GLOBAL EXISTENCE FOR PARTIALLY DISSIPATIVE HYPERBOLIC SYSTEMS IN THE L^p FRAMEWORK, AND RELAXATION LIMIT

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ABSTRACT. Here we investigate global strong solutions for a class of partially dissipative hyperbolic systems in the framework of critical homogeneous Besov spaces. Our primary goal is to extend the analysis of our previous paper [10] to a functional framework where the low frequencies of the solution are only bounded in L^p-type spaces with p larger than 2. This enables us to prescribe weaker smallness conditions for global well-posedness and to get a more accurate information on the qualitative properties of the constructed solutions. Our existence theorem in particular applies to the multi-dimensional isentropic compressible Euler system with relaxation, and provide us with bounds that are *independent* of the relaxation parameter. As a consequence, we justify rigorously the relaxation limit to the porous media equation and exhibit explicit rates of convergence for suitable norms, a completely new result to the best of our knowledge.

INTRODUCTION

We are concerned with multi-dimensional first order n-component systems in $\mathbb{R}^d$ for $d \geq 1$ of the type:

$$\frac{\partial V}{\partial t} + \sum_{k=1}^{d} A^k(V) \frac{\partial V}{\partial x_k} + \frac{LV}{\varepsilon} = 0 \tag{1}$$

where ε stands for a positive relaxation parameter and the unknown $V = V(t,x) \in \mathbb{R}^n$ depends on the time variable $t \in \mathbb{R}_+$ and on the space variable $x \in \mathbb{R}^d$. The symmetric matrices valued maps A^k $(k = 1, \cdots, d)$ are assumed to be linear. We suppose that *partial* dissipation occurs, that is to say, the $n \times n$ matrix L is such that $L + {}^T L$ is nonnegative, but not necessarily definite positive (in other words, the term LV concerns only on a part of the solution). In order to achieve global-in-time results, we shall suppose that the so-called Shizuta-Kawashima condition is satisfied (see details in the next section) so that dissipation acts – indirectly – on all the components of the solution.

We supplement System (1) with an initial data V_0 at time $t = 0$ and are concerned with the existence of global strong solutions in the case where V_0 is close to some constant state $\bar{V}$ such that $L\bar{V} = 0$, and to the relaxation limit $\varepsilon \to 0$.

Although our structure assumptions on (1) will be a bit more restrictive than in [10], our approach applies to the following compressible Euler system with relaxation:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla P + \dfrac{1}{\varepsilon} \rho v = 0, \end{cases} \tag{2}$$

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with $\varepsilon > 0$ and for a pressure law P satisfying

(3)
$$P(\rho) = A\rho^\gamma \quad \text{for } \gamma > 1 \quad \text{and} \quad A > 0.$$

Above, $v = v(t, x) \in \mathbb{R}^d$ designates the velocity field and $\rho = \rho(t, x) > 0$ the density of the fluid.

It is well known that classical systems of conservation laws (that is with $LV = 0$) supplemented with smooth data admit local-in-time strong solutions that may develop singularities (shock waves) in finite time even if the initial data are small perturbations of a constant solution (see for instance the works by Majda in [21] and Serre in [25]). A sufficient condition of global existence for small perturbations of a constant solution $\bar{V}$ of (1) is the *total dissipation hypothesis*, namely the dissipation term LV acts directly on each component of the system (that is $L + {}^T L$ is positive definite), making the whole solution to tend to $\bar{V}$ exponentially fast, cf. the work of Li in [17].

However, in most physical situations that may be modelled by systems of the form (1), some components of the solution satisfy conservation laws and only *partial* dissipation occurs, that is to say, the term LV acts only on a part of the solution. Typically, this happens in gas dynamics where the mass density and entropy are conserved, or in numerical schemes involving conservation laws with relaxation.

In his 1984 PhD thesis [15], Kawashima found out a sufficient structure condition on a class of hyperbolic-parabolic systems containing (1) guaranteeing the global existence of strong solutions for perturbations of a constant state $\bar{V}$ (see also [26]). This criterion, now called the (SK) condition, has been revisited later by Yong in [32] who noticed that the existence of a (dissipative) entropy which provides a suitable symmetrization of the system compatible with 0-th order term allows to get a global existence result for small data in H^s spaces with $s > \frac{d}{2} + 1$. Later, by taking advantage of the properties of the Green kernel of the linearized system around $\bar{V}$ and on the Duhamel formula, Bianchini, Hanouzet and Natalini in [3] pointed out the convergence of global solutions to $\bar{V}$ in L^p, with the rate $\mathcal{O}(t^{-\frac{d}{2}(1-\frac{1}{p})})$ when $t \to \infty$, for all $p \in [\min\{d, 2\}, \infty]$. In [16], Kawashima and Yong proved decay estimates in Sobolev spaces and, a few years ago, Kawashima and Xu in [30] obtained a global existence result for small data in critical non-homogeneous Besov spaces.

In [2], Beauchard and Zuazua developed a new and systematic approach that allows to establish global existence and to describe large time behavior of solutions to partially dissipative systems. Looking at the linearization of System (1) around a constant solution $\bar{V}$, namely (denoting from now on $\partial_t \triangleq \frac{\partial}{\partial t}$ and $\partial_k \triangleq \frac{\partial}{\partial x_k}$),

(4)
$$\partial_t Z + \sum_{k=1}^m \bar{A}^k \partial_k Z = -\frac{LZ}{\varepsilon} \quad \text{with} \quad \bar{A}^k \triangleq A^k(\bar{V}),$$

they showed that the (SK) condition is equivalent to the Kalman maximal rank condition on the matrices $\bar{A}^k$ and L. More importantly, they introduced a Lyapunov functional equivalent to the L^2 norm that encodes enough information to recover all the dissipative properties of (4). Considering such a functional is motivated by the classical (linear) control theory of ODEs, and has some connections with Villani's work on hypocoercivity [27]. Back to the nonlinear system (1), Beauchard and Zuazua obtained the existence of global smooth solutions for perturbations in H^s with $s > d/2 + 1$ of a constant equilibrium $\bar{V}$ that satisfies Condition (SK). Furthermore, using arguments borrowed from Coron's return method [8], they were able to achieve certain cases where (SK) does not hold.

Using the techniques from [2] and a deep analysis of the low frequencies of the solution, in [11, 10], we investigated the global existence issue for these systems in critical homogeneous Besov spaces first in the one-dimensional, and then in the multi-dimensional setting.

Our aim here is to extend these results to a more general functional framework where the low frequencies of the solution are only bounded in some L^p-type space with $p > 2$. In this way, one can take initial data that are less decaying at infinity than those that have been considered so far. To compare with the prior works, one can keep in mind the following chain of embedding that holds true for all $s > 1 + d/2$ and $p \geq 2$:

$$H^s \hookrightarrow \mathbb{B}_{2,1}^{\frac{d}{2}+1} \hookrightarrow \dot{\mathbb{B}}_{2,1}^{\frac{d}{2}} \cap \dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1} \hookrightarrow \dot{\mathbb{B}}_{p,2,1}^{\frac{d}{p},\frac{d}{2}+1} \hookrightarrow \mathcal{C}_b^1.$$

The left space corresponds to the classical theory, generalized to the nonhomogeneous Besov space $\mathbb{B}_{2,1}^{\frac{d}{2}+1}$ by Kawashima and Xu in [30]. The theory in $\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}} \cap \dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}$ has been performed by us in [10] while $\dot{\mathbb{B}}_{p,2,1}^{\frac{d}{p},\frac{d}{2}+1}$ – the aim of the present paper – amounts to assuming that the low (resp. high) frequencies of the data are in $\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}$ (resp. $\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}$).

Looking for global well-posedness results with low and high frequencies of the data belonging to different type of Besov spaces originates from the study of the compressible Navier-Stokes system in [5, 7]. There, the authors proved a global well-posedness result with the high frequencies of the solution belonging to some homogeneous Besov space based on L^p with $p \geq 2$ while the low frequencies are in a space related to L^2. Essentially, this is possible because the high frequencies' eigenvalues of the linearized system (in the Fourier variables) are real-valued. In our case, the situation is reversed: the low frequencies' eigenvalues are real and can thus be handled in a L^p-type framework, while the high frequencies are complex-valued.

To achieve our results, we need to reconsider the way of treating the low frequencies compared to [10]. Similarly to what we did in [11], we shall exhibit a damped mode with better decay properties than the whole solution and use it to diagonalize and decouple the system into a purely damped equation and a parabolic one. This, as well as a link between the (SK) condition and the ellipticity of a certain operator will be the keys to obtaining a priori estimates in the L^p framework in low frequencies. To handle the high frequencies, we shall work out a Lyapunov functional in Beauchard-Zuazua's style, that is equivalent to the norm that we aim at controlling. We shall proceed as in [10] but will take advantage of new commutator and composition lemma to handle the low frequencies part of the nonlinearities that do not belong to L^2. This leads to a global well-posedness result for (1) (Theorem 1.1).

The advantage of using homogeneous norms is that it allows us to keep track of the dependency of the estimates of the solution with respect to ε by a mere rescaling, and to get uniform estimates in the asymptotics $\varepsilon \to 0$. This is a crucial step for investigating the infinite relaxation limit for general systems of type (1). As an application, we shall consider the isentropic compressible Euler with damping, and justify its relaxation limit toward the porous media equation after performing the following so-called diffusive rescaling:

(5) $$(\widetilde{\rho^\varepsilon}, \widetilde{v^\varepsilon})(t, x) \triangleq (\rho, \varepsilon^{-1}v)(\varepsilon^{-1}t, x).$$

Then, System (2) becomes

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}\,(\rho^\varepsilon v^\varepsilon) = 0, \\ \varepsilon^2 \partial_t(\rho^\varepsilon v^\varepsilon) + \varepsilon^2 \mathrm{div}(\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon) + \nabla P(\rho^\varepsilon) + \rho^\varepsilon v^\varepsilon = 0. \end{cases}$$

As $\varepsilon \to 0$, we expect ρ^ε and u^ε to tend to the solution of the following porous media equations:

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0, \\ \mathcal{N}v + \nabla P(\mathcal{N}) = 0, \end{cases}$$

where the second equation corresponds to Darcy's law.

The rigorous derivation of the relaxation limit for systems of conservation laws can be tracked back to the work of Chen, Levermore and Liu in [6]. For the damped Euler equations, it was performed in [22, 9, 23, 13, 14, 31, 20] in various settings.

In [14], Junca and Rascle were able to justify the relaxation process from the damped Euler equations to the porous media equation in the one-dimensional setting for large global-in-time BV solution and to provide an explicit convergence rate. Their approach was based on a stream function technique which is related to the Lagrangian mass coordinates. More recently, following the same approach as in [14], Peng et al. in [18] justified the convergence of partially dissipative hyperbolic systems to parabolic systems globally-in-time in one space dimension and derived a convergence rate of the relaxation process. Using similar techniques, Liang and Shuai in [19] generalized the previous result to the multi-dimensional periodic setting (in the torus $\mathbb{T}^3$). As their result relies on the Poincaré inequality to control mixed partial derivative terms that do not appear in the one dimensional case, their method cannot handle global-in-space norms.

In the context of smooth solutions, Coulombel, Goudon and Lin in [20, 9] proved the strong convergence locally-in-space in Sobolev spaces with regularity index $s > \frac{d}{2} + 1$. These works were extended by Xu, Kawashima and Wang in [31, 29] to the setting of non-homogeneous critical Besov spaces.

In the present paper, our focus is on critical solutions with regularity $\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}$ in low frequencies and $\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}$ in high frequencies. For the compressible Euler system (2), we shall justify the *strong* convergence to the porous media equation with *an explicit convergence rate*. This will be possible thanks to the uniform bounds from Theorem 1.1 on the damped mode, a suitable threshold between the low and high frequencies depending on ε and the diffusive rescaling (5). This result is given in Theorem 1.3.

The rest paper is arranged as follows. In the first section, we specify the structure of the class of partially dissipative hyperbolic systems we aim at considering and state our main results. In the next section, we give some insight on the strategies of the proofs. Section 3 is devoted to the proof of our global existence result for a class of partially dissipative systems satisfying the Shizuta-Kawashima condition. In section 4, we justify rigorously the relaxation limit of the compressible Euler system with damping to the porous media equation and derive an explicit convergence rate of the process. Some technical results are proved or recalled in Appendix.

1. Hypotheses and results

Before presenting our main results, motivating the structure assumptions that will be made on System (1) is in order. To this end, fix some constant reference solution $\bar{V}$ (hence satisfying $L\bar{V} = 0$). Setting $Z \triangleq V - \bar{V}$, System (1) becomes

$$(6) \qquad \partial_t Z + \sum_{k=1}^{d} A^k(V) \partial_k Z + \frac{LZ}{\varepsilon} = 0. \cdot$$

We assume that:

(i) The range and kernel of operator L satisfy

$$\mathbb{R}^n = \mathrm{Ker}(L) \overset{\perp}{\oplus} \mathrm{Im}(L).$$

In what follows, we shall assume with no loss of generality that $\ker L = \mathbb{R}^{n_1} \times \{0\}$ and $\mathrm{Im}(L) = \{0\} \times \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$.

(ii) The restriction L_2 of L to $\mathrm{Im}(L)$ (identified to $\mathbb{R}^{n_2}$) satisfies, for some $c > 0$, the positivity condition:

$$(7) \qquad \forall V_2 \in \mathbb{R}^{n_2}, \ (L_2 V_2 | V_2) \geq c|V_2|^2.$$

Note that, using the decompositions,

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \qquad LV = \begin{pmatrix} 0 \\ L_2 V_2 \end{pmatrix} \quad \text{and} \quad A^k = \begin{pmatrix} A^k_{1,1} & A^k_{1,2} \\ A^k_{2,1} & A^k_{2,2} \end{pmatrix},$$

System (6) can be rewritten:

$$(8) \qquad \begin{cases} \partial_t Z_1 + \displaystyle\sum_{k=1}^d \left(A^k_{1,1}(V) \partial_k Z_1 + A^k_{1,2}(V) \partial_k Z_2 \right) = 0, \\ \partial_t Z_2 + \displaystyle\sum_{k=1}^d \left(A^k_{2,1}(V) \partial_k Z_1 + A^k_{2,2}(V) \partial_k Z_2 \right) + \dfrac{L_2 Z_2}{\varepsilon} = 0. \end{cases}$$

Recall that we assumed from the very beginning that:

$$(9) \qquad \text{all the maps} \ \ Z \to A^k(\bar{V} + Z) \ \ \text{are linear.}$$

In order to obtain our results, we shall additionally assume that for all $k \in \{1, \cdots, d\}$,

$$(10) \qquad A^k_{1,1}(\bar{V}) = 0 \ \ \text{and} \ \ Z \mapsto A^k_{1,1}(\bar{V} + Z) \ \ \text{is linear with respect to} \ \ Z_2.$$

Remark 1.1. *The condition of linearity for $A^k(V)$ may be just technical. However, in our functional framework, we do not know how to handle more than quadratic nonlinearities when performing estimates in the high frequencies regime.*

At the same time, in contrast with [10, Th. 2.3], we do not have to assume here that $A^k_{1,2}$ and $A^k_{2,1}$ are linear with respect to Z_2. This is due to a better treatment of the corresponding terms (use of suitable commutator estimates recalled in Appendix).

In order to explain the supplementary conditions that we need in order to get our main results, let us consider the linearization of (6) about $\bar{V}$, namely, in the case $\varepsilon = 1$:

$$(11) \qquad \partial_t Z + \sum_{k=1}^d \bar{A}^k \partial_k Z + LZ = 0 \quad \text{with} \quad \bar{A}^k := A^k(\bar{V}) \ \ \text{for} \ \ k = 1, \cdots, d.$$

Denoting by $\xi \in \mathbb{R}^d$ the Fourier variable corresponding to $x \in \mathbb{R}^d$, we have

$$\partial_t \widehat{Z} + i \sum_{k=1}^d \bar{A}^k \xi_k \widehat{Z} + L\widehat{Z} = 0.$$

Setting $\xi = \rho\omega$ with $\omega \in \mathbb{S}^{d-1}$ and $\rho = |\xi|$, the above system rewrites

$$(12) \qquad \partial_t \widehat{Z} + i\rho M_\omega \widehat{Z} + L\widehat{Z} = 0 \quad \text{with} \quad M_\omega \triangleq \sum_{k=1}^d \omega_k \bar{A}^k.$$

Clearly, (7) implies that

$$(13) \qquad (L\widehat{Z}|\widehat{Z})_{L^2} \geq c\|\widehat{Z}_2\|^2_{L^2} \quad \text{for all} \ \ Z \in L^2(\mathbb{R}^d; \mathbb{R}^n).$$

This provides dissipation on the directly damped component Z_2. In order to have dissipation for *all* the components of the solution Z to (11), Shizuta and Kawashima [15, 26] proposed the following:

Definition 1.1. *System (1) verifies the (SK) condition at $\bar{V}$ if, for all $\omega \in \mathbb{S}^{d-1}$, we have at the same time $L\phi = 0$ and $\lambda\phi + M_\omega\phi = 0$ for some $\lambda \in \mathbb{R}$, if and only if $\phi = 0_{\mathbb{R}^n}$.*

As we shall see in the last section of the paper, the compressible Euler system with damping, rewritten in suitable variables, satisfies Condition (SK) about any constant state with positive density and null velocity.

