Hyperbolic approximation of the Navier-Stokes-Fourier system

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"Critical phenomena in nonlinear partial differential equations, harmonic analysis and functional inequalities"

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• A shortcoming of Fourier's law is that it leads to a parabolic equation for the temperature field: any initial disturbance is felt instantly throughout the entire medium.

→ Such behavior contradicts the principle of causality.

- To correct this unrealistic feature one can use the Maxwell-Cattaneo law:

$$\varepsilon^2 \partial_t q + q = -\kappa \nabla T,$$

where ε is the thermal relaxation characteristic time

- However, this leads to a non-Galilean invariant model. In '09, Christov formulated the following law

$$\varepsilon^2 \left(\partial_t q + u \cdot \nabla q - q \cdot \nabla u + (\nabla \cdot u) q \right) + q = -\kappa \nabla T. \tag{2}$$

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- $\rightarrow$ Finite speed of propagation for the temperature.
- Question: How to justify rigorously the limit $\varepsilon \to 0$?
- Element of response to the *paradox of heat conduction*.
- Useful for numerics.

First-order partially dissipative coupling

Porous media approximation

- The compressible Euler equations with damping reads:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \varepsilon^2(\partial_t u + u \cdot \nabla u) + \dfrac{\nabla P(\rho)}{\rho} + u = 0. \end{cases} \tag{E}$$

This system can be understood as a hyperbolic approximation, as $\varepsilon \to 0$, of the solution of the porous media equation:

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- Weak convergence result in the multi-dimensional case: Coulombel-Goudon-Lin '07 '13, Fang-Xu '09, Kawashima-Xu '14
- Strong convergence in $\mathbb{R}^d$ with $d \geq 1$ for global-in-time strong solutions being small perturbations of $(\bar{\rho}, \bar{u}) = (\bar{\rho}, 0)$ with $\bar{\rho} > 0$: Danchin-CB '22.

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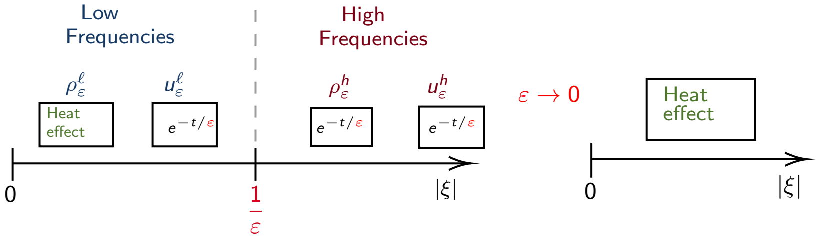
- **Tools**: Littlewood-Paley, Shizuta-Kawashima's theory and hypocoercivity theory.

Cattaneo approximation:

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x u_\varepsilon = 0 \\ \varepsilon^2 \partial_t u_\varepsilon + \partial_x \rho_\varepsilon + u_\varepsilon = 0 \end{cases} \quad \xrightarrow[\varepsilon \to 0]{} \quad \partial_t \rho - \Delta \rho = 0$$



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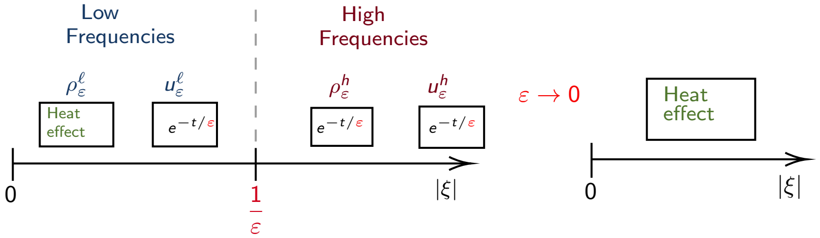

- We proved the strong relaxation limit in $\mathbb{R}^d$ in various contexts
  - Compressible Euler equations with damping (Danchin-CB, Math. Ann.).
  - Jin-Xin System (Shou-CB, JDE).
  - 2D-Boussinesq system (Bianchini-Paicu-CB, ARMA).
- How to show it for the Navier-Stokes-Cattaneo system?

A (partially) hyperbolic Navier-Stokes system

We have just seen that the equation

$$\partial_t u - \Delta u = 0$$

can be approximated, for a small ε, by the following hyperbolic system

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Performing such approximation for the compressible Navier-Stokes system, one has

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div} \tau, \\ \partial_t(\rho T) + \operatorname{div}(\rho u T + up) + \operatorname{div} q - \operatorname{div}(\tau \cdot u) = 0, \\ \varepsilon^2 (\partial_t q + u \cdot \nabla q - q \cdot \nabla u + (\nabla \cdot u) q) + q + \kappa \nabla T = 0, \end{cases} \tag{4}$$

Let us now see how to justify that the solution of this system converges to the solution of the classical Navier-Stokes equations.

