left( I_{N_1} + \tau \left( \frac{\sin(\xi h)}{h} \right)^2 A_{1,2} \widetilde{B}^{-1} A_{2,1} \right)^{-1} \widehat{U}_{1,j}^k(\xi) \tag{5.32}$$

Then, thanks to Lemma A.1, $A_{1,2}\widetilde{B}^{-1}A_{2,1}$ is symmetric and positive definite and thus:

$$\left\| \left( I_{N_1} + \tau \left( \frac{\sin(\xi h)}{h} \right)^2 A_{1,2} \widetilde{B}^{-1} A_{2,1} \right)^{-1} \right\| \leq \frac{1}{1 + \lambda_0 \tau \left( \frac{\sin(\xi h)}{h} \right)^2} \tag{5.33}$$

Using the Definition 4.4 of the localization operator $\delta_h^j$, we have

$$|\widehat{U}_{1,j}^{k+1}(\xi)| \leq \frac{1}{1 + \alpha 2^{2j} \tau} |\widehat{U}_{1,j}^k(\xi)|,$$

where we denote $\alpha := \frac{9}{16}\lambda_0$. Further, by Parseval's equality and the fact that $j > J_\varepsilon$, we obtain

$$\|U_{1,j}^{k+1}\|_{\ell_h^2} \leq \left( \frac{1}{1 + \alpha \tau \frac{\kappa^2}{\varepsilon^2}} \right)^{k+1} \|U_{1,j}^0\|_{\ell_h^2}.$$

Thus, we have

$$\|U_{1,j}\|_{\ell_{\tau,K}^1(\ell_h^2)} \leq \|U_{1,j}^0\|_{\ell_h^2} \tau \sum_{k=0}^{K-1} \left( \frac{1}{1 + \alpha \tau \frac{\kappa^2}{\varepsilon^2}} \right)^{k+1} \leq \|U_{1,j}^0\|_{\ell_h^2} \frac{\tau}{1 - \frac{1}{1 + \alpha \tau \frac{\kappa^2}{\varepsilon^2}}} = \|U_{1,j}^0\|_{\ell_h^2} \varepsilon^2 \frac{1 + \alpha \tau \frac{\kappa^2}{\varepsilon^2}}{\alpha \kappa^2}.$$

Therefore, using that $\tau \leq M\varepsilon^2$, we obtain

$$\|U_{1,j}\|_{\ell_{\tau,K}^1(\ell_h^2)} \lesssim \varepsilon^2 \|U_{1,j}^0\|_{\ell_h^2}.$$

As a result

$$\|U_1\|_{\ell_{\tau,K}^1(\dot{B}_h^s)}^H \lesssim \varepsilon^2 \|U_1^0\|_{\dot{B}_h^s}^H. \tag{5.34}$$

Combining (5.31), (5.34) and (5.30) we obtain

$$\|\bar{U}_1^{\varepsilon,K}\|_{\dot{B}_h^{s-2}}^H + \|\bar{U}_1^\varepsilon\|_{\ell_{\tau,K}^1(\dot{B}_h^s)}^H \lesssim \varepsilon^2 \|(U_1^0, \varepsilon U_2^0)\|_{\dot{B}_h^s}^H \tag{5.35}$$

Summing (5.29) with (5.35) and using Theorem 2.4 concludes the proof of Theorem 2.6.

## 6. Proof of Theorem 2.4: Uniform Besov estimates with respect to the grid width

In this section, we prove Theorem 2.4 concerning uniform Besov estimates with respect to the grid width $h$, for regular enough functions.

*Proof of Theorem 2.4.* We recall that the discrete $\dot{B}_h^s$-norm of $\mathcal{T}_h v$, by definition (4.10), reads

$$\|\mathcal{T}_h v\|_{\dot{B}_h^s} = \sum_{j\in\mathbb{Z}} 2^{js} \|\delta_h^j \mathcal{T}_h v\|_{\ell_h^2}. \tag{6.1}$$

Taking into account that, by definition (2.7), the function $v$ and the bilateral sequence $\mathcal{T}_h u$ have essentially the same Fourier transform, we use Parseval's equality to write

$$\begin{aligned} \|\delta_h^j \mathcal{T}_h v\|_{\ell_h^2}^2 &= \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} (\hat{v}(\xi))^2 (\varphi_j(\xi))^2 \,\mathrm{d}\xi \\ &= \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} (\hat{v}(\xi))^2 (1+|\xi|^{2s'}) (\varphi_j(\xi))^2 \frac{1}{1+|\xi|^{2s'}} \,\mathrm{d}\xi. \end{aligned} \tag{6.2}$$

Now, since $\mathrm{supp}(\varphi_j) \subseteq F_h(j)$, it means by (4.5) that, if $\varphi_j(\xi) \neq 0$, then

$$\left|\frac{\sin(\xi h)}{\xi h}\right| |\xi| \geq \frac{3}{4} 2^j.$$

Since, $\left|\frac{\sin(x)}{x}\right| \leq 1$, for all $x \in [-\pi, \pi]$, we obtain

$$\varphi_j(\xi) \neq 0 \Rightarrow |\xi| \geq \frac{3}{4} 2^j.$$

This fact, together with (6.2) and $\varphi_j(\xi) \in [0,1]$, for all $\xi \in [-\pi/h, \pi, h]$, leads to

$$\|\delta_h^j \mathcal{T}_h v\|_{\ell_h^2}^2 \leq \frac{1}{1+\left(\frac{3}{4}\right)^{2s'} 2^{2js'}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} (\hat{v}(\xi))^2 (1+|\xi|^{2s'}) \,\mathrm{d}\xi. \tag{6.3}$$