Condition (SK) turns out to be equivalent to the strong stability of System (11) and it has been shown in [26] that if it is satisfied, then solutions to (11) satisfy for all $t \geq 0$,

(14)
$$\|Z^h(t)\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)} \leq Ce^{-\lambda t}\|Z_0\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)},$$
$$\|Z^\ell(t)\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^n)} \leq Ct^{-\frac{d}{2}}\|Z_0\|_{L^1(\mathbb{R}^d,\mathbb{R}^n)},$$

where Z^h and Z^ℓ correspond, respectively, to the high and low frequencies of the solution. Thus, at the linear level, we observe that, provided (SK) is satisfied:

- in high frequencies, the solution decays exponentially to 0;
- in low frequencies, the solution behaves as the solutions of the heat equation.

Even though Condition (SK) ensures strong linear stability, it does not tell us how to achieve the corresponding quantitative estimates. In [2], Beauchard and Zuazua proposed a new approach to build an explicit Lyapunov functional that allows to control suitable norms of the solution. We took advantage of it in [10] to prove global existence and decay estimates for solutions to (1) in a critical L^2-type framework. In the present paper, we will combine Beauchard and Zuazua's strategy to control the high frequencies of the solution, with estimates for a suitable 'damped mode' so as to achieve L^p-type estimates for the low frequencies of the solution.

At this stage, introducing a few notations is in order. First, we fix a homogeneous Littlewood-Paley decomposition $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$ that is defined by

$$\dot{\Delta}_j \triangleq \varphi(2^{-j}D) \quad \text{with} \quad \varphi(\xi) \triangleq \chi(\xi/2) - \chi(\xi)$$

where χ stands for a smooth function with range in $[0,1]$, supported in the open ball $B(0,4/3)$ and such that $\chi \equiv 1$ on the closed ball $\bar{B}(0,3/4)$. We further state:

$$\dot{S}_j \triangleq \chi(2^{-j}D) \quad \text{for all} \quad j \in \mathbb{Z}$$

and define $\mathcal{S}'_h$ to be the set of tempered distributions z such that

$$\lim_{j\to-\infty} \|\dot{S}_j z\|_{L^\infty} = 0.$$

Following [1], we introduce the homogeneous Besov semi-norms:

$$\|z\|_{\dot{\mathbb{B}}^s_{p,r}} \triangleq \left\| 2^{js}\|\dot{\Delta}_j z\|_{L^p(\mathbb{R}^d)} \right\|_{\ell^r(\mathbb{Z})},$$

then define the homogeneous Besov spaces $\dot{\mathbb{B}}^s_{p,r}$ (for any $s \in \mathbb{R}$ and $(p,r) \in [1,\infty]^2$) to be the subset of z in $\mathcal{S}'_h$ such that $\|z\|_{\dot{\mathbb{B}}^s_{p,r}}$ is finite.

Use from now on the shorthand notation

(15)
$$\dot{\Delta}_j z \triangleq z_j.$$

For any fixed threshold $J \in \mathbb{Z}$, we define the low and high frequency parts of any $z \in \mathcal{S}'_h$ to be:

$$z^{\ell,J} := \sum_{j \leq J-1} z_j \quad \text{and} \quad z^{h,J} := \sum_{j \geq J} z_j.$$

Likewise, we set[1] if $r < \infty$,

$$\|z\|^{\ell,J}_{\dot{\mathbb{B}}^s_{p,r}} \triangleq \left(\sum_{j \leq J+1} \left(2^{js}\|\dot{\Delta}_j z\|_{L^p} \right)^r \right)^{\frac{1}{r}} \quad \text{and} \quad \|z\|^{h,J}_{\dot{\mathbb{B}}^s_{p,r}} \triangleq \left(\sum_{j \geq J} \left(2^{js}\|\dot{\Delta}_j z\|_{L^p} \right)^r \right)^{\frac{1}{r}}.$$

Whenever the value of J is clear from the context, it is omitted in the notations.

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[1]For technical reasons, we need a small overlap between low and high frequencies.

For any Banach space X, index ρ in $[1, \infty]$ and time $T \in [0, \infty]$, we use the notation $\|z\|_{L_T^\rho(X)} \triangleq \big\| \|z\|_X \big\|_{L^\rho(0,T)}$. If $T = \infty$, then we just write $\|z\|_{L^\rho(X)}$. Finally, in the case where z has n components z_k in X, we slightly abusively keep the notation $\|z\|_X$ to mean $\sum_{k \in \{1, \cdots, n\}} \|z_k\|_X$.

The first part of the paper is devoted to proving the following global existence result with uniform estimates with respect to the relaxation parameter for System (6).

Theorem 1.1. *Let $p \in [2, 4]$ if $1 \le d \le 4$, or $p \in [2, \frac{2d}{d-2}]$ if $d \ge 5$. Assume that the (SK) condition and (9)-(10) are satisfied. There exist $k_p \in \mathbb{Z}$ and $\eta_0 > 0$ such that for all $\varepsilon > 0$, if we assume that $Z_0^{\ell, J_\varepsilon} \in \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}$ and $Z_0^{h, J_\varepsilon} \in \dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}$ with*

$$\|Z_0\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^{\ell, J_\varepsilon} + \varepsilon \|Z_0\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}}^{h, J_\varepsilon} \le \eta_0 \quad and \quad J_\varepsilon \triangleq \lfloor -\log_2 \varepsilon \rfloor + k_p,$$

then System (6) admits a unique global solution Z in the space $E_p^{J_\varepsilon}$ defined by

$$Z_1^{\ell, J_\varepsilon} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2}), \quad Z^{h, J_\varepsilon} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}) \cap L^1(\mathbb{R}^+, \dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}),$$

$$Z_2^{\ell, J_\varepsilon} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}), \quad W_\varepsilon \in L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}) \quad and \quad Z_2 \in L^2(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})$$

$$with \quad W_\varepsilon \triangleq \frac{Z_2}{\varepsilon} + \sum_{k=1}^{d} L_2^{-1}\big(A_{2,1}^k(V)\partial_k Z_1 + A_{2,2}^k(V)\partial_k Z_2\big).$$

Moreover, we have the following bound:

$$X_{p,\varepsilon}(t) \lesssim \|Z_0\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^{\ell, J_\varepsilon} + \varepsilon \|Z_0\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}}^{h, J_\varepsilon} \quad for \ all \ t \ge 0,$$

where

$$X_{p,\varepsilon}(t) \triangleq \|Z\|_{L_t^\infty(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})}^{\ell, J_\varepsilon} + \varepsilon \|Z\|_{L_t^\infty(\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1})}^{h, J_\varepsilon} + \varepsilon \|Z_1\|_{L_t^1(\dot{\mathbb{B}}_{p,1}^{\frac{1}{p}+2})}^{\ell, J_\varepsilon} + \|Z\|_{L_t^1(\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1})}^{h, J_\varepsilon} + \|Z_2\|_{L_t^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1})}^{\ell}$$

$$+ \|W_\varepsilon\|_{L_t^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})} + \varepsilon^{-\frac{1}{2}} \|Z_2\|_{L_t^2(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})}.$$

Remark 1.2. *The choice of the threshold J_ε corresponds[2] to the place where the 0-order terms and the 1-order terms have the same strength (parameter included). It can be easily deduced from a spectral analysis of the linearized system.*

The above theorem can be applied to the isentropic compressible Euler equation with damping (2) after suitable symmetrization. Indeed, introduce the following 'sound speed':

$$(16) \qquad\qquad c \triangleq \frac{2}{\gamma - 1}\sqrt{\frac{\partial P}{\partial \rho}} = \frac{(4\gamma A)^{\frac{1}{2}}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}}.$$

Then, setting $\check{\gamma} = \dfrac{\gamma - 1}{2}$ and $\widetilde{c} = c - \bar{c}$, we can rewrite (2) under the form :

$$(17) \qquad \begin{cases} \partial_t \widetilde{c} + v \cdot \nabla \widetilde{c} + \check{\gamma}(\widetilde{c} + \bar{c})\mathrm{div}\, v = 0, \\[2mm] \partial_t v + v \cdot \nabla v + \check{\gamma}(\widetilde{c} + \bar{c})\nabla \widetilde{c} + \dfrac{1}{\varepsilon}v = 0. \end{cases}$$

The above system is symmetric and satisfies both Conditions (SK) and (9) - (10). Then, applying Theorem 1.1 yields:

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[2]The value of k_p is given by our low frequencies analysis. At some point, we need the threshold to be small enough in order to close the estimates. As pointed out in [10], for $p = 2$, one can take $k_p = 0$.

Theorem 1.2. *Fix some positive constant density $\bar\rho$. Let $p \in [2,4]$ if $1 \le d \le 4$, or $p \in [2, \frac{2d}{d-2}]$ if $d \ge 5$. There exist $k_p \in \mathbb{Z}$ and $\eta_0 > 0$ such that for all $\varepsilon > 0$, if we set $J_\varepsilon \triangleq \lfloor -\log_2\varepsilon \rfloor + k_p$ and assume that $(c_0 - \bar c)^{\ell,J_\varepsilon}, v_0^{\ell,J_\varepsilon} \in \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}$ and $(c_0 - \bar c)^{h,J_\varepsilon}, v_0^{h,J_\varepsilon} \in \dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}$ with*

$$\|(c_0 - \bar c, v_0)\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^{\ell,J_\varepsilon} + \varepsilon \, \|(c_0 - \bar c, v_0)\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}}^{h,J_\varepsilon} \le \eta_0,$$

then System (17) admits a unique global solution (c,v) with $(c - \bar c, v)$ in the space $E_p^{J_\varepsilon}$ defined by

$$(c - \bar c)^{\ell,J_\varepsilon} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2}), \quad (c - \bar c)^{h,J_\varepsilon} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}),$$

$$v^{\ell,J_\varepsilon} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}), \quad v^{h,J_\varepsilon} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}),$$

$$\frac{v}{\varepsilon} + \check\gamma c\nabla c \in L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}) \quad and \quad v \in L^2(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}).$$

Moreover, we have the following a priori bound:

$$X_{p,\varepsilon}(t) \lesssim \|(c_0 - \bar c, v_0)\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^{\ell,J_\varepsilon} + \varepsilon \, \|(c_0 - \bar c, v_0)\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}}^{h,J_\varepsilon} \quad for \ all \ t \ge 0,$$

where

$$X_{p,\varepsilon}(t) \triangleq \|(c - \bar c, v)\|_{L_t^\infty(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})}^{\ell,J_\varepsilon} + \varepsilon \, \|(c - \bar c, v)\|_{L_t^\infty(\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1})}^{h,J_\varepsilon} + \varepsilon \, \|c - \bar c\|_{L_t^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2})}^{\ell,J_\varepsilon} + \|v\|_{L_t^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1})}^\ell$$

$$+ \|(c - \bar c, v)\|_{L_t^1(\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1})}^{h,J_\varepsilon} + \left\|\frac{v}{\varepsilon} + \check\gamma c\nabla c\right\|_{L_t^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})} + \varepsilon^{-\frac{1}{2}} \|v\|_{L_t^2(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})}.$$

Remark 1.3. *According to Theorem 1.1, the damped mode should be $W_\varepsilon \triangleq v \cdot \nabla v + \frac{v}{\varepsilon} + \check\gamma c\nabla c$. However, the above estimate combined with product laws ensure that $\|v \cdot \nabla v\|_{L_t^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})} \lesssim \eta_0^2$, hence is negligible compared to $\varepsilon^{-1}v$. Consequently, $W_\varepsilon' \triangleq \frac{v}{\varepsilon} + \check\gamma c\nabla c$ can be seen as a damped quantity. This latter function turns out to be more adapted to the study of the relaxation limit.*

The uniform estimates from Theorem 1.2 enable us to obtain the following result pertaining to the relaxation limit of the compressible Euler system.

Theorem 1.3. *Let the hypotheses of Theorem 1.2 be in force and denote by (c,v) the corresponding solution. Let ρ be the density corresponding to c through relation (16).*

Let the positive function $\mathcal{N}_0$ be such that $\mathcal{N}_0 - \bar\rho$ is small enough in $\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}$, and let $\mathcal{N} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2})$ be the global solution of:

$$(18) \qquad\qquad\qquad \partial_t\mathcal{N} - \Delta P(\mathcal{N}) = 0$$

supplemented with initial data $\mathcal{N}_0$ given by Proposition 5.1.

Let $(\widetilde\rho^\varepsilon, \widetilde v^\varepsilon)(t,x) \triangleq (\rho, \varepsilon^{-1}v)(\varepsilon^{-1}t, x)$ and assume that

$$\|\widetilde\rho_0^\varepsilon - \mathcal{N}_0\|_{\mathbb{B}_{p,1}^{\frac{d}{p}-1}} \le C\varepsilon.$$

Then, as $\varepsilon \to 0$, we have

$$\widetilde\rho^\varepsilon - \mathcal{N} \longrightarrow 0 \quad strongly \ in \quad L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}),$$

and

$$\widetilde v^\varepsilon + \frac{\nabla P(\widetilde\rho^\varepsilon)}{\widetilde\rho^\varepsilon} \longrightarrow 0 \quad strongly \ in \quad L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}).$$

Moreover, we have the following quantitative estimate:

$$\|\widetilde{\rho}^{\varepsilon} - \mathcal{N}\|_{L^{\infty}(\mathbb{R}^+;\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-1})} + \|\widetilde{\rho}^{\varepsilon} - \mathcal{N}\|_{L^1(\mathbb{R}^+;\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1})} + \left\|\frac{\nabla P(\widetilde{\rho}^{\varepsilon})}{\widetilde{\rho}^{\varepsilon}} + \widetilde{v}^{\varepsilon}\right\|_{L^1(\mathbb{R}^+;\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})} \leq C\varepsilon.$$

Remark 1.4. *To the best of our knowledge, all the previous works devoted to the relaxation limit were based on compactness methods, which prevents to get strong convergence results globally in space and time and explicit convergence rates (see e.g. [20, 24, 31]). Here our functional framework allows us to bound directly some norms of the difference of the solutions. Having estimates on the damped mode at hand plays a key role.*

Remark 1.5. *In the specific case $p = 2$, we expect to have similar results for the compressible Euler equations supplemented with any pressure law P satisfying*

(19) $$P'(\bar{\rho}) > 0 \quad at \ some \ \ \bar{\rho} > 0.$$

Indeed, it is then possible to use the symmetrization of [10] (instead of the sound speed, we consider the unknown n defined from ρ through the relation: $n(\rho) = \int_{\bar{\rho}}^{\rho} \frac{P'(s)}{s} ds$) and to use the standard composition lemma to treat general nonlinearities that need not be quadratic.

2. A FEW WORDS ON THE PROOFS

As a first, let us observe that performing a suitable time and space rescaling (see (28) below) reduces the proof of global well-posedness for (1) to the case $\varepsilon = 1$. Then, it is essentially a matter of establishing global-in-time a priori estimates for smooth solutions (from them and rather classical arguments, one can obtain the existence of global solutions). The uniqueness part of Theorem 1.1 follows from stability estimates. As in [11], the norms that will be used in the stability estimates are not the standard ones owing to the use of the L^p framework for the low frequencies.

In what follows, we shortly explain how we proceed to prove global a priori estimates, adopting, as pointed out in the introduction, a different strategy to handle the low and the high frequencies, then we explain how to investigate the infinite relaxation limit.