- **Knowledge on the limit system:** Danchin showed the existence of global-in-time solutions by highlighting different properties for $|\xi| \leq K$ and $|\xi| \geq K$ where K is a large constant.

Frequency splitting

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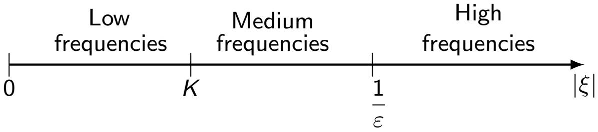

Formally, when $\varepsilon \to 0$, it means that:

- The low frequency regime is not modified.
- The mid-frequency regime becomes larger and larger and recovers the high-frequency regime.
- The high frequency regime disappears.

$\to$ We retrieve the behavior of the compressible Navier-Stokes-Fourier system in the limit.

Tools

- We define homogeneous Besov spaces restricted in frequency as follows:

$$\|f\|^{\ell}_{\dot{B}^s_{2,1}} := \sum_{j\leq J_0} 2^{js}\|f_j\|_{L^2}, \qquad \|f\|^{m,\varepsilon}_{\dot{B}^s_{p,1}} := \sum_{J_0\leq j\leq J_\varepsilon} 2^{js}\|f_j\|_{L^p},$$

$$\|f\|^{h,\varepsilon}_{\dot{B}^s_{2,1}} := \sum_{j\geq J_\varepsilon -1} 2^{js}\|f_j\|_{L^2}$$

  where $J_0 = \log_2(K)$, for $K > 0$ a constant, and $J_\varepsilon = -\kappa\log_2(\varepsilon)$.
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Morale

- The hyperbolic approximation *creates* a temporary high-frequency regime that disappears in the limit.
- The remaining frequency regimes correspond to the behaviour of the limit system.
- Difficulty: justify that the linear and nonlinear analysis can be done in the *new* high-frequency setting.

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Some linear analysis in high frequencies

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• Defining the effective velocity, as introduced by Hoff and Haspot, $w = u + (-\Delta)^{-1}\nabla\rho$, in high frequencies, the linear system we are interested in reads

$$\begin{cases} \partial_t\rho + \rho = \operatorname{div} w, \\ \partial_t w - \Delta w = w - (-\Delta)^{-1}\nabla\rho + \nabla\theta, \\ \partial_t\theta + \operatorname{div} q + \operatorname{div} w = 0, \\ \varepsilon^2\partial_t q + q + \nabla\theta = 0, \end{cases} \tag{5}$$

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• For the Cattaneo part, we introduce the Lyapunov (in the spirit of that of Beauchard and Zuazua and the hypocoercivity theory)

$$\mathcal{L}_j^h = \|(\theta_j, q_j)\|_{L^2}^2 + 2^{-2j}\int_{\mathbb{R}^d} q_j \cdot \nabla\theta_j \quad \text{for } j \geq J_\varepsilon. \tag{6}$$

$\rightarrow$ The blue term allows to recover dissipation for θ. Using that $\mathcal{L}_j^h \sim \|(\theta_j, q_j)\|_{L^2}^2$, direct computations gives

$$\frac{d}{dt}\mathcal{L}_j^h + \mathcal{L}_j^h \leq \|\mathrm{div}\, w_j\|_{L^2}\|\theta_j\|_{L^2}.$$

Some nonlinear analysis

- We are working in the $L^2 - L^p$ framework:

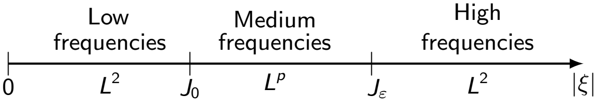

Figure: Frequency domain splitting for Navier-Stokes Cattaneo

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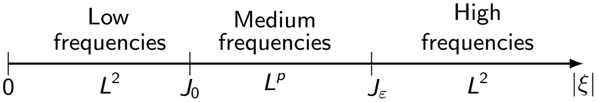

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- Due to the lack of embedding of the type $B^s_{p,1} \hookrightarrow B^s_{2,1}$ if $p > 2 \rightarrow$ it is difficult to absorb nonlinearities in the high and low-frequency regimes.
- Indeed, the medium frequencies are only bounded in L^p-based spaces.
- $\rightarrow$ Need to develop advanced product laws.