Applying Parseval's equality again, we deduce that

$$\|\delta_h^j \mathcal{T}_h v\|_{\ell_h^2} \leq \frac{C_{s'}}{1+2^{js'}} \|v\|_{H^{s'}(\mathbb{R})}. \tag{6.4}$$

Inserting this inequality into (6.1), we get

$$\|\mathcal{T}_h v\|_{\dot{B}_h^s} \leq C_{s'} \|v\|_{H^{s'}(\mathbb{R})} \sum_{j\in\mathbb{Z}} \frac{2^{js}}{1+2^{js'}}. \tag{6.5}$$

We now claim that the hypotheses of Theorem 2.4 imply that the series above is convergent. Indeed, one has

$$\sum_{j\in\mathbb{Z}} \frac{2^{js}}{1+2^{js'}} = \sum_{j\leq 0} \frac{2^{js}}{1+2^{js'}} + \sum_{j>0} \frac{2^{js}}{1+2^{js'}} \lesssim \sum_{j\leq 0} 2^{js} + \sum_{j>0} 2^{j(s-s')}, \tag{6.6}$$

which converges provided that $s \in (0, s')$. □

## 7. Numerical simulations

In this section, we showcase a set of numerical experiments validating our theoretical findings. The simulations in Section 7.1, carried out using the `NumPy` and `Matplotlib` Python libraries Harris et al. (2020), Hunter (2007), confirm the sharpness of the polynomial decay verified by the solutions of the system (2.1) (as per Theorem 2.1). Furthermore, the experiments carried out in Section 7.2 show that the order of convergence $\mathcal{O}(\varepsilon^2)$ obtained in Theorem 2.6 (more specifically in Corollary 2.8) appears, in turn, to be sharp.

### 7.1. The numerical hypocoercivity property.

The plot depicted in Figure 3 validates the polynomial large-time decay estimate (2.3), for a particular instance of (2.1) – namely the linearization of the compressible Euler system (1.7) – exhibiting a decay rate of exactly $(1+t)^{-\frac{1}{2}}$. The initial data that we used in the simulation is obtained by a cut-off near infinity of the function:

$$\widetilde{\rho}^0(x) = \widetilde{u}^0(x) = \frac{1}{\sqrt[4]{x^2+10^{-6}}}. \tag{7.1}$$

In Figure 4, we provide the same simulation for the $3 \times 3$ system:

$$\begin{cases} \partial_t \rho^\varepsilon + a\partial_x u^\varepsilon + b\partial_x v^\varepsilon = 0, \\ \varepsilon^2 \partial_t u^\varepsilon + a\partial_x \rho^\varepsilon + u^\varepsilon = 0, \\ \varepsilon^2 \partial_t v^\varepsilon + b\partial_x \rho^\varepsilon + v^\varepsilon = 0, \end{cases} \tag{7.2}$$

with $a = 2, b = 3$ and all three initial data $\widetilde{\rho}^0$, $\widetilde{u}^0$, $\widetilde{v}^0$ given by (7.1), confirming the expected decay rate.

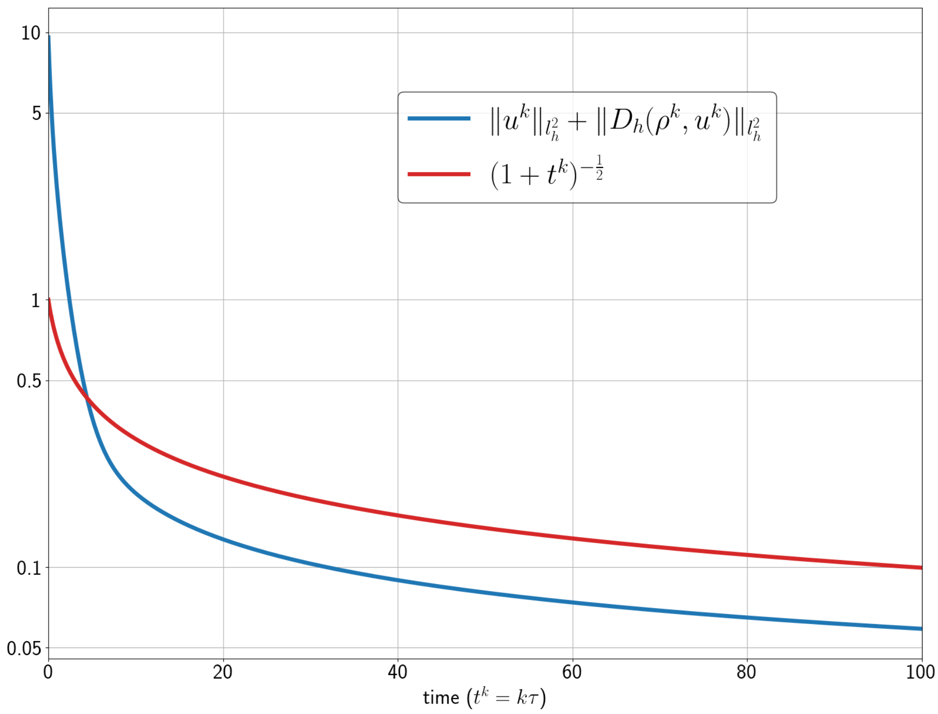


Figure 3. The semi-log plot of the large time behaviour of the solution of the Euler system (1.7) with parameters $\varepsilon = 1$, $h = 2^{-4}$ and $\tau = 2^{-5}$.