2.1. **Low frequencies: the damped mode.** Although Kawashima's decomposition (14) gives the overall behavior of the solution of the linearized system, it does not fully reflect that a part of the solution has better decay properties in the low frequencies regime. The essential ingredient of our low frequencies analysis is to look at the time evolution of a 'damped mode' corresponding to the part of the solution that experiences maximal dissipation in low frequencies. In the case $\varepsilon = 1$, it may be defined as follows :

(20) $$W \triangleq -L_2^{-1}\partial_t Z_2 = Z_2 + \sum_{k=1}^{d} L_2^{-1}\left(A_{2,1}^k(V)\partial_k Z_1 + A_{2,2}^k(V)\partial_k Z_2\right).$$

Hence W satisfies

(21) $$\partial_t W + L_2 W = L_2^{-1}\sum_{k=1}^{d}\partial_t\left(A_{2,1}^k(V)\partial_k Z_1 + A_{2,2}^k(V)\partial_k Z_2\right).$$

On the left-hand side, Property (7) ensures maximal dissipation on W. As the right-hand side of (21) contains only at least quadratic terms, or linear terms with one derivative, it can be expected to be negligible in low frequencies. Furthermore, the second equality of (20) reveals that W is comparable to Z_2 in low frequencies if Z is small enough. This will allow us to recover better integrability properties on Z_2 than for the whole solution Z.

In order to get optimal information, it is more suitable to look at the evolution equations of Z_1 and W, and we thus have to rewrite the equation of Z_1 in terms of W instead of Z_2. Let us denote for all $k \in \{1, \cdots, d\}$ and $(p, m) \in \{1, 2\}^2$,

$$\bar{A}_{p,m}^k \triangleq A_{p,m}^k(\bar{V}) \quad \text{and} \quad \widetilde{A}_{p,m}^k(Z) \triangleq \widetilde{A}_{p,m}^k(\bar{V} + Z) - \bar{A}_{p,m}^k.$$

Then, we rewrite the equation of Z_1 as follows:

$$(22) \qquad \partial_t Z_1 + \sum_{k=1}^d \bar{A}_{1,1}^k \partial_k Z_1 - \sum_{k=1}^d \sum_{\ell=1}^d \bar{A}_{1,2}^k L_2^{-1} \bar{A}_{2,1}^\ell \partial_k \partial_\ell Z_1 = f_1 + f_2 + f_3 + f_4 + f_5,$$

where

$$f_1 = \sum_{k=1}^d \sum_{\ell=1}^d A_{1,2}^k(V) \partial_k \left(L_2^{-1} A_{2,2}^\ell(V) \partial_\ell Z_2 \right),$$

$$f_2 = -\sum_{k=1}^d A_{1,2}^k(V) \partial_k W,$$

$$f_3 = \sum_{k=1}^d \sum_{\ell=1}^d A_{1,2}^k(V) \partial_k (L_2^{-1} \widetilde{A}_{2,1}^\ell(Z)) \partial_\ell Z_1),$$

$$f_4 = \sum_{k=1}^d \sum_{\ell=1}^d \widetilde{A}_{1,2}^k(Z)) L_2^{-1} \bar{A}_{2,1}^\ell \partial_k \partial_\ell Z_1,$$

$$f_5 = -\sum_{k=1}^d \widetilde{A}_{1,1}^k(Z) \partial_k Z_1.$$

As in [24], in order to ensure the parabolic behavior of the unknown Z_1 in low frequencies, we make the following hypothesis:

$$(23) \qquad \forall k \in \{1, \cdots, d\}, \ \bar{A}_{1,1}^k = 0 \quad \text{and} \quad \mathcal{A} \triangleq -\sum_{k=1}^d \sum_{\ell=1}^d \bar{A}_{1,2}^k L_2^{-1} \bar{A}_{2,1}^\ell \partial_k \partial_\ell \text{ is strongly elliptic.}$$

It turns out that if the first condition of (23) is satisfied then the strong ellipticity of $\mathcal{A}$ is equivalent to Condition (SK) (see the proof in Appendix).

Remark 2.1. *In the context of fluid mechanics, the first condition of (23) is satisfied up to a Galilean change of frame. Indeed, Z_1 then corresponds to conserved quantities like e.g. the density or the entropy, and $\sum_{k=1}^d A_{1,1}^k(V) \partial_k Z_1$ represents the corresponding transport term by the velocity field.*

Further remark that even if the matrices $\bar{A}_{1,1}^k$ are nonzero, the above sum has no contribution in the energy arguments, and can be treated in a purely L^2 framework like in [10].

Finally, in the general L^p framework, our results still hold if all the matrices $\bar{A}_{1,1}^k$ are diagonal.

Now, as (23) is satisfied, one can take advantage of parabolic maximal regularity estimates (see Lemma 5.2) to bound Z_1 in Besov spaces of type $\dot{\mathbb{B}}_{p,1}^s$. Then, pointing out that the equations of W and Z_1 are coupled by at least quadratic terms, or by linear terms of higher order (in terms of derivatives) that are thus negligible provided that the threshold J_1 between low and high frequencies is negative enough, one can establish the a priori estimates in the low frequencies region.

2.2. **High frequencies: Lyapunov functional.** We proceed as in [10], but our nonstandard functional framework regarding the low frequencies will complicate the treatment of some nonlinear terms.

Recall that, in Fourier variables, System (11) reads

$$(24) \qquad \partial_t \widehat{Z} + i\rho M_\omega \widehat{Z} + L\widehat{Z} = 0 \quad \text{with} \quad M_\omega \triangleq \sum_{k=1}^{d} \omega_k \bar{A}^k.$$

To compensate the lack of coercivity depicted in the previous section, we introduce a "correction" term to exhibit the time integrability properties for the un-damped components. Fix $n-1$ positive parameters $\varepsilon_1, \cdots \varepsilon_{n-1}$ (bound to be small) and set

$$(25) \qquad \mathcal{I} \triangleq \Im \sum_{k=1}^{n-1} \varepsilon_k \big(LM_\omega^{k-1}\widehat{Z} \cdot LM_\omega^k \widehat{Z} \big)$$

where $\cdot$ designates the Hermitian scalar product in $\mathbb{C}^n$.

In [2], the authors proved the following result

Lemma 2.1. *There exist positive parameters $\varepsilon_1, \cdots, \varepsilon_{n-1}$ so that*

$$(26) \qquad \frac{d}{dt}\mathcal{L} + \mathcal{H} \leq 0 \quad \text{with} \quad \mathcal{H} \triangleq \int_{\mathbb{R}^d} \sum_{k=0}^{n-1} \varepsilon_k \min(1, |\xi|^2) |LM_\omega^k \widehat{Z}(\xi)|^2 \, d\xi$$

$$\text{and} \quad \mathcal{L} \triangleq \|Z\|_{L^2}^2 + \int_{\mathbb{R}^d} \min(|\xi|, |\xi|^{-1}) \mathcal{I}(\xi) \, d\xi,$$

with, in addition, $\mathcal{L} \simeq \|Z\|_{L^2}^2$.

The question now is whether $\mathcal{H}$ may be compared to $\|Z\|_{L^2}^2$. The answer depends on the properties of the support of $\widehat{Z}_0$ and on the possible cancellation of the following quantity:

$$(27) \qquad \mathcal{N}_{\bar{V}} := \inf\left\{\sum_{k=0}^{n-1} \varepsilon_k |LM_\omega^k x|^2; \ x \in \mathbb{S}^{n-1}, \ \omega \in \mathbb{S}^{d-1}\right\}.$$

In order to pursue our analysis, we need the following key result (see the proof in e.g. [2]).

Proposition 2.1. *Let M and L be two matrices in $\mathcal{M}_n(\mathbb{R})$. The following assertions are equivalent:*

(1) *$L\phi = 0$ and $\lambda\phi + M\phi = 0$ for some $\lambda \in \mathbb{R}$ implies $\phi = 0$;*
(2) *For every $\varepsilon_0, \cdots, \varepsilon_{n-1} > 0$, the function*

$$y \longmapsto \sqrt{\sum_{k=0}^{n-1} \varepsilon_k |LM^k y|^2}$$

defines a norm on $\mathbb{R}^n$.

Thanks to the above proposition and observing that the unit sphere $\mathbb{S}^{d-1}$ is compact, one may conclude that the (SK) condition is satisfied by the pair (M_ω, L) for all $\omega \in \mathbb{S}^{d-1}$ if and only if $\mathcal{N}_{\bar{V}} > 0$. Furthermore, we note that:

- if $\widehat{Z}_0$ is compactly supported then $\mathcal{H} \gtrsim \|\nabla Z\|_{L^2}^2$ (parabolic behavior of the low frequencies of the solution);
- if the support of $\widehat{Z}_0$ is away from the origin, then $\mathcal{H} \gtrsim \|Z\|_{L^2}^2$ (exponential decay for the high frequencies).

We thus readily recover Kawashima's decomposition (14).

As in [10], an important part of the proof consists in studying the evolution of the functional $\mathcal{L}$. One cannot repeat exactly the computations therein however, since the nonlinearities contain a little amount of low frequencies which are not bounded in L^2-based spaces. Taking advantage of appropriate product laws and commutator estimates (see the Appendix) will enable us to overcome the difficulty.

2.3. **The relaxation limit.** We finally give some insight on our study of the limit $\varepsilon \to 0$ for the isentropic compressible Euler equations with relaxation. At first sight, when the damping parameter $1/\varepsilon$ increases, we expect the dissipation to dominate more and more. This is not quite the case however, owing to the so-called *overdamping* phenomenon: somehow, the 'overall' damping rate behaves like $\min(\varepsilon, 1/\varepsilon)$. This can be seen from a spectral analysis of the Euler system for example (see also [33] for the case of the damped harmonic oscillator). Looking at the one-dimensional case for simplicity, the linearized Euler equations with relaxation parameter ε has the following matrix in the Fourier side:

$$\begin{pmatrix} 0 & i\xi \\ i\xi & \varepsilon^{-1} \end{pmatrix}.$$

It is straightforward that:

- for frequencies ξ such that $|\xi| \leq 1/(2\varepsilon)$, the above matrix has two real eigenvalues asymptotically equal to $1/\varepsilon$ and $\varepsilon\xi^2$, respectively as ξ tends to 0;
- for frequencies ξ such that $|\xi| \geq 1/(2\varepsilon)$, it has two complex conjugated eigenvalues with real part equal to $1/2\varepsilon$.

It seems that in the prior literature dedicated to the relaxation limit, the dissipative aspect of the low frequencies has been overlooked. In fact, only the overall behavior of the solution was taken into account and the threshold between the low and high frequencies was not taken depending on ε.

In the approach that we offer here, we take advantage of the uniform bounds on the damped mode which corresponds to the eigenvalue asymptotically equal to $1/(2\varepsilon)$ for $\xi \to 0$, and allow the threshold between low and high frequencies to depend on ε by putting it at the place where the 0-order terms and the 1-order terms have the same weight (parameter included). This corresponds approximately to $1/\varepsilon$, in accordance with our spectral analysis above. Somehow, as $\varepsilon \to 0$, the low frequencies 'invade' the whole space of frequencies. Hence the fact that the limit system for one of the components of the system (namely the density) has to be purely parabolic does not come up as a surprise.

With this heuristics in hand and after performing the diffusive rescaling (5) and using the uniform bounds on the damped mode and on the solution from the uniform existence theorem, it is rather easy to estimate the difference between the solutions of the compressible Euler System and the porous media equation. In this way, we get Theorem 1.3.

3. Global existence in the L^p framework

This section is devoted to proving Theorem 1.1. We shall focus on the case $\varepsilon = 1$, which is not restrictive owing to the rescaling:

$$(28) \qquad Z(t,x) \triangleq \check{Z}\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right).$$

Indeed, Z satisfies (1) with relaxation parameter ε if and only if $\check{Z}$ satisfies (1) with relaxation parameter 1, and performing the inverse scaling gives us the desired dependency with respect to ε in Theorem 1.1 since (see e.g. [1, Chap.2]) for all $s \in \mathbb{R}$ and $m \in [1, \infty]$, we have

$$(29) \qquad \|z(\varepsilon\cdot)\|_{\dot{\mathbb{B}}^s_{m,1}} \simeq \varepsilon^{s-d/m} \|z\|_{\dot{\mathbb{B}}^s_{m,1}}$$

and, by the same token, we have

(30) $$\|z(\varepsilon\cdot)\|_{\dot{\mathbb{B}}_{p,1}^s}^{\ell,J_1} \simeq \varepsilon^{s-d/p}\|z\|_{\dot{\mathbb{B}}_{p,1}^s}^{\ell,J_\varepsilon} \quad \text{and} \quad \|z(\varepsilon\cdot)\|_{\dot{\mathbb{B}}_{2,1}^s}^{h,J_1} \simeq \varepsilon^{s-d/2}\|z\|_{\dot{\mathbb{B}}_{2,1}^s}^{h,J_\varepsilon}.$$

We shall use repeatedly that for $s \leq s'$, the following inequalities hold true:

(31) $$\|z^{\ell,J_1}\|_{\dot{\mathbb{B}}_{2,1}^{s'}} \lesssim \|z\|_{\dot{\mathbb{B}}_{2,1}^{s'}}^{\ell,J_1} \lesssim 2^{J_1(s'-s)}\|z\|_{\dot{\mathbb{B}}_{2,1}^s}^{\ell,J_1} \quad \text{and} \quad \|z^{h,J_1}\|_{\dot{\mathbb{B}}_{2,1}^s} \lesssim \|z\|_{\dot{\mathbb{B}}_{2,1}^s}^{h,J_1} \lesssim 2^{J_1(s-s')}\|z\|_{\dot{\mathbb{B}}_{2,1}^{s'}}^{h,J_1}.$$

3.1. **A priori estimates.** Throughout this part we set the threshold between low and high frequencies at some integer J_1 the value of which will be chosen during the computations. For better readability, the exponent J_1 on the Besov norms will be omitted.

We assume that we are given a smooth (and decaying) solution Z of (6) on $[0, T] \times \mathbb{R}^d$ with Z_0 as initial data, satisfying

(32) $$\sup_{t\in[0,T]} \|Z(t)\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^{\ell} + \sup_{t\in[0,T]} \|Z(t)\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}}^{h} \ll 1.$$

We shall use repeatedly that, owing to (31) and to the embedding $\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}} \hookrightarrow L^\infty$, we have

(33) $$\sup_{t\in[0,T]} \|Z(t)\|_{L^\infty} \ll 1$$

and also that for all $\sigma \in \mathbb{R}$, $q \in [1,\infty]$ and $\alpha \geq 0$,

(34) $$\|z\|_{\dot{\mathbb{B}}_{q,1}^{\sigma+\alpha}}^{\ell} \lesssim \|z\|_{\dot{\mathbb{B}}_{q,1}^{\sigma}}^{\ell} \quad \text{and} \quad \|z\|_{\dot{\mathbb{B}}_{q,1}^{\sigma-\alpha}}^{h} \lesssim \|z\|_{\dot{\mathbb{B}}_{q,1}^{\sigma}}^{h}.$$

From now on, $C > 0$ designates a generic harmless constant, the value of which depends on the context and we denote by $(c_j)_{j\in\mathbb{Z}}$ various nonnegative sequences such that $\sum_{j\in\mathbb{Z}} c_j = 1$.

3.1.1. *Low frequencies analysis in L^p spaces.* In this part, we fix the threshold J_1 between the low and high frequencies to be a negative enough integer. We aim at proving:

Proposition 3.1. *Assume that $2 \leq p \leq 4$ if $d \leq 4$, or $d \in [2, 2d/(d-2)]$ if $d \geq 5$. There exists a positive real number κ_0 such that for all $t \in [0, T]$,*

(35) $$\|(Z,W)(t)\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^{\ell} + \kappa_0 \int_0^t \left(\|Z_1\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2}}^{\ell} + \|Z_2\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}}^{\ell} + \|W\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^{\ell} \right)$$
$$\lesssim \|(Z_0,W_0)\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^{\ell} + \int_0^t \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|Z_2\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}} + \int_0^t \|Z_2\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}}$$
$$+ \int_0^t \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}}^2 + \int_0^t \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}}^{h} + \int_0^t \|W\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}} \cap \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}}.$$

First, we state a result that will be used on several occasions.