For instance: let $2 \leq p \leq 4$ and $p^* \triangleq 2p/(p-2)$. For all $s > 0$, we have

$$\begin{aligned}\|ab\|^{h,\varepsilon}_{\dot B^s_{2,1}} \lesssim{}& \|a\|_{\dot B^{\frac{d}{p}}_{p,1}} \|b\|^{h,\varepsilon}_{\dot B^s_{2,1}} + \|b\|_{\dot B^{\frac{d}{p}}_{p,1}} \|a\|^{h,\varepsilon}_{\dot B^s_{2,1}} \\ &+ \|a\|^{\ell,\varepsilon}_{\dot B^{\frac{d}{p}}_{p,1}} \|b\|^{\ell,\varepsilon}_{\dot B^{s+\frac{d}{p}-\frac{d}{2}}_{p,1}} + \|b\|^{\ell,\varepsilon}_{\dot B^{\frac{d}{p}}_{p,1}} \|a\|^{\ell,\varepsilon}_{\dot B^{s+\frac{d}{p}-\frac{d}{2}}_{p,1}}.\end{aligned}$$

Tools: Bony paraproduct decomposition and precise frequency analysis.

Ill-prepared relaxation result in a critical framework

Theorem (Kawashima-Xu-Zuazua-CB '23)

Let $d \geq 3$, $p \in [2,4]$ and $P(\rho,\theta) = \pi(\rho)\theta$, $\bar{\rho}, \bar{\theta} > 0$

- *Let $(\rho^\varepsilon - \bar{\rho}, v^\varepsilon, \theta^\varepsilon - \bar{\theta}, q^\varepsilon)$ be the global solution of Navier-Stokes-Cattaneo (constructed with the previous arguments) with initial data $(\rho_0^\varepsilon, v_0^\varepsilon, \theta_0^\varepsilon, q_0^\varepsilon)$.*
- *Let $(\rho - \bar{\rho}, v, \theta - \bar{\theta})$ be the global solution of Navier-Stokes-Fourier with initial data (ρ_0, v_0, θ_0).*

We define the error unknowns $(\widetilde{\rho}, \widetilde{v}, \widetilde{\theta})$ as

$$(\widetilde{\rho}, \widetilde{v}, \widetilde{\theta}) := (\rho^\varepsilon - \rho, v^\varepsilon - v, \theta^\varepsilon - \theta).$$

If we assume that

$$\|(\widetilde{\rho}_0, \widetilde{v}_0, \widetilde{\theta}_0)\|^\ell_{B^{\frac{d}{2}-1}_{2,1}} + \|\widetilde{\rho}_0\|^h_{B^{\frac{d}{p}-1}_{p,1}} + \|(\widetilde{v}_0, \widetilde{\theta}_0)\|^h_{B^{\frac{d}{p}-1}_{p,1}} \lesssim \varepsilon. \quad (7)$$

Then, we have the strong convergence result:

$$\|(\widetilde{\rho}, \widetilde{v}, \widetilde{\theta})\|^\ell_{L^\infty_T(B^{\frac{d}{2}-2}_{2,1})} + \|(\widetilde{\rho}, \widetilde{v}, \widetilde{\theta})\|^\ell_{L^1_T(B^{\frac{d}{2}}_{2,1})} + \|q^\varepsilon + \kappa\nabla\theta^\varepsilon\|_{L^1_T(B^{\frac{d}{p}-1}_{p,1})}$$
$$+ \|\widetilde{\rho}\|^h_{L^\infty_T \cap L^1_T(B^{\frac{d}{p}-1}_{p,1})} + \|(\widetilde{v}, \widetilde{\theta})\|^h_{L^\infty_T(B^{\frac{d}{p}-2}_{p,1})} + \|(\widetilde{v}, \widetilde{\theta})\|^h_{L^1_T(B^{\frac{d}{p}}_{p,1})} \lesssim \varepsilon$$

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  With Roberta Bianchini and Marius Paicu (ARMA '23), we showed that the stably stratified solutions of the incompressible porous media equation:

$$\partial_t \rho - \mathcal{R}_1^2 \rho = 0 \quad \text{with } \mathcal{R}_1 = \frac{\partial_1}{\sqrt{-\Delta}}$$

  can be approximated by the 0-th order stratified Boussinesq system:

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  Such justification involves anisotropic Besov spaces so as to recover crucial $L^1_T(W^{1,\infty})$ bounds on the solution.
- Question: under what conditions can an operator be approximated in this fashion?
- Interplay of partial dissipation, anisotropy and special structure of the nonlinearities.

Thank you for your attention!