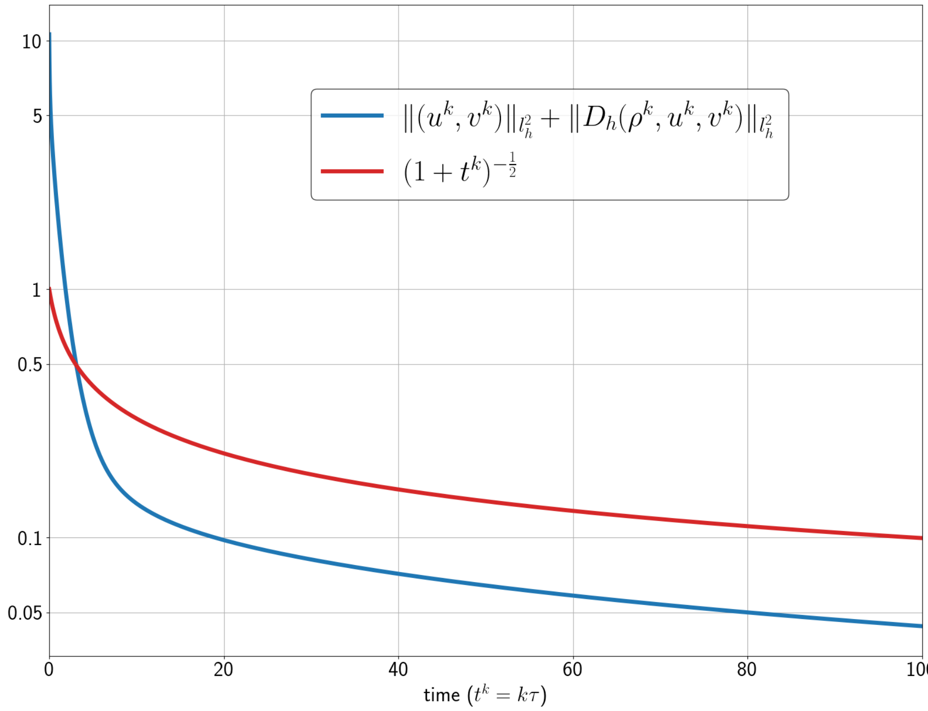


FIGURE 4. The semi-log plot of the large time behaviour of the solution of the system (7.2) with parameters $\varepsilon = 1$, $h = 2^{-4}$ and $\tau = 2^{-5}$.

7.2. **The relaxation limit – error estimates.** The objective of the overlapped plot in Figure 5 is to emphasize that the solutions of the system (1.7) effectively approximate the discrete heat equation $(1.8)_1$ for small $\varepsilon$.

The plot in Figure 6 serves as experimental evidence, indicating that for the initial data

$$\widetilde{\rho}^0(x) = e^{-\frac{1}{1-(x-1)^2}}\chi_{(0,2)}(x) \quad \text{and} \quad \widetilde{u}^0(x) = e^{-\frac{1}{1-(x-1.5)^2}}\chi_{(0.5,2.5)}(x), \tag{7.3}$$

the convergence rate of both the first and the third left-hand side term in (1.7) is exactly $\mathcal{O}(\varepsilon^2)$, thus suggesting the sharpness of the rate in Theorem 2.6. Moreover, the table in Figure 8 confirms that the relaxation is uniform with respect to the grid width $h$. Figure 7 contains the analoguous evidence for system (7.2), with the extra initial data given by:

$$\widetilde{v}^0(x) = e^{-\frac{1}{1-(x-0.5)^2}}\chi_{(-0.5,1.5)}(x).$$

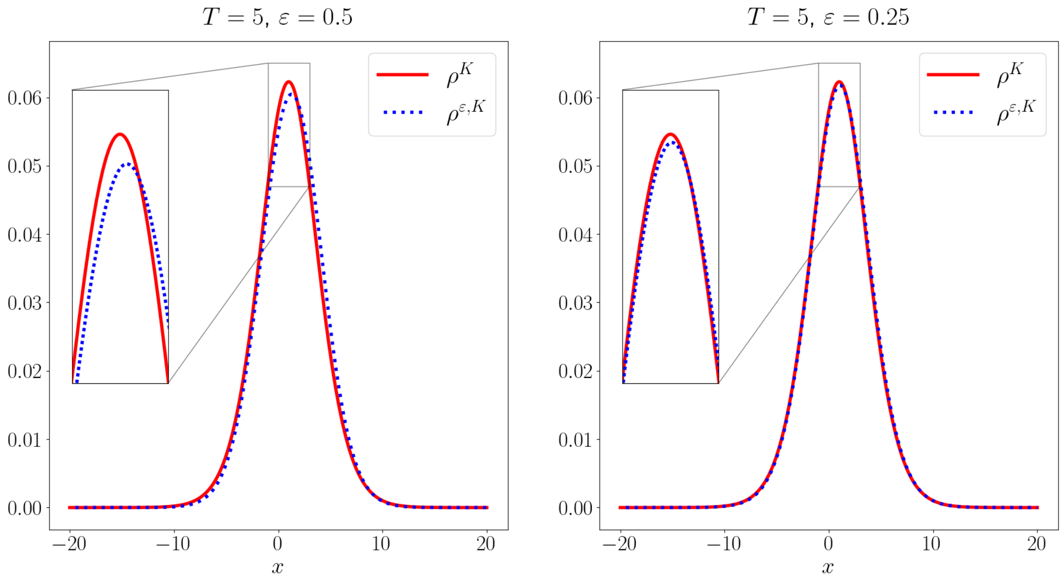


FIGURE 5. The first component $\rho^\varepsilon$ of the solution of (1.7) (blue) approximates the solution $\rho$ of the heat equation (1.8) (red) as $\varepsilon \to 0$. The plots were generated for $h = 2^{-4}$, $\tau = 12\varepsilon^2$ and $T = K\tau = 5$.

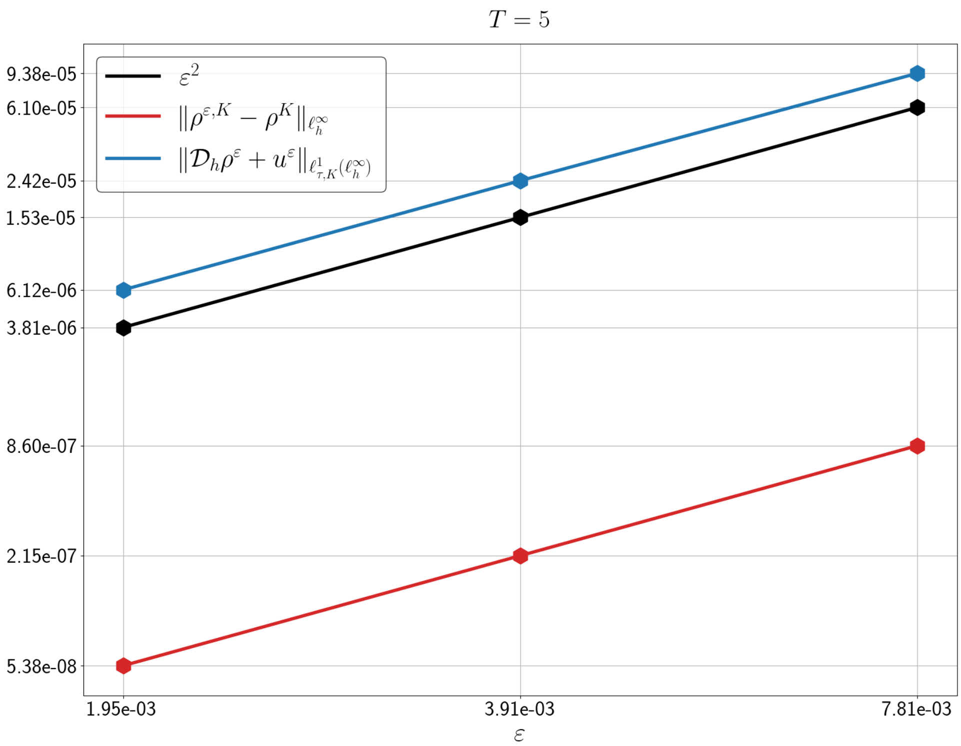


FIGURE 6. The log-log plot of the approximation error and Darcy law in $\ell_h^\infty$, obtained in Corollary 2.8 for the Euler system (1.7), as a function of $\varepsilon$, for fixed $h = 2^{-4}$ and $T = K\tau = 5$. The time discretization parameter is $\tau = 12\varepsilon^2$.