Lemma 3.1. *Under hypotheses (10) and (32), we have for all $\sigma \in]d/p - d/p^*, d/p]$,*

$$\|\partial_t Z_1\|_{\dot{\mathbb{B}}_{p,1}^{\sigma}} \lesssim \|\nabla Z_2\|_{\dot{\mathbb{B}}_{p,1}^{\sigma}} + \|Z_2\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|\nabla Z_1\|_{\dot{\mathbb{B}}_{p,1}^{\sigma}},$$

$$\|\partial_t Z_2\|_{\dot{\mathbb{B}}_{p,1}^{\sigma}} \lesssim \|W\|_{\dot{\mathbb{B}}_{p,1}^{\sigma}}.$$

Proof. The second inequality just stems from the fact that the definition of W in (20) is equivalent to

(36) $$\partial_t Z_2 = -L_2 W.$$

The proof of the first item relies on the explicit expression of $\partial_t Z_1$: since $\bar{A}_{1,1}^k = 0$, we have for all $k \in \{1, \cdots, d\}$,

(37) $$\partial_t Z_1 + \sum_{k=1}^d \bar{A}_{1,2}^k \partial_k Z_2 = -\sum_{k=1}^d \left(\widetilde{A}_{1,1}^k(Z_2)\partial_k Z_1 + \widetilde{A}_{1,2}^k(Z)\partial_k Z_2 \right).$$

All the terms of the right-hand side are at least quadratic, and (10), (32) thus ensure the desired inequality for $\partial_t Z_1$. $\qquad\square$

We now turn to the proof of Proposition 3.1.

Step 1: Estimate for W. We have the following statement.

Proposition 3.2. *Denoting by c the constant in (7), we have*

$$\|W(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + c\int_0^t \|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \leq \|W_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + C\int_0^t \|(\nabla Z_2, W)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}$$

$$(38) \qquad\qquad + C\int_0^t \|(\nabla Z_2, W)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}\cap\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + C\int_0^t \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}.$$

Proof. From (21), we gather that

$$(39) \qquad \partial_t W + L_2 W = h \triangleq L_2^{-1} h_1 \quad \text{with} \quad h_1 \triangleq \sum_{k=1}^d \partial_t\big(A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2\big).$$

Applying $\dot{\Delta}_j$ to (39) and taking the scalar product with $W_j|W_j|^{p-2}$ (where $W_j \triangleq \dot{\Delta}_j W$) yields, thanks to (7),

$$(40) \qquad\qquad \frac{1}{p}\frac{d}{dt}\|W_j\|^p_{L^p} + c\|W_j\|^p_{L^p} \leq C\|\dot{\Delta}_j h_1\|_{L^p}\|W_j\|_{L^p}.$$

For bounding h_1, we use that for all $k \in \{1,\cdots,d\}$,

$$\partial_t(A^k_{2,1}(V)\partial_k Z_1 + A^k_{2,2}(V)\partial_k Z_2) = D_V A^k_{2,1}(V)\partial_t Z\partial_k Z_1 + \bar{A}^k_{2,1}\partial_t\partial_k Z_1$$
$$+ \widetilde{A}^k_{2,1}(Z)\partial_t\partial_k Z_1 + D_V A^k_{2,2}(V)\partial_t Z\partial_k Z_2 + \bar{A}^k_{2,2}\partial_t\partial_k Z_2 + \widetilde{A}^k_{2,2}(Z)\partial_t\partial_k Z_2.$$

For $m = 1,2$, we have, according to Proposition 5.3, Lemma 3.1 and the fact that $D_V A^k_{2,m}$ for $m = 1,2$ is constant,

$$\|D_V A^k_{2,m}(V)\partial_t Z\partial_k Z_m\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \lesssim \|\partial_t Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$$

$$\lesssim (\|(\nabla Z_2, W)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}})\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}.$$

Bounding the terms $\widetilde{A}^k_{2,m}(Z)\partial_t\partial_k Z_m$ involves a commutator estimate. We write

$$\dot{\Delta}_j(\widetilde{A}^k_{2,m}(Z)\partial_t\partial_k Z_m) = \widetilde{A}^k_{2,m}(Z)\partial_t\partial_k Z_{m,j} - R^{m,1}_j \quad \text{with} \quad R^{m,1}_j = [\widetilde{A}^k_{2,m}(Z), \dot{\Delta}_j]\partial_t\partial_k Z_m.$$

Now, combining Hölder's inequality, embedding and (31) yields

$$\sum_{j\leq J_1} 2^{j\frac{d}{p}}\|\widetilde{A}^k_{2,m}(Z)\partial_t\partial_k Z_{m,j}\|_{L^p} \lesssim \|Z\|_{L^\infty} \sum_{j\leq J_1} 2^{j\frac{d}{p}}\|\partial_t\partial_k Z_{m,j}\|_{L^p}$$

$$\lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\partial_t Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$$

$$\lesssim (\|(\nabla Z_2, W)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}})\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$$

and using (96), we obtain

$$\sum_{j\leq J_1} 2^{j\frac{d}{p}}\|R^{m,1}_j\|_{L^p} \lesssim \|\partial_t Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$$

$$\lesssim (\|(\nabla Z_2, W)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}})\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}.$$

Differentiating (36) and (37), then using product laws (namely Proposition 5.3) and (34) yields

$$\|(\partial_t \nabla Z_1, \partial_t \nabla Z_2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \lesssim \|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|\nabla Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z_1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} .$$

Hence, using Lemma 5.1, multiplying the resulting inequality by $2^{j\frac{d}{p}}$, summing up on $j \le J_1+1$ and taking advantage of (32), we end up with (38). $\qquad\square$

Step 2: Estimates for Z_1. We have the following proposition.

Proposition 3.3. *There exists come $\kappa_0 > 0$ such that*

$$(41) \quad \|Z_1(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \kappa_0 \int_0^t \|Z_1\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} \le \|Z_{1,0}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$$

$$+ C\left(\int_0^t \|(W, \nabla Z_2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \|W\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \int_0^t \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \right).$$

Proof. Applying $\dot{\Delta}_j$ to (22), taking the scalar product with $|Z_{1,j}|^{p-2}Z_{1,j}$ and using (95) gives for some $\kappa_0 > 0$,

$$\frac{1}{p}\frac{d}{dt}\|Z_{1,j}\|^p_{L^p} + \kappa_0 2^{2j}\|Z_{1,j}\|^p_{L^p} \lesssim \|(f_{1,j}, f_{2,j}, f_{3,j}, f_{4,j}, f_{5,j})\|_{L^p}\|Z_{1,j}\|^{p-1}_{L^p}$$

where the f_i's are defined in (22).

Using Lemma 5.1, multiplying by $2^{j\frac{d}{p}}$ then summing on $j \le J_1$, this easily leads to

$$(42) \qquad \|Z_1(t)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \kappa_0 \int_0^t \|Z_1\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} \lesssim \|Z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|(f_1, f_2, f_3, f_4, f_5)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} .$$

In order to bound the term corresponding to f_1, we use the decomposition:

$$(43) \quad f_1 = \sum_{k=1}^d \sum_{m=1}^d \Big(\bar{A}^k_{1,2} L_2^{-1} \bar{A}^k_{2,2} \partial_k \partial_m Z_2 + \bar{A}^k_{1,2} \partial_k \big(L_2^{-1} \widetilde{A}^m_{2,2}(Z) \partial_m Z_2 \big)$$

$$+ \widetilde{A}^k_{1,2}(Z) L_2^{-1} \bar{A}^m_{2,2} \partial_k \partial_m Z_2 + \widetilde{A}^k_{1,2}(Z) \partial_k \big(L_2^{-1} \widetilde{A}^m_{2,2}(Z) \partial_m Z_2 \big) \Big).$$

The first term in the right-hand side term obviously satisfies

$$\|\bar{A}^k_{1,2} L_2^{-1} \bar{A}^k_{2,2} \partial_k \partial_m Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \lesssim \|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}}.$$

For the other terms of (43), applying directly the product laws of Proposition 5.3 would entail a loss of one derivative. To avoid it, we will take advantage once more of the commutator estimate provided by Lemma 5.4. Let us explain how to proceed for the last term of (43) (which is the most complicated). We have

$$\widetilde{A}^k_{1,2}(Z) \partial_k (L_2^{-1} \widetilde{A}^m_{2,2}(Z) \partial_m Z_2) = \widetilde{A}^k_{1,2}(Z) L_2^{-1} \partial_k (\widetilde{A}^m_{2,2}(Z)) \partial_m Z_2 + \widetilde{A}^k_{1,2}(Z) L_2^{-1} \widetilde{A}^\ell_{2,2}(Z) \partial_k \partial_m Z_2.$$

The first term can be bounded directly as follows:

$$\|\widetilde{A}^k_{1,2}(Z) L_2^{-1} \partial_k (\widetilde{A}^m_{2,2}(Z)) \partial_m Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} .$$

To handle the second term, one introduces a commutator with $\dot{\Delta}_j$ as follows:

$$\dot{\Delta}_j \Big(\widetilde{A}^k_{1,2}(Z) L_2^{-1} \widetilde{A}^\ell_{2,2}(Z) \partial_k \partial_m Z_2 \Big) = \widetilde{A}^k_{1,2}(Z) L_2^{-1} \widetilde{A}^m_{2,2}(Z) \partial_k \partial_m Z_{2,j} - R^{m,2}_j$$

where $R^{m,2}_j \triangleq \Big[\widetilde{A}^k_{1,2}(Z) L_2^{-1} \widetilde{A}^m_{2,2}(Z), \dot{\Delta}_j \Big] \partial_k \partial_m Z_2.$

We have

$$\sum_{j \leq J_1} 2^{j\frac{d}{p}} \|\widetilde{A}_{1,2}^k(Z) L_2^{-1} \widetilde{A}_{2,2}^\ell(Z) \partial_k \partial_\ell Z_{2,j}\|_{L^p} \lesssim \sum_{j \leq J_1} 2^{j\frac{d}{p}} \|Z\|_{L^\infty}^2 \|\nabla^2 Z_{2,j}\|_{L^p}$$

$$\lesssim \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^2 \|Z_2\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2}}^\ell,$$

and (96) yields

$$\sum_{j \leq J_1} 2^{j\frac{d}{p}} \|R_j^{\ell,2}\|_{L^p} \lesssim \|\nabla(Z \otimes Z)\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|\nabla Z_2\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}$$

$$\lesssim \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}} \|Z_2\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}}.$$

The other terms of (43) may be handled similarly. In the end, using (32), we obtain

$$\|f_1\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^\ell \lesssim \|Z_2\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2}}^\ell + \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}}^2.$$

Noticing that the component Z_2 did not play a special role in the above computations, we can reproduce the procedure for f_3 and f_4, and eventually get:

$$\|f_3\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^\ell = \|\sum_{k=1}^d \sum_{m=1}^d A_{1,2}^k(V) \partial_k (L_2^{-1} \widetilde{A}_{2,1}^m(Z) \partial_m Z_1)\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^\ell$$

$$\lesssim \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}}^2 + \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|Z_1\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2}}^\ell,$$

$$\|f_4\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^\ell \lesssim \|\sum_{k=1}^d \sum_{m=1}^d \widetilde{A}_{1,2}^k(Z) L_2^{-1} A_{2,1}^m \partial_k \partial_m Z_1\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^\ell$$

$$\lesssim \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}}^2 + \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|Z_1\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2}}^\ell.$$

For f_2, we write that

$$f_2 = -\sum_{k=1}^d \left(\bar{A}_{1,2}^k \partial_k W - \widetilde{A}_{1,2}^k(Z) \partial_k W\right) \cdot$$

Hence,

$$\dot{\Delta}_j f_2 = -\sum_{k=1}^d \bar{A}_{1,2}^k \partial_k W_j - \widetilde{A}_{1,2}^k(Z) \partial_k W_j + [\widetilde{A}_{1,2}^k(Z), \dot{\Delta}_j] \partial_k W.$$

This allows to get

$$\|f_2\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^\ell \lesssim \|\sum_{k=1}^d \bar{A}_{1,2}^k \partial_k W^\ell\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} + \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|W\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}}^\ell + \|\nabla Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|W\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}$$

$$\lesssim \|W\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}}^\ell + \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}} \|W\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}.$$

The structure condition (10) comes into play only for bounding f_5. Thanks to it and to (32), we get

$$\|f_5\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^\ell \lesssim \|Z_2\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|\nabla Z_1\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}.$$

Inserting the estimates pertaining to f_1, f_2, f_3, f_4, f_5 in (42) and using (32) completes the proof of the proposition. $\qquad\square$

Now, choosing J_1 small enough (so that the higher-order linear terms of the right-hand side are absorbed by the left-hand side) and putting together (38) and (41), we obtain

$$(44) \quad \|(W, Z_1)\|^{\ell}_{L^{\infty}_t(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \kappa_0 \int_0^t \left(\|Z_1\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} + \|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \right) \leq \|(W_0, Z_{1,0})\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$$

$$+ C \left(\int_0^t \|Z_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} + \int_0^T \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|(\nabla Z_2, W)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1} \cap \dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^T \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \right).$$

Step 3: Recovering information for Z_2. In order to bound Z_2 from W, one can use the identity:

$$(45) \qquad W - Z_2 = L_2^{-1} \sum_{k=1}^{d} \left(\bar{A}^k_{2,1} \partial_k Z_1 + \widetilde{A}^k_{2,1}(Z) \partial_k Z_1 + \bar{A}^k_{2,2} \partial_k Z_2 + \widetilde{A}^k_{2,2}(Z) \partial_k Z_2 \right).$$

It implies that

$$\|W - Z_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \lesssim \|\nabla Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$$

$$\lesssim (1 + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}) \|\nabla Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}.$$

Hence, owing to (32), if one takes J_1 negative enough,

$$(46) \qquad \|Z_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \lesssim \|W\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|\nabla Z_1\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}.$$

This already ensures that $\|Z_2(t)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$ may be bounded for all $t \in [0, T]$ by the right-hand side of (35). Next, since for $m = 1, 2$, owing to (34) and the linearity of A^k, it holds that

$$\|\widetilde{A}^k_{2,1}(Z) \partial_k Z_m\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \lesssim \|\widetilde{A}^k_{2,m}(Z) \partial_k Z_m^{\ell}\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|\widetilde{A}^k_{2,m}(Z) \partial_k Z_m^{h}\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$$

$$\lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z_m^{\ell}\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \|\nabla Z_m^{\ell}\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z_m^{h}\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$$

we have

$$(47) \qquad \|W - Z_2\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \lesssim \|\nabla Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} (1 + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}) + \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}.$$

Hence, one can also include $\|Z_2\|^{\ell}_{L^1_t(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})}$ in the left-hand side of (35), which completes the proof of Proposition 3.1.

3.1.2. *High frequencies analysis.* Although the functional framework for high frequencies is the same as in [10], one cannot repeat exactly the computations therein since the non-linear terms contain a little amount of low frequencies of Z that are only in spaces of the type $\dot{\mathbb{B}}^s_{p,1}$ for some $p > 2$ and thus not in some $\dot{\mathbb{B}}^{s'}_{2,1}$. To overcome the difficulty, we will have to resort to more elaborate product laws and commutator estimates (see the Appendix).

The goal of this part is to prove the following proposition.

Proposition 3.4. *Let $p \in [2, 4]$ if $d \leq 4$ ($p \in [2, 2d/(d-2)]$ if $d \geq 5$) and define p^* by the relation $1/p + 1/p^* = 1/2$. Then, the following a priori estimate holds:*

$$\|Z(t)\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \lesssim \|Z_0\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \int_0^t \left((\|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}) \|Z\|^{h}_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^{\ell}_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} \right).$$

Proof. The starting point is to differentiate in time the following functional:

$$
\mathcal{L}_j \triangleq \|Z_j\|_{L^2}^2 + 2^{-j}\mathcal{I}_j, \qquad j \geq J_1, \tag{48}
$$

with

$$
\mathcal{I}_j \triangleq \int_{\mathbb{R}^d} \sum_{q=1}^{n-1} \varepsilon_q \Im\left((LM_\omega^{q-1}\widehat{Z_j})\cdot (LM_\omega^q\widehat{Z_j})\right), \tag{49}
$$

and where $\varepsilon_1, \cdots, \varepsilon_{n-1} > 0$ are chosen small enough.