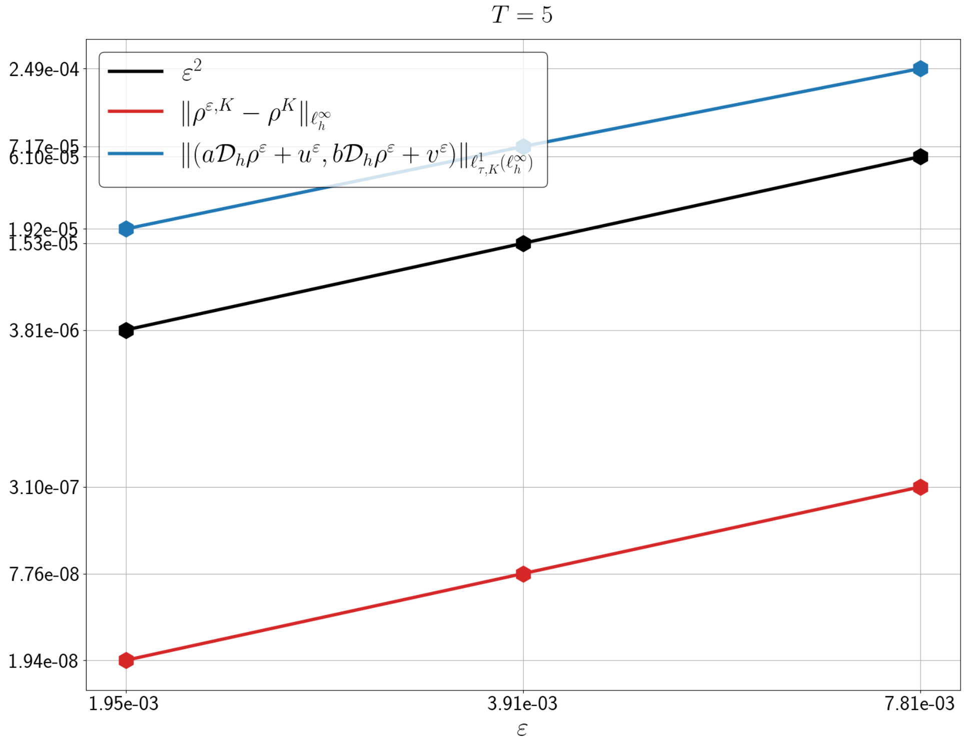


FIGURE 7. The log-log plot of the approximation error and Darcy law in $\ell_h^\infty$, obtained in Corollary 2.8 for the system (7.2), as a function of $\varepsilon$, for fixed $h = 2^{-4}$ and $T = K\tau = 5$. The time discretization parameter is $\tau = 12\varepsilon^2$.

| $h$ | $\|\rho^{\varepsilon,K} - \rho^K\|_{\ell_h^\infty}$ | $\|\mathcal{D}_h\rho^\varepsilon + u^\varepsilon\|_{\ell^2_{\tau,K}(\ell_h^\infty)}$ |
|---|---|---|
| $2^{-4}$ | 1.381531714e−05 | 1.176454042e−03 |
| $2^{-5}$ | 1.381330054e−05 | 1.183401971e−03 |
| $2^{-6}$ | 1.381294718e−05 | 1.185514414e−03 |

FIGURE 8. The approximation error and the Darcy law in $\ell_h^\infty$, obtained in Corollary 2.8 for the Euler system (1.7), in terms of $h$, for fixed $\varepsilon = 2^{-5}$, $\tau = 12\varepsilon^2$ and $T = K\tau = 5$.

## 8. Conclusion and extensions

Our theoretical and experimental evidence demonstrates that the decay estimates and relaxation properties inherent to partially dissipative hyperbolic systems can be effectively captured by one of the simplest and unconditionally stable numerical techniques: the implicit central finite difference scheme. We have thus introduced a novel approach for numerically approximating the solutions of a class of parabolic equations utilizing only first-order discrete operators. Furthermore, the new discrete Littlewood-Paley theory we propose may serve as a foundation for addressing other problems related to discrete equations, particularly those in which frequency decomposition techniques play a central role.

Dedicated to broadening the scope of Theorems 2.1 and 2.6, we discuss some additional research directions.

1. *More general finite difference operators.* The reasoning in the present paper can be applied for more general discrete first-order operators, namely to multi-point central finite difference operators. For instance, replacing $\mathcal{D}_h$ with the four-point central finite difference operator

$$(\widetilde{D}_h v)_n := \frac{-v_{n+2} + 8v_{n+1} - 8v_{n-1} + v_{n-2}}{12h},$$

in the implicit scheme (2.1), one obtains another unconditionally stable numerical scheme for the hyperbolic system (1.1). Moreover, since the Fourier symbol of $\widetilde{D}_h$ is the following:

$$\widehat{(\widetilde{D}_h v)}(\xi) = i\frac{\sin(\xi h)}{h}\frac{4-\cos(\xi h)}{3} = \widehat{(D_h v)}(\xi)\frac{4-\cos(\xi h)}{3},$$

0 and thus it is comparable to the Fourier symbol of the two-point finite difference operator $\mathcal{D}_h$, the results in this paper can be easily generalized to the multi-point scheme. More precisely, the required Bernstein estimate in Proposition 4.6 can be obtained immediately, since $\frac{4-\cos(\xi h)}{3} \in [1, \frac{5}{3}]$, for every $\xi$ and $h$.
2. *The Jin-Xin approximation.* For a conservation law:

$$\partial_t \rho + \partial_x f(\rho) = 0, \tag{8.1}$$

its diffusive Jin-Xin approximation reads:

$$\begin{cases} \partial_t \rho^\varepsilon + \partial_x u^\varepsilon = 0, \\ \varepsilon^2 \partial_t u^\varepsilon = -\partial_x \rho^\varepsilon + u^\varepsilon - f(\rho^\varepsilon). \end{cases} \tag{8.2}$$

This approximation was introduced in Jin & Xin (1995) and further examined through a frequency-decomposition approach in Crin-Barat & Shou (2023). It should be possible to justify the limit from the discrete approximation of (8.2) to the discrete counterpart of (8.1) as $\varepsilon$ approaches zeros using the discrete frequency framework established here. The challenge further involves formulating product laws to handle the nonlinearity $f(\rho^\varepsilon)$ which, in the simplest case, reads as $(\rho^\varepsilon)^2$.