We note that

$$
\frac{d}{dt}\mathcal{L}_j = \frac{d}{dt}\|Z_j\|_{L^2}^2 + 2^{-j}\frac{d}{dt}\mathcal{I}_j. \tag{50}
$$

Step 1: Energy estimates. To bound the first term in the right-hand side of (50), we localize System (6) by means of $\dot{\Delta}_j$ as follows:

$$
\partial_t Z_j + \sum_{k=1}^d \dot{S}_{j-1}A^k(V)\partial_k Z_j + LZ_j = R_j^1 \quad \text{with} \quad R_j^1 \triangleq \sum_{k=1}^d \dot{S}_{j-1}A^k(V)\partial_k Z_j - \dot{\Delta}_j(A^k(V)\partial_k Z).
$$

For $j \geq J_1$, taking the scalar product in $L^2(\mathbb{R}^d; \mathbb{R}^n)$ with Z_j, then integrating by parts and using (7) leads to:

$$
\frac{1}{2}\frac{d}{dt}\|Z_j\|_{L^2}^2 + \kappa_0\|Z_{2,j}\|_{L^2}^2 \lesssim \|\nabla Z\|_{L^\infty}\|Z_j\|_{L^2}^2 + \|R_j^1\|_{L^2}\|Z_j\|_{L^2}. \tag{51}
$$

For the remainder term R_j^1, using Lemma 5.4 with $w = \widetilde{A}^k(Z)$, $z = \partial_k Z$, $k = 0$, $\sigma_1 = \frac{d}{p}+2$ and $\sigma_2 = \frac{d}{p}+1$, we have (here we use our assumptions on p):

$$
\begin{aligned}
\|R_j^1\|_{L^2} \leq Cc_j 2^{-j(\frac{d}{2}+1)} \sum_{k=1}^d \Big(& \left\|\nabla\widetilde{A}^k(Z)\right\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}\|Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}}^h + \|\nabla Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-\frac{d}{p*}}}^\ell \left\|\widetilde{A}^k(Z)\right\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2}}^\ell \\
& + \|\nabla Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}\left\|\widetilde{A}^k(Z)\right\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}}^h + \|\nabla Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}}^\ell \left\|\nabla\widetilde{A}^k(Z)\right\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-\frac{d}{p*}}}^\ell \Big),
\end{aligned}
$$

whence, observing that $p^* \geq d$ and thus by (31),

$$
\|\nabla Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-\frac{d}{p*}}}^\ell \lesssim \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^\ell, \tag{52}
$$

then using the linearity of $\widetilde{A}^k$, we get

$$
\|R_j^1\|_{L^2} \leq Cc_j 2^{-j(\frac{d}{2}+1)}\left(\|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}}\|Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}}^h + \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^\ell\|Z\|_{\mathbb{B}_{p,1}^{\frac{d}{p}+2}}^\ell\right). \tag{53}
$$

Step 2: Cross estimates. To recover the dissipation for all the components of Z, we have to look at the time derivative of $\mathcal{I}_j$ defined in (49).

To start with, let us rewrite (6) as follows:

$$
\partial_t Z + \sum_{k=1}^d \bar{A}^k\partial_k Z + LZ = -\sum_{k=1}^d \widetilde{A}^k(Z)\partial_k Z. \tag{54}
$$

Hence, localizing the above equation yields

$$
\partial_t Z_j + \sum_{k=1}^d \bar{A}^k\partial_k Z_j + LZ_j = G_j \triangleq -\sum_{k=1}^d \widetilde{A}^k(Z)\partial_k Z_j + \sum_{k=1}^d [\widetilde{A}_k(Z), \dot{\Delta}_j]\partial_k Z. \tag{55}
$$

Following the computations we did in [10] leads for a suitable choice of $\varepsilon_1, \cdots, \varepsilon_{n-1}$ to

$$(56) \qquad \frac{d}{dt}\mathcal{I}_j + \frac{2^j}{2}\sum_{q=1}^{n-1}\varepsilon_q\int_{\mathbb{R}^d}|LM_\omega^q\widehat{Z}_j|^2\,d\xi \leq \frac{2^{-j}\kappa_0}{2}\|LZ_j\|_{L^2}^2 + C\|\dot{\Delta}_jG\|_{L^2}\|Z_j\|_{L^2}.$$

We have

$$2^{j\frac{d}{2}}\|G_j\|_{L^2} \lesssim \|\widetilde{A}^k(Z)\|_{L^\infty}\big(2^{j\frac{d}{2}}\|\nabla Z_j\|_{L^2}\big) + \sum_{k=1}^d 2^{j\frac{d}{2}}\|[\widetilde{A}_k(Z),\dot{\Delta}_j]\partial_k Z\|_{L^2}.$$

Hence, using the embedding $\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty$, applying Lemma 5.4 with $k = 0$, $\sigma_1 = d/p + 2$ and $\sigma_2 = \frac{d}{p} + 1$ and remembering that all the $\widetilde{A}^k$'s are linear, we get

$$\sum_{j\geq J_1} 2^{j\frac{d}{2}}\|G_j\|_{L^2} \lesssim \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}\|\nabla Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}}^h + \|\nabla Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}\|Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}}^h$$
$$+ \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-\frac{d}{p*}}}^\ell\|\nabla Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}}^\ell + \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2}}^\ell\|\nabla Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-\frac{d}{p*}}}^\ell,$$

whence, owing to (52),

$$(57) \qquad \|G\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}}^h \lesssim \|Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}}^h\big(\|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^\ell + \|Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}}^h\big) + \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^\ell\|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2}}^\ell.$$

Remember that since the (SK) condition is satisfied, the quantity $\mathcal{N}_{\bar{V}}$ defined in (27) is positive for any choice of positive parameters $\varepsilon_0, \cdots, \varepsilon_{n-1}$. Consequently, if we set

$$\mathcal{H}_j := \frac{\kappa_0}{2}\|LZ_j\|^2 + \eta\sum_{q=1}^{n-1}\varepsilon_q\int_{\mathbb{R}^d}|LM_\omega^q\widehat{Z}_j|^2\,d\xi$$

and use Fourier-Plancherel theorem and that $\mathcal{L}_j \simeq \|Z_j\|_{L^2}$, we see that (up to a change of κ_0 and choosing η small enough to kill the first term of the right-hand side of (56)), we have for all $j \geq J_1$,

$$(58) \qquad\qquad\qquad\qquad \mathcal{H}_j \geq \kappa_0\mathcal{L}_j.$$

Combining Inequalities (51) and (53), the cross estimate (56), (57) and (58), we get

$$\frac{d}{dt}2^{j(\frac{d}{2}+1)}\mathcal{L}_j + \kappa_0 2^{j(\frac{d}{2}+1)}\mathcal{L}_j \leq Cc_j\Big(\|(\nabla Z, Z)\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}\|Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}}^h\|Z_j\|_{L^2} + \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}\|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2}}^\ell\|Z_j\|_{L^2}\Big).$$

Hence, using that $\mathcal{L}_j \simeq \|Z_j\|_{L^2}$ and Lemma 5.1, and summing up on $j \geq J_1$ yields

$$(59) \quad \|Z(t)\|_{\dot{\mathcal{B}}_{2,1}^{\frac{d}{2}+1}}^h + \kappa_0\int_0^t\|Z\|_{\dot{\mathcal{B}}_{2,1}^{\frac{d}{2}+1}}^h$$

$$\leq \|Z_0\|_{\dot{\mathcal{B}}_{2,1}^{\frac{d}{2}+1}}^h + C\int_0^t\Big(\big(\|Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}}^h + \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^\ell\big)\|Z\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1}}^h + \|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}\|Z\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2}}^\ell\Big)$$

where we used the notation

$$\|Z\|_{\dot{\mathcal{B}}_{2,1}^\sigma}^h \triangleq \sum_{j\geq J_1} 2^{j\sigma}\sqrt{\mathcal{L}_j}.$$

As $\|Z\|_{\dot{\mathcal{B}}_{2,1}^\sigma}^h \simeq \|Z\|_{\dot{\mathbb{B}}_{2,1}^\sigma}^h$, this completes the proof of the proposition. $\qquad\square$

3.1.3. *The final a priori estimate.* As a first, observe that (45) implies that

$$\|W - Z_2\|^h_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \lesssim \|\nabla Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \sum_{m=1}^{2} \big(\|\widetilde{A}^k_{2,m}(Z)\partial_k Z^\ell_m\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|\widetilde{A}^k_{2,1}(Z)\partial_k Z^h_m\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \big).$$

Hence, using product laws in Besov spaces (that is Proposition 5.3), embedding, the smallness condition (32) and (34), we get

$$\|W - Z_2\|^h_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \lesssim \|\nabla Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} + \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$$

(60)
$$\lesssim \|\nabla Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} + \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}.$$

Let us introduce the functionals

(61) $\quad \mathcal{L} \triangleq \|(Z,W)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|^h_{\mathcal{B}^{\frac{d}{2}+1}_{2,1}} \quad$ and $\quad \mathcal{H} \triangleq \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z_1\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} + \|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|W\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}.$

As seen in (46), $\mathcal{L}$ is equivalent to $\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}.$

Adding up the inequality of Proposition 3.1 with Inequalities (59) and (60), remembering (32) and using several times the fact that

$$\|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \lesssim \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}},$$

we get for all $t \in [0,T]$,

$$\mathcal{L}(t) + \int_0^t \mathcal{H} \lesssim \|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|(W_0, Z_0)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \int_0^t (\|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}) \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}$$

$$+ \int_0^t \|Z\|^2_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|Z_2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1} \cap \dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \|W\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}.$$

Owing to the definition of $\mathcal{L}$, an obvious embedding and (32), the above inequality may be simplified into:

(62) $\quad \mathcal{L}(t) + \int_0^t \mathcal{H} \lesssim \|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|(W_0, Z_0)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \int_0^t \big(\|W\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \big) \mathcal{L}$

$$+ \int_0^t (\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}})^2 + \int_0^t \|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}.$$

On the one hand, by interpolation and (31), we have

$$(\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}})^2 \leq \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} \lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \big(\|Z_1\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}} + \|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \big) \lesssim \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \mathcal{H},$$

and the time integral of this term may thus be absorbed by the left-hand side of (62).

On the other hand, (46) guarantees that

$$\|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \lesssim \|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \big(\|Z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}\big)^2 + \big(\|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}\big)^2 + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}.$$

The second term of the right-hand side may be handled as above, and we thus end up owing to (32) with

$$\|Z_2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \lesssim \big(\|W\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}\big) \mathcal{L} + \|Z\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \mathcal{H}.$$

Hence, reverting to (62), we conclude that

$$\mathcal{L}(t) + \int_0^t \mathcal{H} \leq C \bigg(\|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|(W_0, Z_0)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \int_0^t \mathcal{H}\mathcal{L} \bigg).$$

It is now clear that if $\|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|(W_0, Z_0)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$ or, equivalently, $\|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$ is small enough, then we have

$$(63) \qquad \mathcal{L}(t) + \int_0^t \mathcal{H} \le C\big(\|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\big) \quad \text{for all } t \in [0, T].$$

In order to complete the proof of the estimate in Theorem 1.1, it suffices to observe that, in light of (46), one can recover a L^2-in-time control of Z_2 as follows:

$$\|Z_2\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{2,1})} \lesssim \|W\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \|\nabla Z\|^\ell_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \|Z\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}\|\nabla Z\|^h_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}$$

$$\lesssim \big(\|W\|^\ell_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}\|W\|^\ell_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}\big)^{1/2} + \big(\|Z_1\|^\ell_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}\|Z_1\|^\ell_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})}\big)^{1/2}$$

$$+ \|Z\|_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}\big(\|\nabla Z\|^h_{L^\infty_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}\|\nabla Z\|^h_{L^1_T(\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1})}\big)^{1/2}.$$

Hence we have

$$(64) \qquad \|Z_2\|_{L^2_T(\dot{\mathbb{B}}^{\frac{d}{p}}_{2,1})} \le C\big(\|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \|Z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}\big).$$

3.2. **Proof of the existence part of Theorem 1.1.** Proving the existence of a global solution under the hypotheses of Theorem 1.1 is an adaptation of [11] to the multi-dimensional case. First, we multiply the low frequencies of the data by a cut-off function in order to have a converging sequence $(Z_0^n)_{n\in\mathbb{N}}$ of approximate data in the *nonhomogeneous* Besov space $\mathbb{B}^{\frac{d}{2}+1}_{2,1}$. This enables us to take advantage of a classical existence statement (recalled in Appendix) to construct a sequence $(Z^n)_{n\in\mathbb{N}}$ of solutions to (6) with $\varepsilon = 1$ and initial data $(Z_0^n)_{n\in\mathbb{N}}$. Then, from the a priori estimates of the previous subsection, embedding and a continuation criterion, we gather that the approximate solutions are actually global and that $(Z_n)_{n\in\mathbb{N}}$ is bounded in the space E_p (that is $E_p^{J_\varepsilon}$ with $\varepsilon = 1$). At this stage, one may use compactness arguments in the spirit of Aubin-Lions lemma so as to prove convergence, up to subsequence to a global solution of (6) supplemented with initial data Z_0, with the desired properties.

First step: Construction of approximate solutions. Let Z_0 be such that $Z_0^\ell \in \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$ and $Z_0^h \in \dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}$. Since Z_0 need not be in $\mathbb{B}^{\frac{d}{2}+1}_{2,1}$, we set for all $n \ge 1$,

$$Z_0^n \triangleq \chi_n \dot{S}_{J_1-5} Z_0 + (\text{Id} - \dot{S}_{J_1-5}) Z_0 \quad \text{with} \quad \chi_n \triangleq \chi(n^{-1}\cdot),$$

where χ stands (for instance) for a smooth function with range in $[0, 1]$, supported in $]-4/3, 4/3[$ and such that $\chi \equiv 1$ on $[-3/4, 3/4]$.

It is obvious that Z_0^n tends to Z_0 in the sense of distributions, when n tends to infinity. Moreover, as Z_0^ℓ is in $\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$, the low frequencies of the data are in L^∞, and the spatial truncation thus guarantees that $Z_0^n \in \mathbb{B}^{\frac{d}{2}+1}_{2,1}$. Furthermore, an easy adaptation of the proof of the one-dimensional case in [11] reveals that

$$(65) \qquad \|Z_0^n\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z_0^n\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} \lesssim \|Z_0\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} + \|Z_0\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}}.$$

Now, applying Theorem 5.1, we get a unique maximal solution Z^n in $\mathcal{C}([0, T_n[; \mathbb{B}^{\frac{d}{2}+1}_{2,1}) \cap \mathcal{C}^1([0, T_n[; \mathbb{B}^{\frac{d}{2}}_{2,1})$ to System (1).

Second step: Uniform estimates. Since, for all $T > 0$, the space $\mathcal{C}([0,T]; \mathbb{B}_{2,1}^{\frac{d}{2}+1}) \cap \mathcal{C}^1([0,T]; \mathbb{B}_{2,1}^{\frac{d}{2}})$ is included in our 'solution space' $E_p(T)$ (that is, E_p restricted to $[0,T]$), one can take advantage of the computations for the previous sequence to bound our sequence. In the end, denoting

$$X_p^n(t) \triangleq \|Z^n\|_{L_t^\infty(\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1})}^h + \|Z^n\|_{L_t^\infty(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})}^\ell + \|Z^n\|_{L_t^1(\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}+1})}^h + \|Z_1^n\|_{L_t^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2})}^\ell$$

$$+ \|Z_2^n\|_{L_t^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1})}^\ell + \|W^n\|_{L_t^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})}^\ell + \|Z_2^n\|_{L_t^2(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})}^\ell,$$

we get thanks to (63), (64) and (65),

(66) $$X_p^n(t) \leq C X_{p,0} \quad \text{for all} \quad t \in [0, T_n[.$$

In order to show that the above inequality implies that the solution is global (namely that $T_n = \infty$), one can argue by contradiction, assuming that $T_n < \infty$, and use the blow-up criterion of Theorem 5.1. However, we first have to justify that the nonhomogeneous Besov norm $\mathbb{B}_{2,1}^{\frac{d}{2}+1}$ of the solution is under control *up to time* T_n. Using the classical energy method for (6) and the Gronwall lemma, we get that for all $t < T_n$,

$$\|Z^n(t)\|_{L^2} \leq C \|Z_0^n\|_{L^2} \exp\left(C \int_0^t \|\nabla Z^n\|_{L^\infty} \right).$$

Since (66) and the embedding of $\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}$ and $\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}$ in L^∞ ensure that ∇Z^n is in $L_{T_n}^1(L^\infty)$, we do have $Z^n \in L_{T_n}^\infty(L^2)$. Combining with (66) yields $L_{T_n}^\infty(\mathbb{B}_{2,1}^{\frac{d}{2}+1})$.

It is now easy to conclude : for all $t_{0,n} \in [0, T_n[$, Theorem 5.1 provides us with a solution of (TM) with the initial data $Z(t_{0,n}$ on $[t_{0,n}, T + t_{0,n}]$ for some T that may be bounded from below independently of $t_{0,n}$. Consequently, choosing $t_{0,n}$ such that $t_{0,n} > T_n - T$, we see that the solution Z^n can be extended beyond T_n, which contradicts the maximality of T_n. Hence $T_n = \infty$ and the solution corresponding to the initial data Z_0^n is global in time and satisfies (66) for all time.