## Appendix A. Various lemmata

A.1. **Proof of Lemma 3.3.** From (3.16), we have

$$\begin{aligned}\delta_\tau \mathcal{I}^k + \sum_{q=1}^{N-1} \varepsilon_q \|BA^q\mathcal{D}_h U^{k+1}\|^2_{\ell^2_h} \leq & -\sum_{q=1}^{N-1} \varepsilon_q \big(BA^{q-1}BU^{k+1}, BA^q\mathcal{D}_h U^{k+1}\big)_{\ell^2_h} \\ & -\sum_{q=1}^{N-1} \varepsilon_q \big(BA^{q-1}U^{k+1}, BA^qB\mathcal{D}_h U^{k+1}\big)_{\ell^2_h} \\ & -\sum_{q=1}^{N-1} \varepsilon_q \big(BA^{q-1}U^{k+1}, BA^{q+1}\mathcal{D}_h^2 U^{k+1}\big)_{\ell^2_h} \\ & + \frac{1}{2\tau}\|U^{k+1}-U^k\|^2_{\mathfrak{h}^1_h}\end{aligned} \tag{A.1}$$

To deal with the remainder terms, we proceed as in Beauchard & Zuazua (2011), Crin-Barat & Danchin (2022*a*), Danchin (2016) with some adaptations regarding the discrete setting. First, we fix a positive constant $\varepsilon_0$ and estimate the terms in the right-hand side of (3.16) as follows.

- The terms $\mathcal{I}_q^1 := \varepsilon_q\big(BA^{q-1}BU^{k+1}, BA^q\mathcal{D}_hU^{k+1}\big)_{\ell^2_h}$ with $q \in \{1,\cdots,N-1\}$: due to $BU^{k+1} = \widetilde{B}U_2^{k+1}$ and the fact that the matrices $A$, $B$ are bounded operators, we obtain

$$|\mathcal{I}_q^1| \leq C\varepsilon_q \|\widetilde{B}U_2^{k+1}\|_{\ell^2_h}\|BA^q\mathcal{D}_hU^{k+1}\|_{\ell^2_h} \leq \frac{\varepsilon_0}{4N}\|U_2^{k+1}\|^2_{\ell^2_h} + \frac{C\varepsilon_q^2}{\varepsilon_0}\|BA^q\mathcal{D}_hU^{k+1}\|^2_{\ell^2_h}.$$

- The term $\mathcal{I}_1^2 := \varepsilon_1\big(BU^{k+1}, BAB\mathcal{D}_hU^{k+1}\big)_{\ell^2_h}$: one has

$$|\mathcal{I}_1^2| \leq C\varepsilon_1 \|\widetilde{B}U_2^{k+1}\|_{\ell^2_h}\|\widetilde{B}\mathcal{D}_hU_2^{k+1}\|_{\ell^2_h} \leq \frac{\varepsilon_0}{4N}\|U_2^{k+1}\|^2_{\ell^2_h} + \frac{C\varepsilon_1^2}{\varepsilon_0}\|\mathcal{D}_hU_2(t)\|^2_{\ell^2_h}.$$

- The terms $\mathcal{I}_q^2 := \varepsilon_q\big(BA^{q-1}U^{k+1}, BA^qB\mathcal{D}_hU^{k+1}\big)_{\ell^2_h}$ with $q \in \{2,\cdots,N-1\}$ if $N \geq 3$: we deduce, after integrating by parts, that

$$\begin{aligned}|\mathcal{I}_q^2| &= \varepsilon_q\big|\big(BA^{q-1}\mathcal{D}_hU^{k+1}, BA^qBU^{k+1}\big)_{\ell^2_h}\big| \leq C\varepsilon_q\|BA^{q-1}\mathcal{D}_hU^{k+1}\|_{\ell^2_h}\|BU^{k+1}\|_{\ell^2_h} \\ &\leq \frac{\varepsilon_0}{4N}\|U_2^{k+1}\|^2_{\ell^2_h} + \frac{C\varepsilon_q^2}{\varepsilon_0}\|BA^{q-1}\mathcal{D}_hU^{k+1}\|^2_{\ell^2_h}.\end{aligned}$$

- The terms $\mathcal{I}_q^3 := \varepsilon_q\big(BA^{q-1}U^{k+1}, BA^{q+1}\mathcal{D}_h^2U^{k+1}\big)_{\ell^2_h}$ with $q \in \{1,\cdots,N-2\}$ if $N \geq 3$: a similar argument yields

$$|\mathcal{I}_q^3| = \varepsilon_q\big|\big(BA^{q-1}\mathcal{D}_hU^{k+1}, BA^{q+1}\mathcal{D}_hU^{k+1}\big)_{\ell^2_h}\big| \leq \frac{\varepsilon_{q-1}}{8}\|BA^{q-1}\mathcal{D}_hU^{k+1}\|^2_{\ell^2_h} + \frac{C\varepsilon_q^2}{\varepsilon_{q-1}}\|BA^{q+1}\mathcal{D}_hU^{k+1}\|^2_{\ell^2_h}.$$