Third step: Convergence. In order to show that $(Z^n)_{n \in \mathbb{N}}$ tends, up to subsequence, to some $Z \in E_p$, in the sense of distributions, that satisfies (6), one can use Ascoli Theorem and suitable compact embeddings in a similar fashion as in [11]. We omit the details here.

3.3. **Uniqueness.** Since our functional framework is not the standard one for the low frequencies of the solution, one cannot follow the classical energy method like in e.g. [10]. Here, for all $T > 0$, we shall estimate $\widetilde{Z} := Z^1 - Z^2$ in the space

(67) $$F_p(T) \triangleq \left\{ Z^\ell \in \mathcal{C}([0,T]; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-\frac{d}{p*}}) \ : \ Z^h \in \mathcal{C}([0,T]; \dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}) \right\}.$$

The reason for the exponent $d/2$ for high frequencies is the usual loss of one derivative when proving stability estimates for quasilinear hyperbolic systems. The exponent for low frequencies looks to be the best one for controlling the nonlinearities. To prove the uniqueness, we will need to following lemma.

Lemma 3.2. *Let Z^1 and Z^2 be two solutions of* (6) *on $[0,T]$ supplemented by initial data Z_0^1 and Z_0^2, respectively. Then, $\widetilde{Z} \triangleq Z^1 - Z^2$ satisfies the following a priori estimate for all $0 \leq t \leq T$:*

$$\left\|\widetilde{Z}\right\|_{L_t^\infty(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-\frac{d}{p*}})}^\ell + \left\|\widetilde{Z}\right\|_{L_t^\infty(\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}})}^h \lesssim \left\|\widetilde{Z}_0\right\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-\frac{d}{p*}}}^\ell + \left\|\widetilde{Z}_0\right\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}}^h$$

$$+ \int_0^t \left(\|(Z^1, Z^2)\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}}^\ell + \|(\nabla Z^1, \nabla Z^2)\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}}^h \right) \left(\|\widetilde{Z}\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-\frac{d}{p*}}}^\ell + \|\widetilde{Z}\|_{\dot{\mathbb{B}}_{2,1}^{\frac{d}{2}}}^h \right).$$

Proof. We have to proceed differently for estimating the low and the high frequencies of $\widetilde{Z}$.

Step 1: Estimates for the low frequencies. Let $V^1 \triangleq \bar{V} + Z^1$ and $V^2 \triangleq \bar{V} + Z^2$. Observe that $\widetilde{Z}$ is a solution of

$$\partial_t \widetilde{Z} + L\widetilde{Z} + \sum_{k=1}^{d} A^k(V^1)\partial_k \widetilde{Z} = \sum_{k=1}^{d} \left(A^k(V^2) - A^k(V^1) \right)\partial_k Z^2.$$

Applying $\dot{\Delta}_j$, taking the scalar product with $|\widetilde{Z}_j|^{p-2}\widetilde{Z}_j$, integrating on $\mathbb{R}_+ \times \mathbb{R}^d$ and using (7) and Lemma 5.1, we get for all $j \in \mathbb{Z}$,

$$\left\| \widetilde{Z}_j(t) \right\|_{L^p} + \kappa_0 \int_0^t \left\| L\widetilde{Z}_j \right\|_{L^p} \leq \left\| \widetilde{Z}_{0,j} \right\|_{L^p} + \sum_{k=1}^{d} \int_0^t \left\| \nabla A^k(V^1) \right\|_{L^\infty} \left\| \widetilde{Z}_j \right\|_{L^p}$$

$$+ \int_0^t \left\| \dot{\Delta}_j \sum_{k=1}^{d} (A^k(V^2) - A^k(V^1))\partial_k Z^2 \right\|_{L^p} + \int_0^t \sum_{k=1}^{d} \|[\dot{\Delta}_j, A^k(V^1)]\partial_k \widetilde{Z}\|_{L^p}.$$

Multiplying this inequality by $2^{j(\frac{d}{p} - \frac{d}{p*})}$ and using the embedding $\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}} \hookrightarrow L^\infty$ as well as the commutator estimate (96), we get

$$(68) \quad 2^{j(\frac{d}{p} - \frac{d}{p*})}\left\| \widetilde{Z}_j(t) \right\|_{L^p} \leq 2^{j(\frac{d}{p} - \frac{d}{p*})}\left\| \widetilde{Z}_{0,j} \right\|_{L^p} + Cc_j \int_0^t \|\nabla Z^1\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|\widetilde{Z}\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p} - \frac{d}{p*}}}$$

$$+ \int_0^t 2^{j(\frac{d}{p} - \frac{d}{p*})}\left\| \dot{\Delta}_j \sum_{k=1}^{d} \left(A^k(V^2) - A^k(V^1) \right) \partial_k Z^2 \right\|_{L^p}.$$

Therefore, summing up on $j \leq J_1$, we arrive at

$$(69) \quad \left\| \widetilde{Z}(t) \right\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p} - \frac{d}{p*}}}^{\ell} \leq \left\| \widetilde{Z}_0 \right\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p} - \frac{d}{p*}}}^{\ell} + C \int_0^t \|\nabla Z^1\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|\widetilde{Z}\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p} - \frac{d}{p*}}}$$

$$+ C \int_0^t \left\| \sum_{k=1}^{d} \left(A^k(V^2) - A^k(V^1) \right) \partial_k Z^2 \right\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p} - \frac{d}{p*}}}^{\ell}.$$

Since $A^k(V^2) - A^k(V^1) = \widetilde{A}^k(\widetilde{Z})$, bounding the last term just follows from Inequality (98) with $s = d/p - d/p^*$, namely

$$(70) \quad \|a\,b\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p} - \frac{d}{p*}}} \lesssim \|a\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|b\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p} - \frac{d}{p*}}}$$

We conclude that

$$(71) \quad \left\| \widetilde{Z}(t) \right\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p} - \frac{d}{p*}}}^{\ell} \lesssim \left\| \widetilde{Z}_0 \right\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p} - \frac{d}{p*}}}^{\ell} + \int_0^t \|(\nabla Z^1, \nabla Z^2)\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|\widetilde{Z}\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p} - \frac{d}{p*}}}.$$

Step 2: Estimates for the high frequencies. For all $j \in \mathbb{Z}$, the function $\widetilde{Z}_j$ satisfies

$$\partial_t \widetilde{Z}_j + \sum_{k=1}^{d} \dot{S}_{j-1} A^k(V^1)\partial_k \widetilde{Z}_j + L\widetilde{Z}_j = \dot{\Delta}_j \left(\sum_{k=1}^{d} (A^k(V^2) - A^k(V^1))\partial_k Z^2 \right) + \sum_{k=1}^{d} R_k$$

with

$$R_k \triangleq \dot{S}_{j-1}(\widetilde{A}^k(Z^1)) \dot{\Delta}_j \partial_k \widetilde{Z} - \dot{\Delta}_j \left(\widetilde{A}^k(Z^1)\partial_k \widetilde{Z} \right).$$

Hence, performing the classical procedure, we end up with

$$\left\|\widetilde{Z}(t)\right\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \left\|\widetilde{Z}_0\right\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \int_0^t \|\nabla Z^1\|_{L^\infty} \|\widetilde{Z}\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}$$

$$+ \int_0^t \left\|\sum_{k=1}^d \left(A^k(V^2) - A^k(V^1)\right)\partial_k Z^2\right\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \int_0^t \sum_{j \geq J_1} 2^{j\frac{d}{2}} \|\sum_{k=1}^d R_k\|_{L^2}.$$

Applying Lemma 5.4 to $w = \widetilde{A}^k(Z^1)$ and $z = \partial_k \widetilde{Z}$ with $s = \frac{d}{2}$, $k = 1$, $\sigma_1 = \frac{d}{p}+2$ and $\sigma_2 = \frac{d}{p}+1$, and remembering that all the maps $V \mapsto A^k(V)$ are linear, we get

$$\sum_{j \geq J_1}\left(2^{j\frac{d}{2}} \|\sum_{k=1}^d R_k\|_{L^2}\right) \lesssim \|\nabla Z^1\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \left\|\nabla\widetilde{Z}\right\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}-1}_{2,1}} + \left\|\nabla\widetilde{Z}\right\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} \|Z^1\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}}$$

$$+ \left\|\nabla\widetilde{Z}\right\|_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}} \|Z^1\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1}} + \left\|\nabla\widetilde{Z}\right\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} \|\nabla Z^1\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}}.$$

Using (101) with $a = \partial_k Z^2$, $b = A^k(V^2) - A^k(V^1)$, $s = \frac{d}{2}$ and $\sigma = \frac{d}{p}+1$ yields

$$\|\sum_{k=1}^d \left(A^k(V^2) - A^k(V^1)\right)\partial_k Z^2\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}$$

$$\lesssim \|\nabla Z^2\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\widetilde{Z}\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|\widetilde{Z}\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\nabla Z^2\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \|\nabla Z^2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} \|\widetilde{Z}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \|\widetilde{Z}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} \|\nabla Z^2\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}}.$$

Gathering the above estimates and using once more the embedding $\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1} \hookrightarrow L^\infty$, we obtain

$$\left\|\widetilde{Z}(t)\right\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \left\|\widetilde{Z}_0\right\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \int_0^t \|(\nabla Z^1, \nabla Z^2)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\widetilde{Z}\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} + \int_0^t \|(\nabla Z^1, \nabla Z^2)\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\widetilde{Z}\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$$

(72)
$$+ \int_0^t \|(Z^1, Z^2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\widetilde{Z}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|\widetilde{Z}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} \|(Z^1, Z^2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}}.$$

Step 3: Conclusion. Summing (71) and (72) together, we get for all $t \geq 0$,

$$\left\|\widetilde{Z}(t)\right\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} + \left\|\widetilde{Z}(t)\right\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \lesssim \left\|\widetilde{Z}_0\right\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} + \left\|\widetilde{Z}_0\right\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}$$

$$+ \int_0^t \|(\nabla Z^1, \nabla Z^2)\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \left(\|\widetilde{Z}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} + \|\widetilde{Z}\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}}\right) + \int_0^t \|(\nabla Z^1, \nabla Z^2)\|^h_{\dot{\mathbb{B}}^{\frac{d}{2}}_{2,1}} \|\widetilde{Z}\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}}$$

$$+ \int_0^t \|(Z^1, Z^2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|\widetilde{Z}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1}} + \int_0^t \|\widetilde{Z}\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} \|(Z^1, Z^2)\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1}},$$

which, by virtue of (34) yields the desired estimate of Lemma 3.2.   □

In order to prove the uniqueness part of Theorem 1.1, consider two solutions Z^1 and Z^2 of (6) (not necessarily small) in the space E_p, that correspond to the same initial data Z_0. Then, the result follows from Lemma 3.2, *provided we prove that the difference between the two solutions belongs to $F_p(T)$ for all $T > 0$.*

Just denoting by Z one of those two solutions, we have

(73)
$$\partial_t Z = -\sum_{k=1}^d A^k(V)\partial_k Z - LZ.$$

By interpolation in Besov spaces and Hölder inequality with respect to the time variable, since Z^ℓ is in $L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2})$, we get

$$(74) \qquad \nabla Z^\ell \in L^r(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-\frac{d}{p*}}) \quad \text{with} \quad \frac{1}{r} \triangleq \frac{1}{2} - \frac{d}{4} + \frac{d}{2p}.$$

The same property holds for Z^h since it belongs to $L^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1}) \cap L^\infty(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1})$. We also know that $A^k(V) - \bar{A}^k$ is in $L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})$. Therefore, from the product laws in Besov spaces that have been recalled in Proposition 5.3, we have that $\partial_t Z_1$ is in $L^r(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-\frac{d}{p*}})$, and thus

$$(75) \qquad Z_1 - Z_{1,0} \in \mathcal{C}_{loc}^{\frac{1}{r'}}(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-\frac{d}{p*}}).$$

We conclude that $Z_1 - Z_{1,0}$ is in $F_p(T)$ for all finite T.

Owing to the 0-th order term LZ in the equation, in order to justify that $(Z_2 - Z_{2,0}) \in F_p(T)$, we have to proceed slightly differently. Now, we notice that

$$\partial_t(e^{tL_2} Z_2) = -e^{tL_2}\left(\sum_{k=1}^d A_{2,1}^k(V)\partial_k Z_1 + A_{2,2}^k(V)\partial_k Z_2\right).$$

Arguing as above, we see that the right-hand side is in $L^r(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-\frac{d}{p*}})$, which, as before, allows to conclude that $Z_2 - Z_{2,0} \in \mathcal{C}_{loc}^{\frac{1}{r'}}(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-\frac{d}{p*}})$.

Back to our two solutions Z_1 and Z_2, since they coincide initially, the above arguments ensure that $Z_1 - Z_2$ is in $F_p(T)$. Hence, combining Lemma 3.2, Gronwall lemma and the fact that the low frequencies of Z^1 and Z^2 (resp. the high frequencies of ∇Z^1 and ∇Z^2) are bounded in $L^1(0, T; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})$ (resp. in $L^1(0, T; \dot{\mathbb{B}}_{2,1}^{\frac{d}{2}})$) for all $T > 0$ completes the proof of uniqueness.

4. Relaxation limit for the compressible Euler system

In this section we prove Theorem 1.3. We shall often use that, as a consequence of (30), (31) and of the definition of J_ε, there exists $C > 0$ such that for all $s \in \mathbb{R}$ and $\varepsilon > 0$,

$$(76) \qquad \|f\|_{\dot{\mathbb{B}}_{p,1}^s}^{\ell, J_\varepsilon} \leq \frac{C}{\varepsilon}\|f\|_{\dot{\mathbb{B}}_{p,1}^{s-1}}^{\ell, J_\varepsilon} \quad \text{and} \quad \|f\|_{\dot{\mathbb{B}}_{2,1}^s}^{h, J_\varepsilon} \leq C\varepsilon\|f\|_{\dot{\mathbb{B}}_{2,1}^{s+1}}^{h, J_\varepsilon}.$$

4.1. Reformulation of the problem and derivation of the limit system. Let (c, v) be a solution from Theorem 1.2. As in [20, 9], we perform the following 'diffusive' rescaling:

$$(\tilde{c}^\varepsilon, \tilde{v}^\varepsilon)(\tau, x) \triangleq (c, \frac{v}{\varepsilon})(t, x) \quad \text{with} \quad \tau = \varepsilon t.$$

The couple $(\tilde{c}^\varepsilon, \tilde{v}^\varepsilon)$ satisfies:

$$(77) \qquad \begin{cases} \partial_t \tilde{c}^\varepsilon + \tilde{v}^\varepsilon \cdot \nabla \tilde{c}^\varepsilon + \tilde{\gamma}\tilde{c}^\varepsilon \text{div}\,\tilde{v}^\varepsilon = 0, \\ \varepsilon^2\left(\partial_t \tilde{v}^\varepsilon + \tilde{v}^\varepsilon \cdot \nabla \tilde{v}^\varepsilon\right) + \check{\gamma}\tilde{c}^\varepsilon \nabla \tilde{c}^\varepsilon + \tilde{v}^\varepsilon = 0. \end{cases}$$

As a consequence of Theorem 1.2 and of (76), we readily get the following uniform estimate[3] which will be a key ingredient in our study of the relaxation limit:

$$
\|\widetilde{c}^\varepsilon - \bar{c}\|^{\ell,J_\varepsilon}_{L^\infty(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \varepsilon\,\|\widetilde{v}^\varepsilon\|^{\ell,J_\varepsilon}_{L^\infty(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \varepsilon\,\|\widetilde{c}^\varepsilon - \bar{c}\|^{h,J_\varepsilon}_{L^\infty(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \varepsilon^2\,\|\widetilde{v}^\varepsilon\|^{h,J_\varepsilon}_{L^\infty(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|\widetilde{c}^\varepsilon - \bar{c}\|^{\ell,J_\varepsilon}_{L^1(\dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})}
$$

$$
+\frac{1}{\varepsilon}\,\|\widetilde{c}^\varepsilon - \bar{c}\|^{h,J_\varepsilon}_{L^1(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|\widetilde{v}^\varepsilon\|^{h,J_\varepsilon}_{L^1(\dot{\mathbb{B}}^{\frac{d}{2}+1}_{2,1})} + \|\widetilde{v}^\varepsilon\|^{\ell,J_\varepsilon}_{L^1(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} + \|\widetilde{c}^\varepsilon - \bar{c}\|_{L^2(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} + \|\widetilde{v}^\varepsilon\|_{L^2(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}
$$

$$
(78) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\frac{1}{\varepsilon}\,\left\|\widetilde{W}^\varepsilon\right\|_{L^1(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \le C\eta_0
$$

where $\widetilde{W}^\varepsilon = \check{\gamma}\widetilde{c}^\varepsilon\nabla\widetilde{c}^\varepsilon + \widetilde{v}^\varepsilon$ and $J_\varepsilon = -\lfloor \log_2(\varepsilon)\rfloor + k_p$ for some $k_p \in \mathbb{Z}$.

Let us define the density $\widetilde{\rho}^\varepsilon$ and reference density $\bar{\rho}$ from (16). Then, $(\widetilde{\rho}^\varepsilon, \widetilde{v}^\varepsilon)$ obeys the following system:

$$
(79) \qquad
\begin{cases}
\partial_t\widetilde{\rho}^\varepsilon + \operatorname{div}\left(\widetilde{\rho}^\varepsilon\widetilde{v}^\varepsilon\right) = 0, \\[2mm]
\varepsilon^2\left(\partial_t\widetilde{v}^\varepsilon + \widetilde{v}^\varepsilon\cdot\nabla\widetilde{v}^\varepsilon\right) + \dfrac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon} + \widetilde{v}^\varepsilon = 0.
\end{cases}
$$

Owing to (78), $\varepsilon\widetilde{v}^\varepsilon$ and $\nabla\widetilde{v}^\varepsilon$ are uniformly bounded in the spaces $L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})$ and $L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})$, respectively. This implies that

$$
\varepsilon^2\widetilde{v}^\varepsilon\cdot\nabla\widetilde{v}^\varepsilon = \mathcal{O}(\varepsilon) \quad \text{in} \quad L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}).
$$

The uniform estimate (78) also implies that $\varepsilon^2\partial_t\widetilde{v}^\varepsilon$ tends to 0 in the sense of distributions. Plugging this information in the second equation of (79), one may conclude that

$$
(80) \qquad\qquad \widetilde{v}^\varepsilon + \frac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon} \rightharpoonup 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^d).
$$

Let us remember that

$$
(81) \qquad\qquad \widetilde{\rho}^\varepsilon - \bar{\rho} = \left(\frac{\gamma-1}{\sqrt{4A\gamma}}\,\widetilde{c}^\varepsilon\right)^{\frac{2}{\gamma-1}} - \left(\frac{\gamma-1}{\sqrt{4A\gamma}}\,\bar{c}\right)^{\frac{2}{\gamma-1}}.
$$

From (76) and (78), it is easy to see that

$$
\|\widetilde{c}^\varepsilon - \bar{c}\|_{L^\infty(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \|\widetilde{c}^\varepsilon - \bar{c}\|_{L^2(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} \le \eta_0.
$$

Hence, using Proposition 5.4 and (81) gives

$$
(82) \qquad\qquad \|\widetilde{\rho}^\varepsilon - \bar{\rho}\|_{L^\infty(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \|\widetilde{\rho}^\varepsilon - \bar{\rho}\|_{L^2(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} \le \eta_0.
$$

In particular $\widetilde{\rho}^\varepsilon - \bar{\rho}$ is uniformly bounded in $L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})$. Therefore, there exists $\mathcal{N}$ in $\bar{\rho} + L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})$ such that, up to subsequence,

$$
(83) \qquad\qquad \widetilde{\rho}^\varepsilon - \bar{\rho} \overset{*}{\rightharpoonup} \mathcal{N} - \bar{\rho} \quad \text{in} \quad L^\infty(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}).
$$

Now, observing that

$$
(84) \qquad\qquad \widetilde{\rho}^\varepsilon\widetilde{W}^\varepsilon = \nabla P(\widetilde{\rho}^\varepsilon) + \widetilde{\rho}^\varepsilon\widetilde{v}^\varepsilon,
$$

the first equation of (79) may be rewritten

$$
(85) \qquad\qquad \partial_t\widetilde{\rho}^\varepsilon - \Delta P(\widetilde{\rho}^\varepsilon) = \widetilde{S}^\varepsilon \quad \text{with} \quad \widetilde{S}^\varepsilon = -\operatorname{div}\left(\widetilde{\rho}^\varepsilon\widetilde{W}^\varepsilon\right).
$$

_____

[3]The crucial bound on $\|\widetilde{c}^\varepsilon - \bar{c}\|_{L^2(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})}$ can be easily deduced from the other bounds.

Hence, combining (80), (83) and (84), it can be anticipated that $\mathcal{N}$ satisfies

$$(86) \qquad \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0.$$

4.2. **Proving the strong convergence to the limit system.** Having determined the limit system, we are now going to use the uniform estimate (78) to prove the strong convergence of the density to a solution of (86), with an explicit rate of convergence. As a first, let us remember that (78) and (82) imply that[4]

$$(87) \qquad \|\widetilde{\rho}^\varepsilon - \bar{\rho}\|_{L^\infty(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} + \|\widetilde{\rho}^\varepsilon - \bar{\rho}\|_{L^2(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} + \varepsilon^{-1} \left\|\widetilde{W}^\varepsilon\right\|_{L^1(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \leq \eta_0.$$

Next, setting $\bar{\mathcal{N}} = \bar{\rho}$, Proposition 5.1 below guarantees that Equation (86) supplemented with any initial data $\mathcal{N}_0$ such that $\mathcal{N}_0 - \bar{\mathcal{N}} \in \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}$ is small enough admits a unique global solution $\mathcal{N}$ such that $\mathcal{N} - \bar{\mathcal{N}} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})$.

We can now prove Theorem 1.3, assuming that $\varepsilon > 0$ is small, and that

$$(88) \qquad \|\widetilde{\rho}_0^\varepsilon - \mathcal{N}_0\|_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}} \leq \varepsilon.$$

To derive the convergence rate we will estimate the difference of the solutions to the following two equations:

$$(89) \qquad \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0$$

and

$$(90) \qquad \partial_t \widetilde{\rho}^\varepsilon + \text{div}\,(\widetilde{\rho}^\varepsilon \widetilde{v}^\varepsilon) = 0.$$

Recall that (90) may be rewritten in terms of the damped mode $\widetilde{W}^\varepsilon$ as in (85). Hence $\delta D^\varepsilon \triangleq \widetilde{\rho}^\varepsilon - \mathcal{N}$ satisfies

$$\partial_t \delta D^\varepsilon - \Delta(P(\widetilde{\rho}^\varepsilon) - P(\mathcal{N})) = \widetilde{S}^\varepsilon.$$

In light of Taylor formula, there exists a smooth function H_1 vanishing at $\bar{\rho} = \bar{\mathcal{N}}$ such that

$$P(\widetilde{\rho}^\varepsilon) - P(\bar{\rho}) = P'(\bar{\rho})\,(\widetilde{\rho}^\varepsilon - \bar{\rho}) + H_1(\widetilde{\rho}^\varepsilon)\,(\widetilde{\rho}^\varepsilon - \bar{\rho}).$$

and

$$P(\mathcal{N}) - P(\bar{\mathcal{N}}) = P'(\bar{\mathcal{N}})\,(\mathcal{N} - \bar{\mathcal{N}}) + H_1(\mathcal{N})\,(\mathcal{N} - \bar{\mathcal{N}}).$$

Hence we have

$$\partial_t \delta D^\varepsilon - P'(\bar{\rho})\Delta \delta D^\varepsilon = \Delta\,(\delta D^\varepsilon\,H_1(\widetilde{\rho}^\varepsilon)) + \Delta\,((H_1(\widetilde{\rho}^\varepsilon) - H_1(\mathcal{N}))\mathcal{N}) + \widetilde{S}^\varepsilon.$$

Then, using endpoint maximal regularity estimates for the heat equation (see e.g. [1]) yields

$$(91) \quad \|\delta D^\varepsilon\|_{L^\infty(\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} + \|\delta D^\varepsilon\|_{L^1(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} \lesssim \|\delta D_0^\varepsilon\|_{\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1}} + \|\widetilde{S}^\varepsilon\|_{L^1(\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})}$$
$$\|\delta D^\varepsilon\,(H_1(\widetilde{\rho}^\varepsilon) - H_1(\bar{\rho}))\|_{L^1(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} + \|(H_1(\widetilde{\rho}^\varepsilon) - H_1(\mathcal{N}))(\mathcal{N} - \bar{\mathcal{N}})\|_{L^1(\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})}.$$

Basic product laws give us:

$$\|\widetilde{S}^\varepsilon\|_{L^1(\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} \lesssim \|\widetilde{\rho}^\varepsilon \widetilde{W}^\varepsilon\|_{L^1(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \lesssim \|\widetilde{W}^\varepsilon\|_{L^1(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \left(\bar{\rho} + \|\widetilde{\rho}^\varepsilon - \bar{\rho}\|_{L^\infty(\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})}\right).$$

Hence, taking advantage of Inequality (87), we get

$$(92) \qquad \|\widetilde{S}^\varepsilon\|_{L^1(\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} \leq C\varepsilon.$$

---

[4]Unless $\gamma = 3$, we do not know how to deduce specific information on the low (resp. high) frequencies of $\rho - \bar{\rho}$ from that of $c - \bar{c}$. This is due to the *nonlinear* relation between these two functions.

Propositions 5.3 and 5.4 give us

$$\|\delta D^\varepsilon \left(H_1(\widetilde{\rho}^\varepsilon) - H_1(\bar\rho)\right)\|_{L^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1})} \lesssim \|\delta D^\varepsilon\|_{L^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1})} \|(\widetilde{\rho}^\varepsilon - \bar\rho, \mathcal{N} - \bar{\mathcal{N}})\|_{L^\infty(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})},$$

$$\|(H_1(\widetilde{\rho}^\varepsilon) - H_1(\mathcal{N}))(\mathcal{N} - \bar{\mathcal{N}})\|_{L^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1})} \lesssim \|\delta D^\varepsilon\|_{L^2(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})} \|(\widetilde{\rho}^\varepsilon - \bar\rho, \mathcal{N} - \bar{\mathcal{N}})\|_{L^2(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1})}.$$

Thanks to Inequality (87) and Proposition 5.1, we have

$$\|(\widetilde{\rho}^\varepsilon - \bar\rho, \mathcal{N} - \bar{\mathcal{N}})\|_{L^\infty(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})} + \|(\widetilde{\rho}^\varepsilon - \bar\rho, \mathcal{N} - \bar{\mathcal{N}})\|_{L^2(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1})} \le \eta_0 \ll 1.$$

Hence, reverting to (91) yields

$$\|\delta D^\varepsilon\|_{L^\infty(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-1})} + \|\delta D^\varepsilon\|_{L^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+1})} \lesssim \|\delta D_0^\varepsilon\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-1}} + \varepsilon,$$

which concludes the proof of Theorem 1.3.

5. Appendix

Here we gather a few technical results that have been used repeatedly in the paper. We often used the following well known result (see e.g. [11] for the proof).

Lemma 5.1. *Let* $p \ge 1$ *and* $X : [0, T] \to \mathbb{R}^+$ *be a continuous function such that* X^p *is a.e. differentiable. We assume that there exist a constant* $b \ge 0$ *and a measurable function* $A : [0, T] \to \mathbb{R}^+$ *such that*

$$\frac{1}{p}\frac{d}{dt}X^p + bX^p \le AX^{p-1} \quad a.e. \ on \ [0,T].$$

Then, for all $t \in [0, T]$, *we have*

$$X(t) + b\int_0^t X \le X_0 + \int_0^t A.$$

When proving Theorem 1.3, we used the following global existence result for (86).

Proposition 5.1. *Let* $1 \le p < \infty$ *and* $\mathcal{N}_0 - \bar{\mathcal{N}} \in \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}$ *with* $\bar{\mathcal{N}} > 0$. *There exists a constant* $\eta_0 > 0$ *such that if*

$$\text{(93)} \qquad \|\mathcal{N}_0 - \bar{\mathcal{N}}\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \le \eta_0$$

then, System (86) *with a pressure function* P *satisfying* (3) *and supplemented with initial data* $\mathcal{N}_0$ *has a unique global solution* $\mathcal{N}$ *such that* $\mathcal{N} - \bar{\mathcal{N}} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2})$.

Proof. Assume that we have a smooth solution $\mathcal{N}$ of (86). There exists a function H_1 vanishing at $\bar{\mathcal{N}}$ such that:

$$P(\mathcal{N}) - P(\bar{\mathcal{N}}) = P'(\bar{\mathcal{N}})(\mathcal{N} - \bar{\mathcal{N}}) + H_1(\mathcal{N})(\mathcal{N} - \bar{\mathcal{N}}).$$

Therefore one can rewrite (86) as

$$\partial_t \mathcal{N} - P'(\bar{\mathcal{N}})\Delta\mathcal{N} = \Delta\Big(H_1(\mathcal{N})(\mathcal{N} - \bar{\mathcal{N}})\Big).$$

Hence, using classical endpoint maximal regularity estimates for the heat equation (see e.g. [1]), we get for all $T > 0$,

$$\text{(94)} \quad \|\mathcal{N} - \bar{\mathcal{N}}\|_{L_T^\infty(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})} + \|\mathcal{N} - \bar{\mathcal{N}}\|_{L_T^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2})} \lesssim \|\mathcal{N}_0 - \bar{\mathcal{N}}\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} + \|H_1(\mathcal{N})(\mathcal{N} - \bar{\mathcal{N}})\|_{L_T^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2})}.$$

Combining product laws from (97) with composition estimates of Proposition 5.4 yields

$$\|H_1(\mathcal{N})(\mathcal{N} - \bar{\mathcal{N}})\|_{L_T^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2})} \lesssim \|\mathcal{N} - \bar{\mathcal{N}}\|_{L_T^\infty(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}})} \|\mathcal{N} - \bar{\mathcal{N}}\|_{L_T^1(\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}+2})}.$$

Hence the left-hand side of (94) may be bounded for all $T > 0$ in terms of the data provided (93) is satisfied with a small enough η_0. From that point, it is easy to work out a fixed point procedure yielding the global existence of a solution for (86). Uniqueness follows from similar estimates. $\qquad\square$

The first part of the existence proof relied on the following classical local well-posedness result for hyperbolic symmetric systems.

Theorem 5.1. [1, Chap. 4] *Consider the following hyperbolic system:*

$$(QS) \qquad \begin{cases} \partial_t U + \sum_{k=1}^d A_k(U)\partial_k U + A_0(U) = 0, \\ U|_{t=0} = U_0, \end{cases}$$

where A_k, $k = 0, \cdots, d$, are smooth functions from $\mathbb{R}^n$ to the space of $n \times n$ matrices, that are symmetric if $k \neq 0$, supplemented with initial data U_0 in the nonhomogeneous Besov space $\mathbb{B}_{2,1}^{\frac{d}{2}+1}(\mathbb{R}^d; \mathbb{R}^n)$.

Then, (QS) admits a unique maximal solution U in $\mathcal{C}([0, T^*[; \mathbb{B}_{2,1}^{\frac{d}{2}+1}) \cap \mathcal{C}^1([0, T^*[; \mathbb{B}_{2,1}^{\frac{d}{2}})$, and there exists a positive constant c such that

$$T^* \geq \frac{c}{\|U_0\|_{\mathbb{B}_{2,1}^{\frac{d}{2}+1}}}.$$

Furthermore,

$$T^* < \infty \implies \int_0^{T^*} \|\nabla U\|_{L^\infty} = \infty.$$

Next, let us prove the equivalence between Condition (SK) and the strong ellipticity condition for System (8) pointed out in the introduction.