- The term $\mathcal{I}_{N-1}^3 := \varepsilon_{N-1}\big(BA^{N-2}U^{k+1}, BA^N\mathcal{D}_h^2U^{k+1}\big)_{\ell^2_h}$: owing to the Cayley-Hamilton theorem, there exist coefficients $c_*^q$ $(q = \overline{0, N-1})$ such that

$$A^N = \sum_{q=0}^{N-1} c_*^q A^q. \tag{A.2}$$

Consequently, one gets

$$
\begin{aligned}
|\mathcal{I}^3_{N-1}| &\le \varepsilon_{N-1} \sum_{q=0}^{N-1} c_*^q \|BA^{N-2}\mathcal{D}_h U^{k+1}\|_{\ell^2_h} \|BA^q \mathcal{D}_h U^{k+1}\|_{\ell^2_h} \\
&\le \frac{\varepsilon_{N-2}}{8} \|BA^{N-2}\mathcal{D}_h U^{k+1}\|^2_{\ell^2_h} + \sum_{q=1}^{N-1} \frac{C\varepsilon^2_{N-1}}{\varepsilon_{N-2}} \|BA^q \mathcal{D}_h U^{k+1}\|^2_{\ell^2_h} + \frac{C\varepsilon^2_{N-1}}{\varepsilon_{N-2}} \|\mathcal{D}_h U_2^{k+1}\|^2_{\ell^2_h}.
\end{aligned}
$$

In order to absorb the right-hand side terms of $\mathcal{I}^1_q$ and $\mathcal{I}^2_q$ by the left-hand side of (3.16), we take the constant $\varepsilon_q$ small enough so that

$$
C\varepsilon_1^2 \le \frac{\varepsilon_0^2}{8}, \qquad C\varepsilon_q^2 \le \frac{\varepsilon_q \varepsilon_0}{8}, \qquad q = \overline{1, N-1}. \tag{A.3}
$$

To handle the above estimates of $\mathcal{I}^3_q$ with $q = \overline{1, N-2}$, one may let

$$
C\varepsilon_q^2 \le \frac{1}{8}\varepsilon_{q-1}\varepsilon_{q+1}, \qquad q = \overline{1, N-2} \quad \text{if } N \ge 3. \tag{A.4}
$$

In addition, to handle the term $\mathcal{I}^3_{N-1}$, we assume

$$
C\varepsilon^2_{N-1} \le \frac{1}{8}\varepsilon_q \varepsilon_{N-2}, \qquad q = \overline{0, N-1}. \tag{A.5}
$$

Clearly, the inequality (3.17) holds if we find $\varepsilon_1, \cdots, \varepsilon_{N-1}$ fulfilling (A.3) – (A.5). As in Beauchard & Zuazua (2011), one can take $\varepsilon_q = \tilde{\varepsilon}^{m_q}$ with some suitably small constant $\tilde{\varepsilon} \le \varepsilon_0$ and $m_1, \cdots, m_{N-1}$ satisfying for some $\delta > 0$ (that can be taken arbitrarily small):

$$
m_q > 1, \quad m_q \ge \frac{m_{q-1} + m_{q+1}}{2} + \delta \quad \text{and} \quad m_{N-1} \ge \frac{m_q + m_{N-2}}{2} + \delta, \quad q = \overline{1, N-2}.
$$

This concludes the proof of Lemma 3.3. □

Next, we state the equivalence between Kalman rank condition and the strong ellipticity condition for System (1.9) that is proven in Danchin (2023).

**Lemma A.1.** *(Danchin 2023, Lemma A.3) Assume that $A$ and $B$ are symmetric $N \times N$ matrices such that the $N_1 \times N_1$ block of $A$ satisfies $A_{1,1} = 0$ and that (1.2) and (1.3) hold.*

*Then, if $(A, B)$ satisfies the Kalman rank condition (K) (which we recall that, by Beauchard & Zuazua (2011), is equivalent to the SK condition), the matrix $A_{1,2}\widetilde{B}^{-1}A_{2,1}$ is symmetric and positive definite.*

*In particular, this implies the existence of a positive constant $\lambda_0$ such that, for every vector $X \in \mathbb{R}^{N_1}$,*

$$
\left(A_{1,2}\widetilde{B}^{-1}A_{2,1}X, X\right)_{\mathbb{R}^{N_1}} \ge \lambda_0 |X|^2.
$$

A.2. **Technical lemmata.** The first technical lemma provides an estimate for an integral involving exponential and square root functions:

**Lemma A.2.** *Let $\lambda$ be a positive constant. There exists a constant $C = C(\lambda) > 0$ such that, for every $t \in (0, \infty)$,*

$$
I(t) := \int_0^t e^{-\lambda(t-\sigma)}(1+\sigma)^{-\frac{1}{2}} \, \mathrm{d}\sigma \le C(1+t)^{-\frac{1}{2}}.
$$

*Proof.* An integration-by-parts argument leads to

$$I(t) = \frac{1}{\lambda}(1+t)^{-\frac{1}{2}} - \frac{1}{\lambda}e^{-\lambda t} + \frac{1}{2\lambda}\int_0^t e^{-\lambda(t-\sigma)}\frac{1}{(1+\sigma)^{\frac{3}{2}}}\,\mathrm{d}\sigma.$$

The conclusion follows since the last term in the previous inequality can be bounded from above by

$$\int_0^{\frac{t}{2}} e^{-\lambda(t-\sigma)}\,\mathrm{d}\sigma + \int_{\frac{t}{2}}^t \frac{1}{(1+\sigma)^{\frac{3}{2}}}\,\mathrm{d}\sigma.$$

□

Next, we state a time-discrete version of (Crin-Barat & Danchin 2022*a*, Lemma A.1).