Lemma 5.2. *Assume that $\bar{A}_{1,1}^k = 0$ for all $k \in \{1, \cdots, d\}$. Then, the following assertions are equivalent:*

- *System (1) satisfies the condition (SK) at $\bar{V}$;*
- *the operator $\mathcal{A} \triangleq -\sum_{k=1}^d \sum_{\ell=1}^d \bar{A}_{1,2}^k L_2^{-1} \bar{A}_{2,1}^\ell \partial_k \partial_\ell$ is strongly elliptic.*

If one of the above assertions is satisfied and if $Supp(\mathcal{F}Z_1) \subset \{\xi \in \mathbb{R}^d : R_1\lambda \leq |\xi| \leq R_2\lambda\}$ for some $0 < R_1 < R_2$ then, for all $p \in [2, \infty[$, there exists $c = c(p, d, R_1, R_2) > 0$ such that

$$(95) \qquad \int_{\mathbb{R}^d} \sum_{j=1}^{n_1} \sum_{k=1}^d \sum_{\ell=1}^d \bar{A}_{1,2}^k L_2^{-1} \bar{A}_{2,1}^\ell \partial_k \partial_\ell Z_1^j \, |Z_1|^{p-2} Z_1^j \geq c\lambda^2 \|Z_1\|_{L^p}^p.$$

Proof. The direct implication was proved in [32, 24, 28]. For the converse implication, let us still denote by L_2 the $n_2 \times n_2$ (invertible) matrix of L_2 and set

$$\mathcal{A}_{\ell,m}(\xi) \triangleq \sum_{k=1}^d \bar{A}_{\ell,m}^k \xi_k, \quad 1 \leq \ell, m \leq 2.$$

Our assumptions ensure that the symmetric parts of L_2 and of the matrix $\mathcal{A}_{1,2}(\xi)L_2^{-1}\mathcal{A}_{2,1}(\xi)$ for all $\xi \neq 0$ are positive definite. This in particular implies that the ranks of $\mathcal{A}_{1,2}(\xi)$ and $\mathcal{A}_{2,1}(\xi)$ must be equal to n_1 and thus, so does the rank of $L_2\mathcal{A}_{2,1}(\xi)$. Now, the matrices of L and of $L\mathcal{A}(\xi)$ can be written by blocks as follows:

$$L = \begin{pmatrix} 0 & 0 \\ 0 & L_2 \end{pmatrix} \quad \text{and} \quad L\mathcal{A}(\xi) = \begin{pmatrix} 0 & 0 \\ L_2\mathcal{A}_{2,1}(\xi) & L_2\mathcal{A}_{2,2}(\xi) \end{pmatrix}.$$

Hence the rank of $\begin{pmatrix} L \\ L\mathcal{A}(\xi) \end{pmatrix}$ is $n_1 + n_2 = n$, and Condition (SK) is thus satisfied.

To prove Inequality (95), we first observe that L_2^{-1} may be replaced by its symmetric part (this leaves the left-hand side unchanged). Then, performing an appropriate change of orthonormal basis reduces the proof to the case where the matrix $\sum_{k=1}^{d}\sum_{\ell=1}^{d} A_{1,2}^{k} L_2^{-1} \bar{A}_{2,1}^{\ell}$ is diagonal and positive definite. From this point, one can argue exactly as in the proof of Lemma A.5 in [12]. $\qquad\square$

The proof of the following inequality may be found in e.g. [1, Chap. 2].

Lemma 5.3. *There exists a constant C such that for all $1 \leq p,q,r \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, all functions a with gradient in L^p, and b in L^q, we have*

$$\left\|[\dot{\Delta}_j, a]b\right\|_{L^r} \leq C 2^{-j} \|\nabla a\|_{L^q} \|b\|_{L^p} \quad for\ all \ \ j \in \mathbb{Z}.$$

The following result is proved in e.g. [1, Chap. 2].

Proposition 5.2. *For all $1 \leq p \leq \infty$ and $-\min(d/p, d/p') < s \leq d/p$, we have*

$$(96) \qquad 2^{js}\left\|[w, \dot{\Delta}_j]\nabla v\right\|_{L^p} \leq C c_j \|\nabla w\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|v\|_{\dot{\mathbb{B}}_{p,1}^{s}} \quad with \quad \sum_{j \in \mathbb{Z}} c_j = 1.$$

The following product laws in Besov spaces have been used several times.

Proposition 5.3. *Let (s,p,r) be in $]0,\infty[\times[1,\infty]^2$. Then, $\dot{\mathbb{B}}_{p,r}^{s} \cap L^\infty$ is an algebra and we have*

$$(97) \qquad \|ab\|_{\dot{\mathbb{B}}_{p,r}^{s}} \leq C\left(\|a\|_{L^\infty} \|b\|_{\dot{\mathbb{B}}_{p,r}^{s}} + \|a\|_{\dot{\mathbb{B}}_{p,r}^{s}} \|b\|_{L^\infty}\right).$$

If, furthermore, $-\min(d/p, d/p') < s \leq d/p$, then the following inequality holds:

$$(98) \qquad \|ab\|_{\dot{\mathbb{B}}_{p,1}^{s}} \leq C\|a\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|b\|_{\dot{\mathbb{B}}_{p,1}^{s}}.$$

Finally, if $-d/p < \sigma_1 \leq \min(d/p, d/p')$, then the following inequality holds true:

$$(99) \qquad \|ab\|_{\dot{\mathbb{B}}_{p,\infty}^{-\sigma_1}} \leq C \|a\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|b\|_{\dot{\mathbb{B}}_{p,\infty}^{-\sigma_1}}.$$

The following result for left composition can be found in [1].

Proposition 5.4. *Let $p \geq 1$ and f be a function in $\mathcal{C}^\infty(\mathbb{R})$ such that $f(0) = 0$. let $(s_1, s_2) \in]0,\infty[^2$ and $(r_1, r_2) \in [1,\infty]^2$. We assume that $s_1 < d/p$ or that $s_1 = d/p$ and $r_1 = 1$.*

Then, for every real-valued function u in $\dot{\mathbb{B}}_{p,r_1}^{s_1} \cap \dot{\mathbb{B}}_{p,r_2}^{s_2} \cap L^\infty$, the function $f \circ u$ belongs to $\dot{\mathbb{B}}_{p,r_1}^{s_1} \cap \dot{\mathbb{B}}_{p,r_2}^{s_2} \cap L^\infty$ and we have

$$\|f \circ u\|_{\dot{\mathbb{B}}_{p,r_k}^{s_k}} \leq C\left(f', \|u\|_{L^\infty}\right) \|u\|_{\dot{\mathbb{B}}_{p,r_k}^{s_k}} \quad for\ k \in \{1,2\}.$$

As a consequence (see [1, Cor. 2.66]), if g is a $\mathcal{C}^\infty(\mathbb{R})$ function such that $g'(0) = 0$, then, for all u,v in $\dot{\mathbb{B}}_{p,1}^{s} \cap L^\infty$ with $s > 0$, we have

$$(100) \qquad \|g(v) - g(u)\|_{\dot{\mathbb{B}}_{p,1}^{s}} \leq C\left(\|v - u\|_{L^\infty}\|(u,v)\|_{\dot{\mathbb{B}}_{p,1}^{s}} + \|v - u\|_{\dot{\mathbb{B}}_{p,1}^{s}}\|(u,v)\|_{L^\infty}\right).$$

We also need the following more involved product law to handle the high frequencies of some non-linear terms.

Proposition 5.5. *Let $2 \leq p \leq 4$ and $p^* \triangleq 2p/(p-2)$. For all $\sigma \geq s > 0$, we have*

$$(101) \qquad \|ab\|_{\dot{\mathbb{B}}_{2,1}^{s}}^{h} \lesssim \|a\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|b\|_{\dot{\mathbb{B}}_{2,1}^{s}}^{h} + \|b\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} \|a\|_{\dot{\mathbb{B}}_{2,1}^{s}}^{h} + \|a\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p} - \frac{d}{p^*}}}^{\ell} \|b\|_{\dot{\mathbb{B}}_{p,1}^{\sigma}}^{\ell} + \|b\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p} - \frac{d}{p^*}}}^{\ell} \|a\|_{\dot{\mathbb{B}}_{p,1}^{\sigma}}^{\ell}.$$

Proof. Recall the following so-called Bony decomposition (first introduced by J.-M. Bony in [4]) for the product of two tempered distributions f and g:

$$fg = T_f g + T_g' f \quad with \quad T_f g \triangleq \sum_{j \in \mathbb{Z}} \dot{S}_{j-1}f\,\dot{\Delta}_j g \quad and \quad T_g' f \triangleq \sum_{j \in \mathbb{Z}} \dot{S}_{j+2}g\,\dot{\Delta}_j f.$$

Using this decomposition and further splitting a and b into low and high frequencies, we get

$$ab = T_{a^\ell}b^\ell + T'_{b^\ell}a^\ell + T'_b a^h + T_a b^h + T'_{b^h}a^\ell + T_{a^h}b^\ell.$$

All the terms in the right-hand side, except for the last two ones, may be bounded by means of standard results of continuity for operators T and T' (see again [1, Chap. 2]). Provided $\sigma \geq s > 0$, we get,

$$\|T'_{b^\ell}a^\ell\|^h_{\dot{\mathbb{B}}^s_{2,1}} \lesssim \|T'_{b^\ell}a^\ell\|^h_{\dot{\mathbb{B}}^\sigma_{2,1}} \lesssim \|b^\ell\|_{L^{p^*}}\|a^\ell\|_{\dot{\mathbb{B}}^\sigma_{p,1}},$$

$$\|T'_b a^h\|_{\dot{\mathbb{B}}^s_{2,1}} \lesssim \|b\|_{L^\infty}\|a^h\|_{\dot{\mathbb{B}}^s_{2,1}}.$$

Let J_1 be the integer corresponding to the threshold between low and high frequencies. Since $a^\ell = \dot{S}_{J_1+1}a$ and $b^h = (\mathrm{Id} - \dot{S}_{J_1+1})b$, we see that

$$T'_{b^h}a^\ell = \dot{S}_{J_1+2}b^h \, \dot{\Delta}_{J_1+1}a^\ell.$$

Consequently, as $\dot{S}_{J_1+2}b^h = (\dot{\Delta}_{J_1-1} + \dot{\Delta}_{J_1} + \dot{\Delta}_{J_1+1})b^h$,

$$\|T_{b^h}a^\ell\|_{\dot{\mathbb{B}}^s_{2,1}} \lesssim \|\dot{\Delta}_{J_1+1}a^\ell\|_{L^\infty}\|\dot{S}_{J_1+2}b^h\|_{L^2} \lesssim \|a\|_{L^\infty}\|b\|^h_{\dot{\mathbb{B}}^s_{2,1}}.$$

Adding up this latter inequality to the previous one and to the symmetric ones (with just operator T instead of T'), and the embeddings $\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1} \hookrightarrow L^\infty$ and, as $p \leq p*$, $\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1} \hookrightarrow L^{p^*}$ completes the proof of (101). $\qquad\square$

To handle commutators in high frequencies, we need the following lemma.

Lemma 5.4. *Let $p \in [2,4]$ and $s > 0$. Define $p^* \triangleq 2p/(p-2)$. For $j \in \mathbb{Z}$, denote $\mathfrak{R}_j \triangleq \dot{S}_{j-1}w \, \dot{\Delta}_j z - \dot{\Delta}_j(wz)$.*

There exists a constant C depending only on the threshold number J_1 between low and high frequencies and on s, p, d, such that

$$\sum_{j \geq J_1} \left(2^{js}\|\mathfrak{R}_j\|_{L^2}\right) \leq C\Big(\|\nabla w\|_{\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}} \|z\|^h_{\dot{\mathbb{B}}^{s-1}_{2,1}} + \|z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}} \|w\|^\ell_{\dot{\mathbb{B}}^{\sigma_1}_{p,1}}$$

$$+ \|z\|_{\dot{\mathbb{B}}^{\frac{d}{p}-k}_{p,1}} \|w\|^h_{\dot{\mathbb{B}}^{s+k}_{2,1}} + \|z\|^\ell_{\dot{\mathbb{B}}^{\sigma_2}_{p,1}} \|\nabla w\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}}\Big),$$

for any $k \geq 0$, $\sigma_1 \geq s$ and $\sigma_2 \in \mathbb{R}$.

Proof. From Bony's decomposition recalled above and the fact that $\dot{\Delta}_j\dot{\Delta}_{j'} = 0$ for $|j - j'| \geq 2$, we deduce that

$$\mathfrak{R}_j = -\dot{\Delta}_j(T'_z w) - \sum_{|j'-j|\leq 4}[\dot{\Delta}_j, \dot{S}_{j'-1}w]\dot{\Delta}_{j'}z - \sum_{|j'-j|\leq 1}\left(\dot{S}_{j'-1}w - \dot{S}_{j-1}w\right)\dot{\Delta}_j\dot{\Delta}_{j'}z$$

$$\triangleq \mathcal{R}^1_j + \mathcal{R}^2_j + \mathcal{R}^3_j.$$

To estimate $\mathcal{R}^1_j$, we use the decomposition

$$T'_z w = T'_{z^\ell}w^\ell + T'_{z^h}w^\ell + T'_z w^h$$

and proceed as in the proof of Proposition 5.5. In the end, we get

$$\|T'_z w\|^h_{\dot{\mathbb{B}}^s_{2,1}} \lesssim \|z\|_{\dot{\mathbb{B}}^{-k}_{\infty,1}}\|w\|^h_{\dot{\mathbb{B}}^{s+k}_{2,1}} + \|z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}}\|w\|^\ell_{\dot{\mathbb{B}}^{\sigma_1}_{p,1}}.$$

Therefore, since $\dot{\mathbb{B}}^{\frac{d}{p}-k}_{p,1} \hookrightarrow \dot{\mathbb{B}}^{-k}_{\infty,1}$,

$$(102) \qquad \sum_{j\in\mathbb{Z}}\left(2^{js}\|\mathcal{R}^1_j\|_{L^2}\right) \lesssim \|z\|_{\dot{\mathbb{B}}^{\frac{d}{p}-k}_{p,1}}\|w\|^h_{\dot{\mathbb{B}}^{s+k}_{2,1}} + \|z\|^\ell_{\dot{\mathbb{B}}^{\frac{d}{p}-\frac{d}{p*}}_{p,1}}\|w\|^\ell_{\dot{\mathbb{B}}^{\sigma_1}_{p,1}}.$$

Next, taking advantage of Lemma 5.2, we see that if $j' \geq J_1$ and $|j - j'| \leq 4$, then we have

$$2^{js}\|[\dot{\Delta}_j, \dot{S}_{j'-1}w]\dot{\Delta}_{j'}z\|_{L^2} \lesssim \|\nabla \dot{S}_{j'-1}w\|_{L^\infty} 2^{j'(s-1)}\|\dot{\Delta}_{j'}z\|_{L^2}$$

while, if $j' < J_1$, $j \geq J_1$ and $|j - j'| \leq 4$,

$$2^{js}\|[\dot{\Delta}_j, \dot{S}_{j'-1}w]\dot{\Delta}_{j'}z\|_{L^2} \lesssim 2^{J_1(s-\sigma_2-1)}2^{j(\sigma_2+1)}\|[\dot{\Delta}_j, \dot{S}_{j'-1}w]\dot{\Delta}_{j'}z\|_{L^2}$$
$$\lesssim 2^{J_1(s-\sigma_2-1)}\|\nabla \dot{S}_{j'-1}w\|_{L^{p*}} 2^{j'\sigma_2}\|\dot{\Delta}_{j'}z\|_{L^p}.$$

Therefore,

$$(103) \qquad \sum_{j \geq J_1}\left(2^{js}\left\|\mathcal{R}_j^2\right\|_{L^2}\right) \lesssim \|z\|_{\dot{\mathbb{B}}_{2,1}^{s-1}}^h \|\nabla w\|_{L^\infty} + \|z\|_{\dot{\mathbb{B}}_{p,1}^{\sigma_2}}^\ell \|\nabla w\|_{L^{p*}}^\ell.$$

Then with suitable embeddings, one gets

$$(104) \qquad \sum_{j \geq J_1}\left(2^{js}\left\|\mathcal{R}_j^2\right\|_{L^2}\right) \lesssim \|z\|_{\dot{\mathbb{B}}_{2,1}^{s-1}}^h \|\nabla w\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}}} + \|z\|_{\dot{\mathbb{B}}_{p,1}^{\sigma_2}}^\ell \|\nabla w\|_{\dot{\mathbb{B}}_{p,1}^{\frac{d}{p}-\frac{d}{p*}}}^\ell.$$

Finally, for all $j \geq J_1$ and $|j' - j| \leq 1$, we have

$$2^{js}\|(\dot{S}_{j'-1}w - \dot{S}_{j-1}w)\dot{\Delta}_j\dot{\Delta}_{j'}z\|_{L^2} \leq 2^j\|\dot{\Delta}_{j\pm1}w\|_{L^\infty} 2^{j(s-1)}\|\dot{\Delta}_{j'}\dot{\Delta}_j z\|_{L^2}$$
$$\leq C\|\dot{\Delta}_{j\pm1}\nabla w\|_{L^\infty} 2^{j(s-1)}\|\dot{\Delta}_j z\|_{L^2}.$$

Hence

$$(105) \qquad \sum_{j \geq J_1}\left(2^{js}\left\|\mathcal{R}_j^3\right\|_{L^2}\right) \leq C\|\nabla w\|_{L^\infty}\|z\|_{\dot{\mathbb{B}}_{2,1}^{s-1}}^h.$$

Putting (102), (104) and (105) together yields the desired estimate. $\qquad \square$

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