**Lemma A.3.** *Let $M>0$, $K\in\mathbb{N}^*$ and $(v^k)_{k=0}^K$ a sequence such that $v^k\in\ell_h^2$, $\forall k$. Let $\tau>0$ and assume that there exist two constants constants $c>0$ and $\beta\in(0,\frac{Mc}{\tau}]$ and a sequence $(\alpha^k)_{k=1}^K$ of non-negative numbers such that, for every $k\in\overline{1,K-1}$,*

$$\delta_\tau\|v^k\|_{\ell_h^2}^2 + \frac{c}{\tau}\|v^{k+1}-v^k\|_{\ell_h^2}^2 + \beta\|v^{k+1}\|_{\ell_h^2}^2 \le \alpha^{k+1}\|v^{k+1}\|_{\ell_h^2}. \tag{A.6}$$

*Then,*

$$\|v^K\|_{\ell_h^2} + \frac{\beta}{\sqrt{M+4}}\tau\sum_{k=0}^{K-1}\|v^{k+1}\|_{\ell_h^2} \le \|v^0\|_{\ell_h^2} + \tau\sum_{k=0}^{K-1}\alpha^{k+1}. \tag{A.7}$$

*Proof.* If $\|v^{k+1}\|_{\ell_h^2}$ and $\|v^k\|_{\ell_h^2}$ are not both null, we have the following identity:

$$\delta_\tau\|v^k\|_{\ell_h^2} = \frac{\delta_\tau\|v^k\|_{\ell_h^2}^2}{\|v^{k+1}\|_{\ell_h^2}+\|v^k\|_{\ell_h^2}},$$

which, by (A.6), leads to the following inequality:

$$\begin{aligned}\delta_\tau\|v^k\|_{\ell_h^2} + \frac{\beta}{\sqrt{M+4}}\|v^{k+1}\|_{\ell_h^2} &\le \frac{\alpha^{k+1}\|v^{k+1}\|_{\ell_h^2}}{\|v^{k+1}\|_{\ell_h^2}+\|v^k\|_{\ell_h^2}} - \frac{\beta\|v^{k+1}\|_{\ell_h^2}^2 + \frac{c}{\tau}\|v^{k+1}-v^k\|_{\ell_h^2}^2}{\|v^{k+1}\|_{\ell_h^2}+\|v^k\|_{\ell_h^2}}\\ &\quad + \frac{\beta}{\sqrt{M+4}}\|v^{k+1}\|_{\ell_h^2}\\ &\le \alpha^{k+1} - \frac{\left(\beta-\frac{\beta}{\sqrt{M+4}}\right)\|v^{k+1}\|_{\ell_h^2}^2 - \frac{\beta}{\sqrt{M+4}}\|v^{k+1}\|_{\ell_h^2}\|v^k\|_{\ell_h^2} + \frac{c}{\tau}\|v^{k+1}-v^k\|_{\ell_h^2}^2}{\|v^{k+1}\|_{\ell_h^2}+\|v^k\|_{\ell_h^2}}\end{aligned} \tag{A.8}$$

Next, we claim that the numerator above is non-negative, namely

$$\left(\beta-\frac{\beta}{\sqrt{M+4}}\right)\|v^{k+1}\|_{\ell_h^2}^2 + \frac{c}{\tau}\|v^{k+1}-v^k\|_{\ell_h^2}^2 \ge \frac{\beta}{\sqrt{M+4}}\|v^{k+1}\|_{\ell_h^2}\|v^k\|_{\ell_h^2}. \tag{A.9}$$

Indeed, since, by hypothesis $\frac{c}{\tau}\ge\frac{\beta}{M}$, it follows that

$$\begin{aligned}\frac{c}{\tau}\|v^{k+1}-v^k\|_{\ell_h^2}^2 &\ge \frac{\beta}{M}\left[\|v^{k+1}\|_{\ell_h^2}^2 + \|v^k\|_{\ell_h^2}^2 - 2(v^{k+1},v^k)_{\ell_h^2}\right]\\ &\ge \frac{\beta}{M}\left[\|v^{k+1}\|_{\ell_h^2}^2 + \|v^k\|_{\ell_h^2}^2 - 2\|v^{k+1}\|_{\ell_h^2}\|v^k\|_{\ell_h^2}\right].\end{aligned}$$

As a result, to prove (A.9), it is sufficient to show that

$$\left(\beta+\frac{\beta}{M}-\frac{\beta}{\sqrt{M+4}}\right)\|v^{k+1}\|_{\ell_h^2}^2 + \frac{\beta}{M}\|v^k\|_{\ell_h^2}^2 \ge \left(\frac{2\beta}{M}+\frac{\beta}{\sqrt{M+4}}\right)\|v^{k+1}\|_{\ell_h^2}\|v^k\|_{\ell_h^2},$$

which is true by the AM-GM inequality. As a result, (A.8) implies that

$$\delta_\tau \|v^k\|_{\ell^2_h} + \frac{\beta}{\sqrt{M+4}} \|v^{k+1}\|_{\ell^2_h} \leq \alpha^{k+1}, \tag{A.10}$$

which is still true in the case when $\|v^{k+1}\|_{\ell^2_h}$ and $\|v^k\|_{\ell^2_h}$ both vanish. Summing (A.10) for $k = \overline{0, K-1}$, we obtain the conclusion of the lemma. □

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